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THE ESSENCE OF THE VARIATIONAL ITERATION METHOD AND THE HOMOTOPY ANALYSIS METHOD IS DIFFERENT

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Abstract

The recently published paper "The variational iteration method is a special case of the homotopy analysis method" by Robert A. Van Gorder [1], weakly pointed out that the variational iteration method and all of its optimal analogues are specific cases of the more general homotopy analysis method. This assertion was not truly supported by a rigorous mathematical proof, nor by an accessible example from the attributed papers. In this brief, we refute the author's claim by supplementing three simple examples, which do not indicate that the variational iteration method is a special case of the homotopy analysis method. This is justified by a Theorem to compute the rate of convergence of both methods.

1. Introduction

The author of reference [1] claimed that the optimal variational iteration method (VIM) as introduced in [2] can be described completely within the context of the optimal homotopy analysis method (HAM). To be in line with the style and flavor of [1], the author aims concisely at solving the ordinary differential equation

$$L[u] + N[u] = f(x), \tag{1}$$

where $u = u(x)$ is some unknown function to be determined, L is a linear differential operator, N is a nonlinear differential operator, and $f(x)$ is a given source term, or, inhomogeneity. Within the traditional HAM approach, the author approximates the unknown function u with the M-th-order $(M+1)$-term truncated series

$$\hat{u}_M(x) = u_0(x) + \sum_{n=1}^{M} u_n(x), \tag{2}$$

where $u_0(x)$ is the initial guess of the solution satisfying the restrictions due to boundaries and it is well-established that the following linear deformations yield the unknown functions $u_n(x)$ in (2)

$$L[u_0] = 0,$$

$$L[u_{n+1} - u_n] = hH(x)(L[u_n] + N_n[\hat{u}_n] - \delta_{0,\,n} f(x)), \quad n = 0, 1, 2, \cdots, M. \tag{3}$$

Detailed information about h, $H(x)$, N_n and $\delta_{0,n}$ is fully given in [1] and can be found in every HAM applications. We should remark that there is now a huge literature on the use and applications of both methods for solving linear and non-linear problems. However, the motivation here is to focus on the claim of [1] in order to reveal that both methods have different logic of working, hence a comprehensive bibliography is not given. The readers can consult to the renowned book by Liao [3] as well as the recent relevant publications [4], [5, 6, 7, 8] and [9, 10], among many others.

By inverting (3) the author of [1] finds

$$u_{n+1}(x) = (1 - \delta_{0,n})u_n(x) + L^{-1}[hH(x)(L[u_n(x) + N_n[\hat{u}_n(x)] - \delta_{0,n}f(x))]. \tag{4}$$

In the very special case when $L = \frac{d}{dx}$, the author obtains

$$u_{n+1}(x) = u_n(x) + h\int_{x_0}^{x} H(y)(L[u_n(y) + N_n[\hat{u}_n(y)] - \delta_{0,n}f(y))]dy. \tag{5}$$

Having appropriately summing over (4), the author then imposes his final assertion that the obtained (see Equation (9) in [1])

$$\hat{u}_{n+1}(x) = \hat{u}_n(x) + h\int_{x_0}^{x} H(y)(L[\hat{u}_n(y) + N[\hat{u}_n(y)] - f(y))dy, \tag{6}$$

is exactly the optimal variational iteration method [2], if $H(x)$ is set to $\lambda(x)$ of Lagrange multiplier used in the formulation of classical VIM.

It is worth noting that the HAM splits the nonlinear problem into the sub-problems and solves them, and then the solution of the original problem would be the sum of solutions to the sub-problems, while the VIM is a kind of fixed-point iteration method and it obtains the full solution by improving it in each step. Therefore, the essence of these two methods is different as the title reflects.

