A PARTIAL PROOF OF THE ERDŐS-SZEKERES CONJECTURE FOR HEXAGONS

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Abstract

Erdős and Szekeres [5] made the conjecture that, for \( n \geq 3 \), any set of \( 2^{n-2} + 1 \) points in the plane, in general position, contains \( n \) points in convex position. A computer-based proof of this conjecture for \( n = 6 \) appeared in [9] of Peters and Szekeres. The aim of this paper is to give a partial proof of the conjecture for \( n = 6 \), without the use of computers, for the special case when the convex hull of the point set is a pentagon.
1. Introduction

In the early 1930s, Esther Klein asked whether there is an integer $N$, for every $n \geq 3$, such that any planar set of $N$ points in general position contains $n$ points in convex position. Erdős and Szekeres [5] showed the existence of such an integer, and also that there is a solution satisfying $N \leq \left(\frac{2n-4}{n-2}\right) + 1$. This problem is well-known as the "happy ending problem".

The task that arose naturally was to find the smallest value $g(n)$ of $\text{card } S$ with the mentioned property for each $S$. In [5], the authors made the following conjecture.

**Conjecture 1** (Erdős-Szekeres Conjecture). Let $n \geq 3$. Then the smallest number $g(n)$ such that every planar set of $g(n)$ points in general position contains $n$ points in convex position, is $2^{n-2} + 1$.

In [6], Erdős and Szekeres constructed a planar set of $2^{n-2}$ points in general position that does not contain $n$ points in convex position. Presently, the best known bounds are

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 1.$$  

The upper bound is due to Tóth and Valtr [10].

Another attempt is to verify the conjecture for small values of $n$. Note that three points in general position are in convex position. Thus, clearly $g(3) = 3$. The value of $g(4)$ was determined by Esther Klein in the early 1930s.

According to [9], Makai was the first to prove the equality $g(5) = 9$, but he has never published his result. The first published proof appeared in 1970 in [7]. In 1974, Bonnice [2] gave a simple and elegant proof of the same result. In [1], Bisztriczky and Tóth also mention an unpublished proof by Böröczky and Stahl.

The case $n = 6$ seems considerably more complicated. Bonnice [2] makes the following comparison. In a set of nine points, we have $\binom{9}{5}$ =
126 possibilities for five points to be in convex position, whereas in a set of seventeen points, we have \( \binom{17}{6} = 12376 \) possibilities for six points to be in convex position.

For this case, a computer-based proof has been given recently by Peters and Szekeres [9]. In their paper, they used a computer to re-prove the case \( n = 5 \). They remark that to prove the \( n = 5 \) case required less than one second “using a 1.5 GHz workstation”, whereas, for the case of convex hexagons, “the total computing time . . . was approximately 3,000 GHz hours”. For other results related to the Erdős-Szekeres conjecture, the reader is referred to [3] or [8].

In this paper, we examine the \( n = 6 \) case of the Erdős-Szekeres conjecture without the use of computers. If \( S = \{a_i : i = 1, 2, \ldots, k\} \) is a finite point set in \( \mathbb{E}^2 \), we denote the convex hull of \( S \) by \([S]\) or by \([a_1, a_2, \ldots, a_k]\). For \( S_1, S_2, \ldots, S_k \subset \mathbb{E}^2 \), we set \([S_1, S_2, \ldots, S_k] = [S_1 \cup S_2 \cup \ldots \cup S_k]\). By \( V(P) \), we denote the vertex set of the convex polygon \( P \). Our main result is the following theorem.

**Theorem 1.** Let \( S \subset \mathbb{E}^2 \) be a set of seventeen points in general position and \( P = [S] \) be a pentagon. Then \( S \) contains six points in convex position.

We note that a different proof of the same statement appeared in the diploma thesis [4] of one of the authors, Dehnhardt, in 1981.

Figure 1. A point \( b \) beyond exactly three edges of \( P \).
There are two known sets of sixteen points in general position that do not contain the vertices of a convex hexagon: cf. [6] and pp. 331-332 of [3]. We note that in both examples, the convex hulls of the points are pentagons. We present the proof of Theorem 1 in the next section. We note that, using the same tools, it may be shown that every planar set of twenty five points in general position contains six points in convex position. We also observe that, by Lemma 4, our proof yields that if $S \subset \mathbb{R}^2$ is a set of seventeen points in general position, $P = [S]$ is a triangle or a quadrangle, $Q = [S \setminus V(P)]$ is a pentagon, and $R = [S \setminus (V(P) \cup V(Q))]$ is a triangle, then $S$ contains six points in convex position. Thus, according to the classification of planar point sets introduced by Bonnice in [2], our proof yields the Erdős-Szekeres hexagon conjecture for twenty four classes of point sets out of seventy two. We remark that in [2], it is stated incorrectly that the number of classes is seventy. The correct number (and the list of the classes) can be found, for example, in [4].

