Journal of Pure and Applied Mathematics: Advances and Applications
Volume 22, Number 1, 2020, Pages 71-89
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jpamaa_7100122121

PROJECTION ESTIMATES FOR MUTUAL MULTIFRACTAL DIMENSIONS

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Abstract

We study the mutual multifractal spectrum of orthogonal projections of a couple of measures (μ, ν) having a finite s-energy for some $1 \leq m \leq s < n$.

1. Introduction and Preliminaries

Recently the projection behaviour of dimensions and multifractal spectra of measures has generated an interest in the mathematical literature [8, 18-23, 25-28, 34, 39, 41, 43, 45]. This is connected to the question of the relationship between the Hausdorff and packing dimensions of a subset of $\mathbb{R}^n$ or a Borel probability measure and that of its orthogonal projections onto an m-dimensional subspace. The fundamental result on projections concerns Hausdorff dimensions:

2020 Mathematics Subject Classification: 28A80, 28A75, 26A18.
Keywords and phrases: Hausdorff dimension, packing dimension, mutual multifractal analysis, projection.
Received March 5, 2020; Revised May 20, 2020

the projection of an analytic subset E of $\mathbb{R}^n$ onto almost all m-dimensional subspaces has a Hausdorff dimension equal to $\min(m, \dim_H(E))$, where $\dim_H(E)$ denotes the Hausdorff dimension of E. This was proved by Marstrand [30], also in [29], Kaufman employed potential theoretic methods in order to prove Marstrand result's, which has been generalized later by Mattila in [31] and Hu and Taylor [26]. The behaviour of the packing dimension under projections is not as straightforward as that of the Hausdorff dimension. While the Hausdorff dimension of a set or a measure is preserved under almost all projections, its packing dimension may decrease for almost all of them [21, 22]. O'Neil and Selmi [34, 44] compared the generalized Hausdorff and packing dimensions of a set E of $\mathbb{R}^n$ with respect to a measure μ with those of projections onto m-dimensional subspaces. The results of O'Neil were later generalized by Selmi et al. in [18, 41, 43]. In [19, 20, 45], the authors studied the mutual multifractal analysis (see [32, 46-49]) of the orthogonal projections on m-dimensional linear subspaces, more specifically, they investigated the relationship between $f_{\mu, \nu}(\alpha, \beta)$ and $f_{\mu_V, \nu_V}(\alpha, \beta)$, where

$$f_{\mu, \nu}(\alpha, \beta) = \dim_K\big(E_{\mu, \nu}(\alpha, \beta)\big),$$

$$E_{\mu, \nu}(\alpha, \beta) = \left\{ x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu;\ \lim_{r\to 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \ \text{ and } \ \lim_{r\to 0} \frac{\log \nu(B(x, r))}{\log r} = \beta \right\},$$

and $K \in \{H, P\}$, here $\dim_P$ denotes the packing dimension. Recently, there has been a great interest in this subject (the calculus of $f_{\mu, \nu}(\alpha, \beta)$) and positive results have been written in various situations in the dynamic contexts [3-7, 37]. Lately, many authors focused on mutual (mixed) multifractal spectra, see, for example, [1, 2, 9, 10, 12-17, 32, 33, 36, 38].

As a continuity of these researches, we investigate the mutual multifractal spectrum of orthogonal projections of a couple of measures (μ, ν) having a finite s-energy (see (2.1) for the definition).

Casually, we briefly recall some basic definitions and facts which will be repeatedly used in subsequent developments. For an arbitrary Borel probability measures μ and ν on $\mathbb{R}^n$, they introduced two three-parameter families of measures, $\{\mathcal{P}_{\mu,\nu}^{q,t,s};\ q, t, s \in \mathbb{R}\}$ and $\{\mathcal{H}_{\mu,\nu}^{q,t,s};\ q, t, s \in \mathbb{R}\}$ based on certain generalizations of the Hausdorff measure and of the packing measure. For $q, t, s \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$, we define

$$\overline{\mathcal{P}}_{\mu,\nu,\delta}^{q,t,s}(E) = \sup\left\{ \begin{array}{r} \sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s;\ (B_i = B(x_i, r_i))_i \\ \text{is a centered } \delta\text{-packing of } E \end{array} \right\},$$

and

$$\overline{\mathcal{P}}_{\mu,\nu}^{q,t,s}(E) = \inf_{\delta>0} \overline{\mathcal{P}}_{\mu,\nu,\delta}^{q,t,s}(E).$$

The mutual packing measure is then given by

$$\mathcal{P}_{\mu,\nu}^{q,t,s}(E) = \inf_{E\subseteq \bigcup_i E_i} \sum_i \overline{\mathcal{P}}_{\mu,\nu}^{q,t,s}(E_i).$$