Besides, it is true that the formula in (6) represents formally the original VIM introduced by He [11] and the optimal variational iteration method derived from a different context by Turkyilmazoglu in [2], provided that $H(x)$ is substituted by the Lagrange multiplier $\lambda(x)$. However, the steps that the author of [1] adhered to reach (6) from the homotopy concept through (2) to (4) are strictly wrong. Indeed, $N[\hat{u}_n]$ in (6) can not be simply obtained by summing over (4), since $N_n[\hat{u}_n]$ demands derivatives around the embedded parameter in the HAM. The worse, $N[\hat{u}_n]$ in (6) is a fully nonlinear operator in the VIM representation, but it is obtained as a result of linearizing in the HAM analysis of [1]. Furthermore, the claim of the author that $H(x)$ plays the role of Lagrange multiplier $\lambda(x)$ is in general false, since $\lambda(x)$ is determined from the variational iteration formula considering the small element concept and it eventually ends up with two variables, but written as a single variable function in formal representation of the VIM, which can be conceived from the bibliographic sources on the VIM, refer also to the below examples.

In fact the rate of convergence of both HAM and VIM can be controlled by the presence of convergence control parameter h in it. The following Theorem assures this.

Theorem. *Let the M-th-order approximate solution to* (1) *be given by* (2). *Consider the squared residual error*

$$Res(h) = \| N[\hat{u}] + N[\hat{u}] - f \| = \int_D (N[\hat{u}] + N[\hat{u}] - f)^2 dx, \tag{7}$$

where the norm is in the functional space L^2 and D is the domain of interest. Then, the rate of convergence of both methods can be adjusted by optimizing the squared residual error in (7).

Proof. Substituting (2) in (1), and integrating over the domain of interest D will result in a polynomial equation in h as given via (7). The best choice of h, that is the optimal value of convergence control parameter can be selected by minimizing (7), that is, by solving the algebraic equation

$$\frac{dRes(h)}{dh} = 0. \tag{8}$$

This completes the proof. □

2. Examples

We should point out that if the variational iteration method was a special case of HAM, then both schemes would yield the same results at each complete iterations provided that the same auxiliary conditions are imposed. We shall illustrate that the assertion of the author in [1] is totally wrong on three basic examples, whose exact solutions are $u(x) = \tanh x$ and e^{-x}. Full details are given here for the readers to pursue closely.

Example 1 (Example in [2]).

Let us consider the first-order ordinary differential equation from [2]

$$u' + u^2 = 1, \quad u(0) = 0, \quad 0 \leq x \leq 2. \tag{9}$$

Because the Lagrange multiplier corresponding to (9) is $\lambda(x) = 1$, the variational iteration formula for (9) is constructed with $u_0(x) = 0$ by

$$u_{n+1}(x) = u_n(x) + h\int_0^x (u_n'(\tau) + u_n^2(\tau) - 1)d\tau. \tag{10}$$

So, the M-th-order $(M+1)$-term approximate solution of (9) from the iterative algorithm (10) can be listed as

$$\hat{u}_0(x) = 0,$$

$$\hat{u}_1(x) = -hx,$$

$$\hat{u}_2(x) = -h(2+h)x + \frac{h^3x^3}{3},$$

$$\hat{u}_3(x) = -h(3+h(3+h))x + \frac{1}{3}h^3(5+h(5+h))x^3 - \frac{2}{15}h^5(2+h)x^5 + \frac{h^7x^7}{63},$$

$$\hat{u}_4(x) = -h(2+h)(2+h(2+h))x + \frac{1}{3}h^3(14+h(28+h(21+h(7+h))))x^3$$
$$-\frac{2}{15}h^5(17+h(33+h(24+h(8+h))))x^5$$
$$+\frac{1}{315}h^7(2+h)(101+h(131+h(52+5h)))x^7$$
$$-\frac{2h^9(155+h(225+h(103+14h)))x^9}{2835}$$
$$+\frac{2h^{11}(293+h(293+67h))x^{11}}{51975} - \frac{4h^{13}(2+h)x^{13}}{12285} + \frac{h^{15}x^{15}}{59535},$$
$$\vdots \quad , \quad n \le M. \tag{11}$$