In the proof, for two distinct points $a, b \subset \mathbb{R}^2$, $[a, b], L(a, b), L^+(a, b)$ and $L^-(a, b)$ denote, respectively, the closed segment with endpoints $a$ and $b$, the line containing $a$ and $b$, the closed ray emanating from $a$ and containing $b$, and the closed ray emanating from $a$ in $L(a, b)$ that does not contain $b$. Furthermore, if $s \geq 3$, $P$ is a convex $s$-gon, and a point $b \in \mathbb{R}^2$ does not lie on any sideline of $P$ and is strictly separated from $P$ by exactly $m$ sidelines of $P$, we say that $b$ is beyond exactly $m$ edges of $P$ (cf. Figure 1). If these sidelines are the lines passing through the edges $E_1, E_2, \ldots, E_m$ of $P$, we may say that $b$ is beyond exactly the edges $E_1, E_2, \ldots, E_m$ of $P$. For simplicity, a $k$-gon means a convex $k$-gon for every $k \geq 3$, and if a set contains six points in convex position, we say that it contains a hexagon.

2. Proof of Theorem 1

We begin the proof with a series of lemmas.

**Lemma 1.** Let $P$ and $Q$ be polygons with $Q \subset \text{int } P \subset \mathbb{R}^2$. Let $X \subset V(P)$ be a set of points beyond exactly the same edge of $Q$. Then $V(Q) \cup X$ is a set of points in convex position (cf. Figure 2).
Lemma 2. Let \( \{P_i : i = 1, 2, \ldots, m\} \) be a family of \( t \) triangles, \( q \) quadrangles, and \( p \) pentagons such that \( p + q + t = m \) and \( M = [P_1, P_2, P_3, \ldots, P_m] \) be an \( m \)-gon \( [x_1, x_2, x_3, \ldots, x_m] \). Suppose that \( [x_i, x_{i+1}] \) is an edge of \( P_i \), and \( P_i \) and \( P_{i+1} \) have disjoint interiors for \( i = 1, 2, 3, \ldots, m \). Let \( P_0 \) be a \( k \)-gon that contains \( M \) in its interior and assume that the points of \( W = \bigcup_{i=0}^{m} V(P_i) \) are in general position. If \( q + 2t < k \), then \( W \) contains a hexagon.

Proof. Let us denote by \( X_i \) the set of points that are beyond exactly the edge \( [x_i, x_{i+1}] \) of \( P_i \), and observe that every vertex of \( P_0 \) is contained in \( X_i \) for some value of \( i \). If \( \text{card}(X_i \cap V(P_0)) + \text{card}(V(P_i)) \geq 6 \) for some \( P_i \), then the assertion follows from Lemma 1 (cf. Figure 3).

Thus, we may assume that \( \text{card}(X_i \cap V(P_0)) \) is at most two, if \( P_i \) is a triangle, at most one if \( P_i \) is a quadrangle, and zero if \( P_i \) is a pentagon, which yields that \( k = \text{card}(V(P_0)) \leq 0 \cdot p + 1 \cdot q + 2 \cdot t \), a contradiction. \( \square \)

We use Lemma 2 often during the proof with \( k = 5 \). For simplicity, in such cases we use the notation \( P_1 * P_2 * \ldots * P_m \).

Lemma 3. Let \( S \subset \mathbb{E}^2 \) be a set of eleven points in general position such that \( P = [S] \) is a pentagon, \( Q = [S \setminus V(P)] \) is a triangle, and
[S \ (V(P) \cup \{q\})] is a quadrilateral for every q \in V(Q). Then S contains a hexagon.

**Proof.** Note that as card S = 11, P is a pentagon, and Q is a triangle, \( R = [S \setminus (V(P) \cup V(Q))] \) is a triangle. Let \( Q = [q_1, q_2, q_3] \) and \( R = [r_1, r_2, r_3] \). Observe that for any \( i \neq j \), the straight line \( L(r_i, r_j) \) strictly separates the third vertex of \( R \) from a unique vertex of \( Q \). We may label our points in a way that \( q_1, q_2, \) and \( q_3 \) are in counterclockwise cyclic order, and \( L(r_i, r_j) \) separates \( r_k \) and \( q_k \) for any \( i \neq j \neq k \neq i \). Let us denote by \( Q_k \) the open convex domain bounded by \( L^-(q_k, r_i) \) and \( L^-(q_k, r_j) \) for every \( i \neq j \neq k \neq i \). For every \( i \neq j \), let \( Q_{ij} \) denote the open convex domain that is bounded by the rays \( L^-(q_i, r_j), L^-(q_j, r_i) \), and the segment \([q_i, q_j]\) (cf. Figure 4).