On the other hand, the homotopy analysis method with the linear operator $L = \frac{d}{dx}$, initial guess $u_0(x) = 0$ and auxiliary function $H(x) = 1$ for the system (9) results in the subsequent M-th order $(M+1)$-term approximations

$$\hat{u}_0(x) = 0,$$

$$\hat{u}_1(x) = -hx,$$

$$\hat{u}_2(x) = -h(2+h)x,$$

$$\hat{u}_3(x) = \frac{1}{3}hx(-9+h(-9+h(-3+x^2))),$$

$$\hat{u}_4(x) = \frac{1}{3}(-3h(2+h)(2+h(2+h))x + h^3(4+3h)x^3),$$

$$\vdots \quad , \quad n \le M. \tag{12}$$

It is clearly witnessed from the VIM and HAM approximations (11) and (12) that, even though the same operating conditions in both methods are assigned, except the leading and first-order approximates both methods generate totally different approximations; variational method produces much more terms of higher degree within it as compared to the HAM. From this perspective, it may be possible to say that the HAM is a special case of the VIM, but this statement demands further mathematical justification in rigor which is out of the present scope. Actually, by means of the sequared residual error formula

$$Res(h) = \int_0^2 (\hat{u}'_M(x) + \hat{u}_M^2(x) - 1)^2 dx, \tag{13}$$

in accordance with the Theorem provided in Section 1, it is possible to determine the best (optimal) value of h via minimizing (13) at any specified approximation level M, which will certainly control the rate of convergence of the method under consideration. Table 1 tabulates the optimums obtained from the VIM and HAM methods. It is seen that both methods are successful to compute optimal values of h, but they are absolutely different. The VIM seems to be more effective for the current problem, see also Figure 1(a)-(b). It is not surprising that the VIM performs better since it contains much more terms in it as compared to the HAM. Hence, the assertion of the author of [1] fails in this case.

Table 1. The residual errors and optimum values of h versus truncation level M for the VIM and HAM methods for the problem in (9)

M	$\sqrt{Res}$(VIM)	h(VIM)	$\sqrt{Res}$(HAM)	h(HAM)
1	0.4827252177	– 0.5046319040	0.4827252177	– 0.5046319041
2	0.2353078545	– 0.6721397754	0.4827252177	– 0.2961760902
3	0.1342525787	– 0.5651778303	0.4656195031	– 0.2183455346
4	0.0297610101	– 0.6896249726	0.3037392098	– 0.5225662114
5	0.0062554390	– 0.7625555019	0.2612325033	– 0.4259329813
6	0.0011949685	– 0.8113743764	0.2310331108	– 0.3648852428

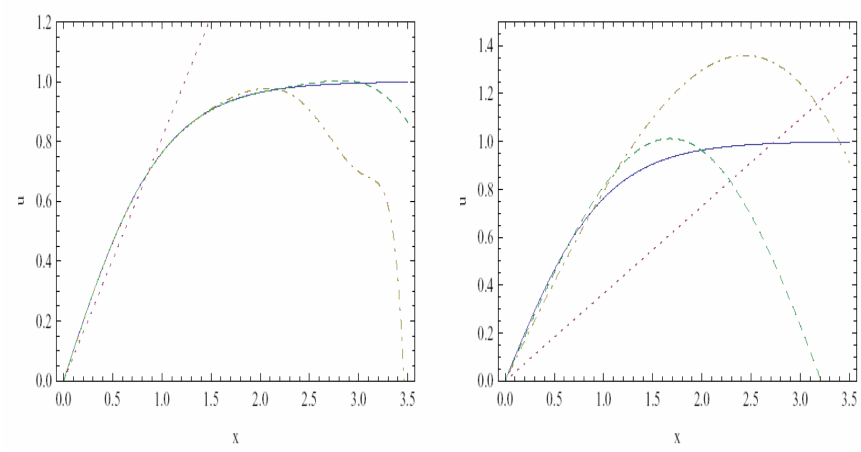

Figure 1. The approximate solutions; first-order (dotted), fourth-order (dot-dashed), sixth-order (dashed), and exact (unbroken) (a) VIM and (b) HAM. All drawn with the optimum h evaluated at the sixth-order approximate solution.