![Figure 4](image)

Observe that if \( Q_{12} \) contains at least two vertices of \( P \), then these vertices together with \( q_1, q_2, r_1, \) and \( r_2 \) are vertices of a hexagon. Similarly, if \( Q_1 \cup Q_{13} \cup Q_3 \) contains at least three vertices of \( P \), or \( Q_2 \cup Q_{23} \cup Q_3 \) contains at least three vertices of \( P \), then \( S \) contains a hexagon. Since \( P \) is a pentagon, we may assume that \( Q_{12} \) contains one, \( Q_1 \cup Q_{13} \) and \( Q_2 \cup Q_{23} \) both contain two, and \( Q_3 \) contains no vertex of \( P \). By
symmetry, we obtain that $S$ contains a hexagon unless $\text{card}(Q_i \cap V(P)) = 0$, and $\text{card}(Q_{ij} \cap V(P)) = 1$ for every $i \neq j$. Since the latter case contradicts the condition that $P$ is a pentagon, $S$ contains a hexagon. \hfill $\square$

**Lemma 4.** Let $S \subset \mathbb{E}^2$ be a set of thirteen points in general position such that $P = [S]$ is a pentagon and $Q = [S \setminus V(P)]$ is a triangle. Then $S$ contains a hexagon.

**Proof.** Let $q_1, q_2$, and $q_3$ be the vertices of $Q$ in counterclockwise cyclic order and let $R = S \setminus (V(P) \cup V(Q))$. Observe that $\text{card } R = 5$.

Using an idea of Klein and Szekeres, we obtain that $R$ contains an empty quadrilateral. In other words, there is a quadrilateral $U$ that satisfies $V(U) \subset R$ and $U \cap R = V(U)$. Let $r_1, r_2, r_3, \text{ and } r_4$ be the vertices of $U$ in counterclockwise cyclic order, and let $r$ be the remaining point of $R$.

We show that if $U$ has no sideline that separates $U$ from an edge of $Q$, then $S$ contains a hexagon. Indeed, if every sideline of $U$ separates $U$ from exactly one vertex of $Q$, then, by the pigeon-hole principle, $Q$ has a vertex, say $q_3$, such that at least two sidelines of $U$ separate $U$ from it. This yields that there are two sidelines passing through consecutive edges of $U$ that separate $U$ from only $q_3$. Let these edges be $[r_{i-1}, r_i]$ and $[r_i, r_{i+1}]$. Then we have $[q_1, r_{i+1}, r_i, r_{i-1}, q_2] \ast [q_2, r_{i-1}, q_3] \ast [q_3, r_{i+1}, q_1]$. Hence, we may assume that $U$ has a sideline that separates $U$ from an edge of $Q$. Without loss of generality, let this sideline pass through the edge $[r_1, r_2]$ and let it separate $U$ from $[q_1, q_2]$. 

![Figure 5](image-url)
For every $3 \neq i \neq j \neq 3$, let $x_i, y_i$, and $z_i$ denote the intersection point of the segment $[q_i, q_3]$ with the line $L(q_j, r_j)$, $L(q_j, r_i)$, and $L(r_i, r_j)$, respectively, and let $w_i$ denote the intersection point of $[r_i, q_3]$ and $L(q_j, r_j)$ (cf. Figure 5). If some point $u \in R$ is beyond exactly the edge $[r_i, r_j]$ of $[q_1, q_2, r_2, r_1]$, then we have $[q_1, r_i, u, r_2, q_2] \ast [q_2, r_2, q_3] \ast [q_3, r_1, q_1]$. If $u \in R$ is beyond exactly the edge $[r_1, q_3]$ of $[q_1, r_1, q_3]$, then $[q_1, r_1, u, q_3] \ast [q_3, s, q_2] \ast [q_2, r_2, r_1, q_1]$ for $s = u$ or $s = r_2$.

Hence, by symmetry, we may assume that $r_3, r_4$, and $r$ are in one of the quadrangles $[r_i, w_i, x_i, z_i]$ for $i = 1$ or 2, or in $[q_1, q_2, z_2, z_1]$.

Assume that $r_3 \in [r_1, w_1, x_1, z_1]$. If $L^+(r_4, r_3) \cap [q_1, q_3] \neq \emptyset$, then $[q_3, r_3, r_4, r_1, r_2] \ast [r_2, r_1, q_1] \ast [q_1, r_3, q_3]$. If $L^-(r_4, r_3) \cap [q_1, q_3] \neq \emptyset$, then $[q_1, r_4, r_3, r_2, q_2] \ast [q_2, r_2, q_3] \ast [q_3, r_4, q_1]$. If $L(r_1, r_3) \cap [q_1, q_3] = \emptyset$, then $[q_1, r_1, r_2, q_2] \ast [q_2, r_2, q_3] \ast [q_3, r_3, r_4, q_1]$. Thus, we may assume that $r_3 \in [r_2, z_2, x_2, w_2]$. Since $r_3 \in [r_2, z_2, x_2, w_2]$ yields $[q_3, r_4, n_1, r_2, r_3] \ast [r_3, r_2, q_1] \ast [q_1, r_4, q_3]$, we may assume that $r_3 \in [r_2, w_2, x_2, y_2]$, and (by symmetry) that $r_4 \in [r_1, w_1, x_1, y_1]$.