Example 2.

Let us consider the second-order ordinary differential equation

$$u'' + 2uu' = 0, \quad u(0) = 0, \quad u'(0) = 1, \quad 0 \leq x \leq 2. \tag{14}$$

Because the Lagrange multiplier for (14) is $\lambda(x, t) = x - t,$ the variational iteration formula for (14) is constructed with $u_0(x) = x$ through

$$u_{n+1}(x) = u_n(x) + h\int_0^x (\tau - x)(u_n''(\tau) + 2u_n(\tau)u_n'(\tau))d\tau. \tag{15}$$

Consequently, the M-th order $(M+1)$-term approximate solution of (14) from the

$\hat{u}_0(x) = x,$

$\hat{u}_1(x) = x - \frac{hx^3}{3},$

$\hat{u}_2(x) = x + \frac{1}{3}(-2+h)hx^3 + \frac{2h^2x^5}{15} - \frac{h^3x^7}{63},$

$$\hat{u}_3(x) = x - \frac{1}{3}h(3 + (-3 + h)h)x^3 + \frac{2}{15}(3 - 2h)h^2x^5 - \frac{1}{315}h^3(37 + 5(-5 + h)h)x^7$$

$$+ \frac{2(33 - 14h)h^4x^9}{2835} + \frac{2h^5(-92 + 25h)x^{11}}{51975} + \frac{4h^6x^{13}}{12285} - \frac{h^7x^{15}}{59535},$$

$$\hat{u}_4(x) = x + \frac{1}{3}(-2 + h)h(2 + (-2 + h)h)x^3 + \tfrac{2}{15}h^2(6 + h(-8 + 3h))x^5$$

$$- \frac{1}{315}h^3(118 + h(-176 + 5h(21 + (-7 + h)h)))x^7$$

$$+ \frac{2h^4(196 + h(-282 + (145 - 28h)h))x^9}{2835}$$

$$- \frac{2h^5(3405 + h(-4793 + 3h(828 + 25(-8 + h)h)))x^{11}}{155925}$$

$$+ \frac{4h^6(17757 - 2h(11613 + 880(-6 + h)h))x^{13}}{6081075}$$

$$- \frac{h^7(1719930 + 13h(-157887 + h(64111 + 75h(-138 + 11h))))x^{15}}{638512875}$$

$$+ \frac{2h^8(2868831 - 2h(1488437 + 25h(-19391 + 2002h)))x^{17}}{10854718875}$$

$$+ \frac{2h^9(-4791843 + 5h(836151 + 325h(-661 + 54h)))x^{19}}{109185701625}$$

$$+ \frac{4h^{10}(122001 + 2h(-41989 + 7025h))x^{21}}{40226311125}$$

$$- \frac{2h^{11}(11666569 + 25h(-234653 + 21515h))x^{23}}{16962094524375}$$

$$- \frac{4h^{12}(-93147 + 28910h)x^{25}}{3016973334375} + \frac{4h^{13}(-29408 + 4225h)x^{27}}{14119435204875}$$

$$+ \frac{8h^{14}x^{29}}{21210236775} - \frac{h^{15}x^{31}}{109876902975},$$

$$\vdots \qquad , \quad n \le M, \tag{16}$$

iterative algorithm (15) can be listed as above.

On the other hand, the HAM with the linear operator $L = \frac{d^2}{dx^2}$, initial guess $u_0(x) = x$ and auxiliary function $H(x) = x$ for the system (14) (to comply with the same input parameters as the VIM) results in the subsequent $(M+1)$-term approximations

$$\hat{u}_0(x) = x,$$

$$\hat{u}_1(x) = x + \frac{hx^4}{6},$$

$$\hat{u}_2(x) = x + \frac{hx^4}{3} + \frac{h^2x^5}{10} + \frac{5h^2x^7}{126},$$

$$\hat{u}_3(x) = \frac{1}{840}x(840 + hx^3(420 + hx(252 + x(100x + h(56 + 43x^2 + 8x^4))))),$$