Assume that $r \in [r_1, w_1, x_1, y_1]$. If $[r_1, r_2, r_4, r]$ is a quadrilateral, then we may apply an argument similar to that in the previous paragraph. Thus, we may assume that $r_4 \in [r_1, r_2, r]$. This yields $[r, r_4, n_1, q_1] \ast [q_1, r_1, r_2, q_2] \ast [q_2, r_3, q_3] \ast [q_3, r_3, r_4, r]$. Hence, $r \in [q_1, q_2, z_2, z_1]$.

If $r \in [q_1, r_1, z_1]$, then $[q_1, r, r_1, r_2, q_2] \ast [q_2, r_2, q_3] \ast [q_3, r, q_1]$. Let $r \in [q_1, q_2, r_2, r_1]$. If $L(q_3, r_4)$ does not separate $q_1$ and $r$, then $[q_1, r, q_2] \ast [q_2, r_1, q_1]$. Otherwise, we may suppose that $L^+(r_4, r_4) \cap [q_1, q_3] \neq \emptyset$. By symmetry, we also obtain that $L^+(r_4, r_3) \cap [q_2, q_3] \neq \emptyset$.

Assume that $r \in [q_1, r_1, r_2]$. Then, we observe that $U' = [r, r_2, r_3, r_1]$ is an empty quadrilateral, and $L(r, r_2)$ separates $U'$ from $[q_1, q_2]$. Since
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Let $R \cap [q_1, r, r_2, q_2] = \emptyset$, an argument applied for $U'$, similar to that applied for $U$, yields a hexagon. Hence, $r \in [q_1, n, q_2] \cap [q_1, r_2, q_2]$. Then $L^+(r_3, r) \cap [q_1, q_2] \neq \emptyset \neq L^+(r_4, r) \cap [q_1, q_2]$. Now, we apply Lemma 3 with $V(P) \cup V(Q) \cup \{r_3, r_4, r\}$ as $S$.

**Definition 1.** Let $A, B \subset \mathbb{E}^2$ be sets of points in general position. Suppose that there is a bijective function $f : A \to B$ such that, for any $a_1, a_2, a_3 \in A$, the ordered triples $(a_1, a_2, a_3)$ and $(f(a_1), f(a_2), f(a_3))$ have the same or the opposite orientation, independently of the choice of $a_1, a_2,$ and $a_3$. Then we say that $A$ and $B$ are identical.

We note that if $A$ and $B$ are identical, then $A' \subset A$ is a $k$-gon, if and only if, $f(A')$ is a $k$-gon.

Let $\tilde{S}$ be a set of less than thirteen points such that $[\tilde{S}]$ is a pentagon, $[\tilde{S} \setminus V(\tilde{S})]$ is a triangle, and $\tilde{S}$ does not contain a hexagon. Using Lemma 4, we may characterize the possible configurations for $\tilde{S} \setminus V(\tilde{S})$. Lemma 5 summarizes our work. We sketch its proof.

**Lemma 5.** Let $\tilde{S} \subset \mathbb{E}^2$ be a set of fewer than thirteen points in general position such that $[\tilde{S}]$ is a pentagon, $Q = [\tilde{S} \setminus V(\tilde{S})]$ is a triangle, and $\tilde{S}$ does not contain a hexagon. Then $Q$ is identical to one of the sets in Figure 6.

**Proof.** Let the vertices of $Q$ be $q_1, q_2,$ and $q_3$ in counterclockwise cyclic order, and let $R = S \cap \text{int } Q$. If card $R \leq 2$, the assertion readily follows. Let us assume that card $R = 3$ and that the vertices of $[R]$ are $r_1, r_2,$ and $r_3$ in counterclockwise cyclic order. By Lemma 3, we may assume that there is a sideline of $R$, that separates exactly two vertices of $Q$ from $R$. Let this line be $L(r_1, r_2)$, and let it separate $q_1$ and $q_2$ from $R$. Note that if $r_3 \in [q_1, r_2, q_3]$, the assertion follows by an argument similar to that in the third paragraph of the proof of Lemma 4. Without
loss of generality, we may assume that \( L(q_3, r_1) \) separates \( r_3 \) from \( q_2 \). If \( L(q_2, r_1) \) separates \( r_3 \) from \( q_3 \), then \( Q \) is a type 3b configuration. Otherwise, \( Q \) is a type 3a configuration.

The proof for the case \( R = 4 \) is similar to the proof in the previous case, and hence we omit it.

This list helps us to exclude some other cases from our investigation. If a set is identical to one of the sets in Figure 6, we say that its type is the type of the corresponding set in the figure.

![Figure 6](image)

**Lemma 6.** Let \( S \subset \mathbb{E}^2 \) be a set of seventeen points in general position such that \( P = [S] \) is a pentagon and \( Q = [S \setminus V(P)] \) is a quadrilateral. Then \( S \) contains a hexagon.
Proof. Consider a diagonal $D$ of $Q$. By Lemma 4, we may assume that $D$ divides $Q$ into two triangles that contain exactly four points of $S$ in their interiors, and both these triangles have to be either type 4a, 4b, or 4c. Let us observe that if both triangles contain a pair of points such that the line passing through them does not intersect $D$, then these two pairs of points and the two endpoints of $D$ are in convex position. Hence, we may assume that, in at least one of the triangles, each line passing through two points intersects $D$.

Since there is, in a type 4c set, no edge of the convex hull that meets all the lines that pass through two of its points, we may assume that the set of the points in one of the triangles is type 4a or 4b, and that $D$ is the left edge of one of the triangles in Figure 6. We also observe that configurations of type 4a or 4b are almost identical, the only difference is that the line passing through the two points closest to the left edge of the triangle intersects the bottom or the right edge of the triangle. Thus, we may handle these two cases together, if we leave it open which edge this line intersects.

![Figure 7](image)

We denote our points as in Figure 7 with $D = [q_1, q_2]$, and let $L = L(n_1, r_2)$. Observe that $L$ divides the set of points, beyond exactly the edge $[q_1, q_2]$ of $[q_1, q_2, q_3]$, into two connected components. If a point $p$ is in the component that contains $q_1$, respectively $q_2$, in its boundary, then we say that $p$ is on the left-hand side, respectively right-hand side, of $L$. Let $B = (Q \cap S) \setminus [q_1, q_2, q_3]$. Observe that, as $\text{card}(Q \cap S) = 12$.
and \( \text{card}(S \cap [q_1, q_2, q_3]) = 7 \), we have that \( \text{card} \ B = 5 \) and every point of \( B \) is either on the left-hand side or on the right-hand side of \( L \). By the pigeon-hole principle, there are three points of \( B \) that are on the same side of \( L \). Let us denote these points by \( s_1 \), \( s_2 \), and \( s_3 \).

Assume that \( s_1 \), \( s_2 \), and \( s_3 \) are on the left-hand side of \( L \). Observe that if \( L(s_i, s_j) \) and \( [q_1, r_1] \) are disjoint for some \( i \neq j \), then \( [q_1, s_i, s_j, r_1, r_2, r_3] \) is a hexagon. Thus, we may relabel \( s_1 \), \( s_2 \), and \( s_3 \) such that \( s_3 \in [q_1, r_1, r_2] \subset [q_1, r_1, s_1] \). This yields that either \([s_1, s_2, s_3, q_1]\) or \([s_1, s_2, s_3, r_1]\) is a quadrilateral. If \([s_1, s_2, s_3, q_1]\) is a quadrilateral, then \([s_1, s_2, s_3, q_1] \ast [q_1, s_3, r_1, r_2, r_3] \ast [r_3, r_4, q_2] \ast [q_2, r_1, s_2, s_1]\). If \([s_1, s_2, s_3, r_1]\) is a quadrilateral, then \([s_1, s_2, s_3, r_1] \ast [q_2, r_3, r_5] \ast [r_3, r_2, s_1]\).

Let \( s_1 \), \( s_2 \), and \( s_3 \) be on the right-hand side of \( L \). Observe that if \( L(s_i, s_j) \) and \( [q_2, r_1] \) are disjoint for some \( i \neq j \), then \( [q_2, s_i, s_j, r_1, r_2, r_3] \) is a hexagon. Hence, we may assume that \( s_3 \in [q_2, r_1, s_2] \subset [q_2, r_1, s_1] \). Then \([s_1, s_2, s_3, q_2]\) or \([s_1, s_2, s_3, r_1]\) is a quadrangle. If \([s_1, s_2, s_3, q_2]\) is a quadrilateral, then \([s_1, s_2, s_3, q_2] \ast [q_2, s_3, r_1, r_2, r_4] \ast [r_4, r_2, q_1] \ast [q_1, r_1, s_2, s_1]\). If \([s_1, s_2, s_3, r_1]\) is a quadrilateral, then \([s_1, s_2, s_3, r_1] \ast [q_2, r_1, r_4] \ast [r_4, r_2, s_1]\).