$$\hat{u}_4(x) = x + \frac{2hx^4}{3} + \frac{3h^2x^5}{5} + \frac{4h^3x^6}{15} + \frac{1}{21}h^2(5 + h^2)x^7 + \frac{43h^3x^8}{210} + \frac{19h^4x^9}{360} + \frac{4h^3x^{10}}{105} + \frac{291h^4x^{11}}{15400} + \frac{671h^4x^{13}}{294840},$$

$$\vdots \quad , \quad n \le M. \tag{17}$$

It is again witnessed from the approximations (16) and (17) that apart from the leading-order terms both methods generate totally different approximations. On the other hand, if the assertion of [1] was correct, then both approximations should coincide at each complete iteration. Table 2 tabulates the optimums obtained from VIM and HAM methods for the current problem. The residuals are seen to rapidly decrease in both methods with strictly distinct values of optimum h. The optimal variational iteration method seems to be again more effective for the current problem, see also Figure 2(a)-(b). Hence, the assertion of the author of [1] again is defeated. We should also emphasize that, despite the fact that Examples 1 and 2 yield the same solutions, the rate of convergence to the unique solution is apparently different from both approaches owing to the selection of different auxiliary variables.

In fact, with the unnecessary choice of $H(x) = x$ in the present problem in an expense to imitate the VIM, the author in [1] degrades performance of the HAM.

Table 2. The residual errors and optimum values of h versus truncation level M for the VIM and HAM methods for the problem in (14)

M	$\sqrt{Res}$(VIM)	h(VIM)	$\sqrt{Res}$(HAM)	h(HAM)
1	0.8705083964	0.2635626812	1.3064445689	− 0.1957239309
2	0.1838178024	0.6614913717	0.7158822677	− 0.2787591293
3	0.1372097665	0.5308949096	0.4289362043	− 0.3246195038
4	0.0350289084	0.7782148025	0.2981486116	− 0.3473771209
5	0.0090832494	0.7318050569	0.2240860298	− 0.3653169987

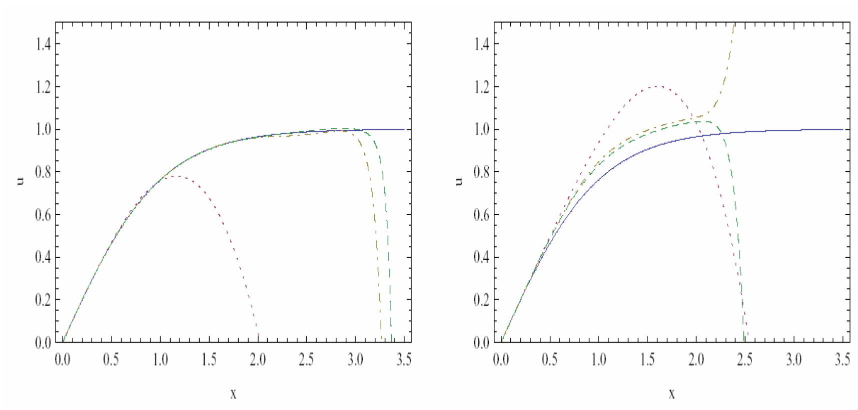

Figure 2. The approximate solutions; first-order (dotted), fourth-order (dot-dashed), fifth-order (dashed), and exact (unbroken) (a) VIM and (b) HAM. All drawn with the optimum h evaluated at the fifth-order approximate solution.

Instead, if it was chosen as $H(x) = 1$, as traditionally assumed unless it requires a functional input), the resulting homotopy series approximates would be

$$\hat{u}_0(x) = x,$$

$$\hat{u}_1(x) = x + \frac{hx^3}{3},$$

$$\hat{u}_2(x) = x + \frac{1}{3}h(2+h)x^3 + \frac{2h^2x^5}{15},$$

$$\hat{u}_3(x) = x + \frac{1}{3}h(3+h(3+h))x^3 + \frac{2}{15}h^2(3+2h)x^5 + \frac{17h^3x^7}{315},$$

$$\hat{u}_4(x) = x + \frac{1}{3}h(2+h)(2+h(2+h))x^3 + \frac{2}{15}h^2(6+h(8+3h))x^5$$

$$+ \frac{17}{315}h^3(4+3h)x^7 + \frac{62h^4x^9}{2825},$$

$$\vdots \quad , \quad n \leq M, \tag{18}$$

and the resulting residual errors and optimal values of h would be as tabulated in Table 3, see also Figure 3.