Lemma 7. Let \( S \subset \mathbb{E}^2 \) be a set of points in general position such that \( P = [S] \) and \( Q = [S \setminus V(P)] \) are pentagons, and \( S \setminus (V(P) \cup V(Q)) \) has a subset of type \( 3a \), or a subset identical to the point set in Figures 9, 10, or 11. Then \( S \) contains a hexagon.

Proof. Let \( R \) denote the subset of \( S \setminus (V(P) \cup V(Q)) \) that is either of type \( 3a \), or is identical to the point set in Figures 9, 10 or 11. Let \( q_1, q_2, q_3, q_4, \) and \( q_5 \) denote the vertices of \( Q \) in counterclockwise cyclic order.
Figure 8. A type 3a set with the
notation of Lemma 7.

Assume that $R$ is of type 3a. Let us denote the points of $R$ as in
Figure 8. Let $R_{12}$, $R_{23}$, and $R_{13}$ denote, respectively, the set of points
that are beyond exactly the edge $[r_1, r_2]$ of $[r_2, t_3, t_2, r_1]$, the edge
$[r_2, r_3]$ of $[r_2, t_1, t_3, r_3]$, and the edge $[r_1, r_3]$ of $[r_1, t_3, r_3]$. If
$\text{card}(R_{12} \cap V(Q)) \geq 2$, $\text{card}(R_{23} \cap V(Q)) \geq 2$, or $\text{card}(R_{13} \cap V(Q)) \geq 3$,
then $S$ contains a convex hexagon. Otherwise, there is a vertex $q_i$ of $Q$ in
the convex domain bounded by the half-lines $L^{-}(r_2, t_1)$ and $L^{-}(r_2, t_2)$,
from which we obtain $[r_1, t_1, r_2, q_i] \ast [q_i, r_2, t_2, r_3] \ast [r_3, t_3, r_1]$.

Let us assume that $R$ is the set in Figure 9 and denote the points of $R$
as indicated. Let $R_{12}$, $R_{23}$, and $R_{13}$ denote, respectively, the set of points
that are beyond exactly the edge $[r_1, r_2]$ of $[r_2, t_2, t_3, t_1]$, the edge
$[r_2, r_3]$ of $[r_2, t_1, t_2, r_3]$, and the edge $[r_1, r_3]$ of $[r_1, t_2, r_3]$. If $\text{card}(R_{12} \cap V(Q))$
$\geq 2$, $\text{card}(R_{23} \cap V(Q)) \geq 3$, or $\text{card}(R_{13} \cap V(Q)) \geq 3$, then $S$ contains a
hexagon. Hence, we may assume that $q_1 \in R_{12}$, $\{q_2, q_3\} \subset R_{23}$, $\{q_4, q_5\}$
$\subset R_{13}$, and there is no vertex of $Q$ in $R_{23} \cap R_{13}$. If $L(q_1, q_4)$ does not
intersect the interior of $[R]$, then the convex hull of $[t_1, t_2, r_2, r_1]$ and $[r_4,
q_1]$ is a hexagon.

Let $r_4 \in [q_1, r_1, r_2]$. If $L(r_4, r_1)$ does not separate $q_5$ and $q_1$, and
$L(r_4, r_2)$ does not separate $q_2$ and $q_1$, then $[q_1, r_4, r_2, q_2] \ast [q_2, r_2, t_2, r_3]$
Thus, we may assume that, say, \( L(r_4, r_1) \) separates \( q_5 \) and \( q_1 \). If \( L(r_2, r_3) \) separates \( q_4 \) and \( R \), then
\[
[q_4, r_3, r_2] \ast [r_2, t_2, t_1, r_1] \ast [r_1, t_1, r_3, q_4].
\]
If \( L(r_2, r_3) \) does not separate \( q_4 \) and \( R \), then \([r_4, r_2, r_3, q_4, q_5, r_1] \) is a hexagon.

Figure 10

Assume that \( R \) is the set in Figure 10 and denote the points of \( R \) as indicated. We may clearly assume that there is no vertex of \( Q \) beyond exactly the edge \([r_1, r_2] \) of \([r_1, r_2, t_2, t_3, t_1]\). Hence, there is an edge, say \([q_1, q_2]\), that intersects both rays \( L^-(r_1, t_1) \) and \( L^-(r_2, t_2) \). If \( L(r_1, r_2) \) separates \( R \) from both \( q_1 \) and \( q_2 \), then
\[
[q_1, r_1, r_2, q_2] \ast [q_2, r_2, t_2, r_3] \ast [r_3, t_2, t_1, r_4] \ast [r_4, t_1, r_1, q_1].
\]
Hence, we may assume that \( L(r_1, r_2) \) does not separate \( R \), say, from \( q_2 \). If \( L(t_2, t_3) \) does not separate \( r_2 \) and \( q_2 \), then \([r_1, r_2, q_2, t_2, t_3, t_1]\) is a hexagon. If \( L(t_2, t_3) \) separates \( r_2 \) and \( q_2 \), then
\[
[q_1, r_2, q_2] \ast [q_2, t_2, t_3, r_4] \ast [r_4, t_1, r_1, q_1].
\]

We are left with the case when \( R \) is the set in Figure 11 with points as indicated. Let \( R_{12}, R_{23}, R_{34}, \) and \( R_{14} \) denote, respectively, the set of points that are beyond exactly the edge \([r_1, r_2] \) of \([r_1, r_2, t_2, t_1]\), the edge \([r_2, r_3]\) of \([r_2, t_2, t_3, r_3]\), the edge \([r_3, r_4]\) of \([r_3, t_3, t_1, r_4]\), and the edge \([r_1, r_4]\) of \([r_4, t_1, r_1]\). If \( \text{card}(R_{i(i+1)} \cap V(Q)) \geq 2 \) for some \( i \in \{1, 2, 3\} \), then \( S \) contains a hexagon. Otherwise, \( R_{14} \) contains at least two vertices of \( Q \), which we denote by \( q_1 \) and \( q_2 \). If both \( q_1 \) and \( q_2 \) are beyond
exactly the edge \([r_1, r_4]\) of \([r_1, t_2, t_3, r_4]\), then \([t_2, t_3, r_4, q_1, q_2, r_1]\) is a hexagon. Thus, we may assume that, say, \(q_1\) is beyond exactly the edge \([r_3, r_4]\) of \([r_3, t_3, r_4]\). From this, it follows that \([r_3, t_3, r_4, q_1]\) * \([q_1, r_4, t_1, r_1]\) * \([r_1, t_1, t_2, r_2]\) * \([r_2, t_2, t_3, r_3]\).

**Proof of Theorem 1.** Let \(Q = [S \setminus V(P)]\), \(R = [S \setminus (V(P) \cup V(Q))]\), and \(T = S \setminus (V(P) \cup V(Q) \cup V(R))\). If \(Q\) is a triangle, then we apply Lemma 4. If \(Q\) is a quadrilateral, then we apply Lemma 6. Let \(Q\) be a pentagon. If \(R\) is a triangle, then, by Lemma 5, \(V(R) \cup T\) has types 4a, 4b, or 4c, and thus, it contains a type 3a subset, and the assertion follows from Lemma 7. If \(R\) is a quadrilateral, then \(V(R) \cup T\) contains a subset identical to the set in Figures 9, 10, or 11, and we may apply Lemma 7.

Let \(R\) be a pentagon. We note that \(T\) contains two points, say, \(t_1\) and \(t_2\).

**Figure 12**

Let \(q_1, q_2, q_3, q_4, q_5, \) and \(r_1, r_2, r_3, r_4, r_5\) denote, respectively, the vertices of \(Q\) and \(R\) in counterclockwise cyclic order. If some \(q_i\) is beyond exactly one edge of \(R\), then \([R, q_i]\) is a hexagon. Thus, we may assume that every vertex of \(Q\) is beyond at least two edges of \(R\). Observe that there is no point on the plane that is beyond all five edges of \(R\). If some \(q_i\) is beyond all edges of \(R\) but one, say \([r_1, r_5]\), then we obtain \([r_1, r_2, r_3, r_4, r_5]\) * \([r_5, r_4, q_i]\) * \([q_i, r_2, r_1]\). Hence, we may assume that every vertex of \(Q\) is beneath at least two edges of \(R\).
For $1 \leq i \leq 5$, let $R_i$ denote the set of points that are beyond the two edges of $R$ that contain $r_i$ and beneath the other three edges of $R$, and let $R_{i(i+1)}$ denote the set of points that are beyond the edges of $R$ that contain $r_i$ or $r_{i+1}$, and beneath the other two edges of $R$ (cf. Figure 12). We call $R_{i(i-1)}$ and $R_{i(i+1)}$ consecutive regions.