Table 3. The residual errors and optimum values of h versus truncation level M for the HAM method for the problem in (14)

M	$\sqrt{Res}$	h
1	0.87050839638044853442	− 0.26356268121860331200
2	0.33458931654425749765	− 0.36533167265687641958
3	0.11612143744671425559	− 0.40815065178092264770
4	0.04628594057704908310	− 0.42546670353310746439
5	0.01889523354837579799	− 0.43990241470518093018

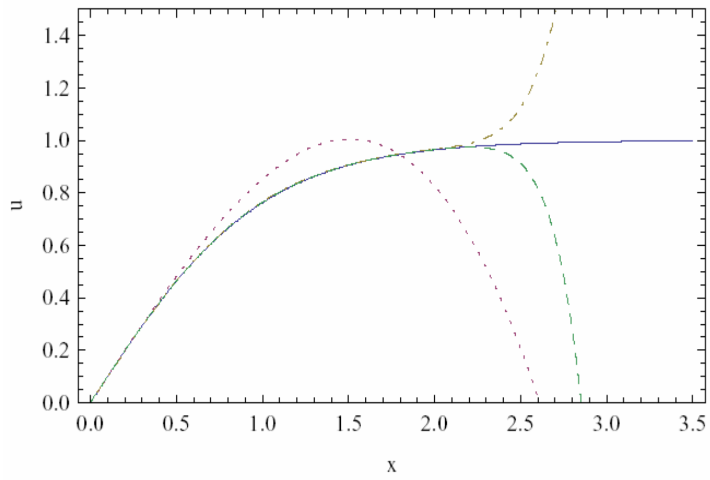

Figure 3. The approximate solutions from HAM; first-order (dotted), fourth-order (dot-dashed), fifth-order (dashed), and exact (unbroken). All drawn with the optimum h at the fifth-order approximate solution.

Example 3.

Finally, we consider the third-order nonlinear ordinary differential equation

$$u''' + u'' + e^{-x}u' + u^2 = 0,$$

$$u(0) = 1, \quad u'(0) = -1, \quad u''(0) = 1, \quad 0 \leq x \leq 1. \tag{19}$$

$\lambda(x, t) = (x - t)^2$ is the corresponding Lagrange multiplier for (19) (the factor ½ can always be absorbed into the convergence control parameter h), and choosing the initial input as $u_0(x) = 1 - x + \frac{x^2}{2}$, we have the following variational iteration formula:

$$u_{n+1}(x) = u_n(x) + h\int_0^x (\tau - x)^2 (u_n'''(\tau) + u_n''(\tau) + e^{-\tau}u_n'(\tau) + u_n^2(\tau))d\tau. \tag{20}$$

The variational iteration formula (20) produces the subsequent approximations up to the approximation level M

$$\hat{u}_0(x) = \frac{1}{2}(2 + (-2 + x)x),$$

$$\hat{u}_1(x) = 1 - x + \frac{x^2}{2} + h\Big(4 - 2x - 2e^{-x}(2 + x),$$

$$+ \frac{1}{420}x^3(280 + x(-70 + x(28 + (-7 + x)x)))\Big),$$

$$\vdots \quad , \quad n \le M. \tag{21}$$

The same operating constraints for the problem (19) by means of the HAM can be accomplished by selecting the auxiliary parameters as

$$L = \frac{d^3}{dx^3}, \quad u_0(x) = 1 - x + \frac{x^2}{2}, \quad H(x) = x^2.$$

Hence, one obtains the HAM approximations

$$\hat{u}_0(x) = \frac{1}{2}(2 + (-2 + x)x),$$

$$\hat{u}_1(x) = \frac{1}{10080}e^{-x}(-10080h(48 + x(30 + x(8 + x)))$$

$$+ e^x(5040(2 + (-2 + x)x) + h(483840 + x(-181440$$

$$+ x(20160 + x^3(336 + x(-168 + x(96 + 5(-6 + x)x))))))))),$$

$$\vdots \quad , \quad , \quad n \le M.$$

It is anticipated again that apart from the first approximations both VIM and HAM generate absolutely distinct analytic approximate expressions for the solution of third-order differential equation (19), that contradicts with the assertion of [1]. Different behaviours of both solutions at different approximation levels can also be observed in Figures 4(a)-(b). In this example, imposition the particular form of auxiliary function H (in an expense to make it look like the form of VIM) is also believed to be the source of relatively bad performance of the HAM method.

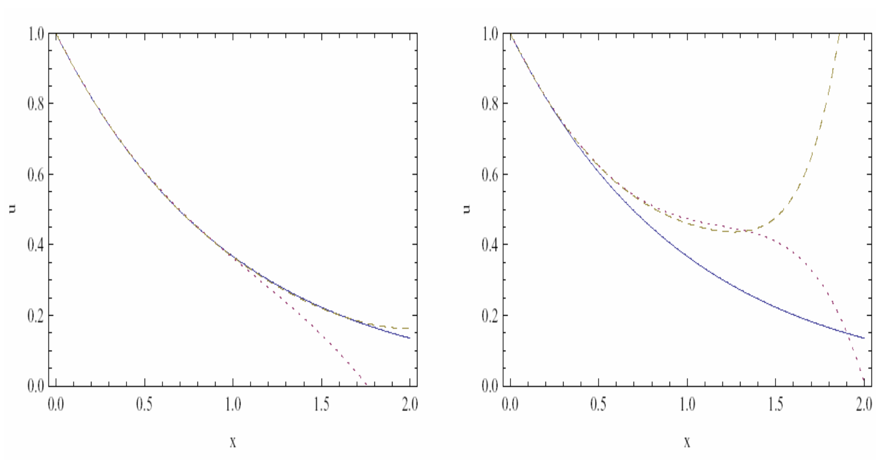

Figure 4. The approximate solutions; first-order (dotted), second-order (dashed), and exact (unbroken) (a) VIM and (b) HAM. All drawn with the optimum h evaluated at the second-order approximate solution.

With these three simple problems, the assertion posed in [1] is hence completely rebutted. It is not understood why the author of [1] did not use such simple examples, at least that easily accessible of [2] to justify his claims. Besides, the careful anonymous reviewers could direct him to do so.

To conclude, it is obvious from the current analysis that the HAM and VIM give rise to approximate solutions that converge at different rates, since the essence of the variational iteration method and the homotopy analysis method is absolutely different. This is in line with the fact that different choices of HAM terms also give different rates of convergence. It is sure that presuming that they are convergent, both techniques must converge to the same solution in the limit where the number of terms goes to infinity. On the other hand, it is mathematically incorrect to interpret that the VIM is a special case of the HAM or vice versa. This assertion strictly requires special restrictions and further mathematical analysis.

3. Concluding Remarks

It is demonstrated in the present work that the variational iteration method is not a special case of the homotopy analysis method as claimed in the recent publication [1]. In fact, the provided basic examples clarify the point that imitating the form of HAM to resemble that of VIM unnecessarily deteriorates the accuracy of HAM. It is worthy to emphasize that although the VIM and HAM methods are different in nature as approved here, and the examples provided here show better performance up to the order of approximations computed, the VIM has the unfortunate deficiency that evaluating integrals for increasing approximations becomes a tedious task, whereas the HAM can generate much higher-order approximate solutions from which better optimums can be gained leading to faster reduction in the residual error formulas. Furthermore, the HAM can operate with more general auxiliary operators/variables, the benefits over the VIM need to be investigated further. The analysis is supported via a Theorem signifying to the rate of convergence of the methods.

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