Assume that two distinct and nonconsecutive regions contain vertices of $Q$, say, $q_k \in R_{51}$ and $q_l \in R_{23}$. Since every vertex of $Q$ is beneath at least two edges of $R$, $q_k$ and $q_l$ are distinct points. If there is a vertex $q_h$ of $Q$ in $R_{34} \cup R_4 \cup R_{45}$, then $[q_l, r_3, r_4, q_h] * [q_h, r_4, r_5, q_k] * [q_k, r_1, r_2, q_l]$. Let $V(Q) \cap (R_{34} \cup R_4 \cup R_{45}) = \emptyset$. Then exactly one edge of $Q$ intersects $R_{34} \cup R_4 \cup R_{45}$. Let us denote this edge by $[q_m, q_{m+1}]$. If $q_m \in R_{23}$, then $[q_{m+1}, r_4, q_m] * [q_m, r_2, q_h] * [q_h, r_1, r_2, r_3, r_4, q_{m+1}]$. Let $q_m \in R_3$ and, by symmetry, $q_{m+1} \in R_5$. If there are at least three vertices of $Q$ in $R_2 \cup R_{23} \cup R_3$ or in $R_1 \cup R_{15} \cup R_5$, then $V(Q) \cup V(R)$ contains a hexagon. Hence, we may assume that a vertex $q_g$ of $Q$ is in $R_{12}$. Since every vertex of $Q$ is beneath at least two edges of $Q$, the sum of the angles of $R$ at $r_1$ and $r_2$ is greater than $\pi$, which implies that $L(r_1, r_2)$ separates $R$ and $q_g$. Thus, we have $[q_g, r_2, r_3, q_m] * [q_m, r_4, q_{m+1}] * [q_{m+1}, r_5, r_1, q_g]$.

Assume that two consecutive regions contain vertices of $Q$, say, $q_k \in R_{51}$ and $q_l \in R_{12}$. If $V(Q) \cap (R_{23} \cup R_{34} \cup R_{45}) = \emptyset$, then we may apply the argument in the previous paragraph. Let $V(Q) \cap (R_{23} \cup R_{34} \cup R_{45}) = \emptyset$. If at least four vertices of $Q$ are beneath the edge $[r_3, r_4]$ of $R$, then these vertices, together with $r_3$ and $r_4$, are six points in convex position. Hence, we may assume that $R_3 \cup R_4$ contains at least two vertices of $Q$. Let us denote these vertices by $q_e$ and $q_f$. If $q_e, q_f \in R_3$, then $[q_1, r_2, q_e, q_f, r_4, r_5]$ is a hexagon. Thus, we may clearly assume that, say, $q_e \in R_3$ and $q_f \in R_4$. Then we have $[q_l, r_2, r_3, q_e] * [q_e, r_3, r_4, q_f] * [q_f, r_1, r_5, q_k] * [q_k, r_5, r_1, q_l]$. 
Assume that $R_{i(i+1)}$ contains a vertex of $Q$ for some $i$, say, $q_1 \in R_{51}$. By the preceding, no vertex of $Q$ is in $R_{12} \cup R_{23} \cup R_{34} \cup R_{45}$. An argument similar to that used in the previous paragraph yields the existence of a hexagon if $R_2, R_3,$ or $R_4$ contains no vertex of $Q$. Let $q_k \in R_2, q_l \in R_3,$ and $q_m \in R_4$. Then $[q_l, r_1, r_2, q_k] \ast [q_k, r_2, r_3, q_l] \ast [q_l, r_3, r_4, q_m] \ast [q_m, r_4, r_5, q_1]$.

Figure 13

We have now arrived at the case that each vertex of $Q$ is beyond exactly two edges of $R$. Clearly, we may assume that $q_i \in R_i$ for each $i$. If $L(t_1, t_2)$ intersects two consecutive edges of $R$, then $S$ contains a hexagon. Hence, we may assume that, say, $L^+(t_1, t_2) \cap [r_2, r_3] \neq \emptyset,$ and $L^-(t_1, t_2) \cap [r_5, r_1] \neq \emptyset$ (cf. Figure 13). If both $q_1$ and $q_2$ are beyond exactly the edge $[r_1, r_2]$ of $[r_1, t_1, t_2, r_2]$, then we have a hexagon. If neither point is beyond exactly that edge, then $[q_1, r_1, r_2, q_2] \ast [q_2, r_2, t_2, r_3] \ast [r_3, t_2, t_1, r_5] \ast [r_5, t_1, r_1, q_1]$. Thus, we may assume that $q_1$ is beyond exactly the edge $[r_1, r_2]$ and $q_2$ is not. If $q_5$ is beyond exactly the edge $[r_4, r_5]$ of $[r_4, r_5, t_1, t_2, r_3]$, then $[q_5, r_5, t_1, t_2, r_3, r_4]$ is a hexagon. Hence, we may assume that $q_5$ is beyond exactly the edge $[r_1, r_5]$ of $[r_1, t_1, r_5]$ and, similarly, that $q_3$ is beyond exactly the edge $[r_2, r_3]$ of $[r_2, t_2, r_3]$. From this, we obtain that $[q_3, r_5, r_4, q_4] \ast [q_4, r_4, r_5, q_5] \ast [q_5, r_5, r_1, q_1] \ast [q_1, r_1, r_2, q_2] \ast [q_2, r_2, t_2, r_3, q_3]$. \[\]
References