In a similar way, we define

$$\overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,t,s}(E) = \inf\left\{ \begin{array}{r} \sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s;\ (B_i = B(x_i, r_i))_i \\ \text{is a centered } \delta\text{-cover of } E \end{array} \right\},$$

and

$$\overline{\mathcal{H}}_{\mu,\nu}^{q,t,s}(E) = \sup_{\delta>0} \overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,t,s}(E).$$

The mutual Hausdorff measure is defined by

$$\mathcal{H}_{\mu,\nu}^{q,t,s}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}_{\mu,\nu}^{q,t,s}(F).$$

The measures $\mathcal{H}_{\mu,\nu}^{q,t,s}$ and $\mathcal{P}_{\mu,\nu}^{q,t,s}$ and the pre-measure $\overline{\mathcal{P}}_{\mu,\nu}^{q,t,s}$ assign, in the usual way, a mutual multifractal dimension to each subset E of $\mathbb{R}^n$. They are respectively denoted by $\dim_{\mu,\nu}^{q,t}(E)$, $\mathrm{Dim}_{\mu,\nu}^{q,t}(E)$, and $\Delta_{\mu,\nu}^{q,t}(E)$ (see [46, 48]). More precisely, we have

$$\begin{aligned} b_{\mu,\nu}^{q,t}(E) &= \inf\left\{s \in \mathbb{R}, \quad \mathcal{H}_{\mu,\nu}^{q,t,s}(E) = 0\right\}, \\ B_{\mu,\nu}^{q,t}(E) &= \inf\left\{s \in \mathbb{R}, \quad \mathcal{P}_{\mu,\nu}^{q,t,s}(E) = 0\right\}, \\ \Delta_{\mu,\nu}^{q,t}(E) &= \inf\left\{s \in \mathbb{R}, \quad \overline{\mathcal{P}}_{\mu,\nu}^{q,t,s}(E) = 0\right\}. \end{aligned}$$

It is clear that

$$b_{\mu,\nu}^{q,t}(E) \leq B_{\mu,\nu}^{q,t}(E) \leq \Delta_{\mu,\nu}^{q,t}(E).$$

Next we define the mutual multifractal dimension functions $b_{\mu,\nu}$, $B_{\mu,\nu}$ and $\Lambda_{\mu,\nu} : \mathbb{R}^2 \to [-\infty, +\infty]$ by

$$\begin{aligned} b_{\mu,\nu} &: (q, t) \to b_{\mu,\nu}^{q,t}(\mathrm{supp}\ \mu \cap \mathrm{supp}\ \nu), \\ B_{\mu,\nu} &: (q, t) \to B_{\mu,\nu}^{q,t}(\mathrm{supp}\ \mu \cap \mathrm{supp}\ \nu), \\ \Lambda_{\mu,\nu} &: (q, t) \to \Delta_{\mu,\nu}^{q,t}(\mathrm{supp}\ \mu \cap \mathrm{supp}\ \nu). \end{aligned}$$

Remark 1.1.

(1) The measure $\mathcal{H}_{\mu,\nu}^{q,t,s}$ is a multifractal generalisation of the centered Hausdorff measure, whereas $\mathcal{P}_{\mu,\nu}^{q,t,s}$ is a multifractal generalisation of the packing measure. In fact, it is easily seen that if $s \geq 0$, then $\mathcal{H}_{\mu,\nu}^{0,0,s} = \mathcal{H}^s$

and $\mathcal{P}_{\mu,\nu}^{0,0,s} = \mathcal{P}^s$, where $\mathcal{H}^s$ denotes the s-dimensional centered Hausdorff measure and $\mathcal{P}^s$ denotes the s-dimensional packing measure (see [35] for more information on $\mathcal{H}^s$ and $\mathcal{P}^s$).

(2) In the special case where $q = 0$ or $t = 0$, the mutual multifractal spectra is strictly related to Olsen's multifractal formalism [35].

(3) The mutual multifractal spectra represents the relative multifractal analysis introduced by Cole [11] (see also [40, 42]) in the case where $s = 0$.

2. The Main Results

We denote by $G_{n,m}$ the Grassmannian manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$ and $\gamma_{n,m}$ its orthogonally invariant Borel probability measure. We write $\mathcal{L}^n$ to denote n-dimensional Lebesgue measure on any n-dimensional plane. We denote by π_V the orthogonal projection onto a linear subspace V of $\mathbb{R}^n$. Now, for a Borel probability measure μ on $\mathbb{R}^n$, supported on the compact set $\operatorname{supp}\mu$ and for $V \in G_{n,m}$ we define μ_V, the projection of μ onto V by

$$\mu_V(A) = \mu(\pi_V^{-1}(A)) \quad \forall A \subseteq V.$$

Since μ has a compact support, $\operatorname{supp}\mu_V = \pi_V(\operatorname{supp}\mu)$ for all $V \in G_{n,m}$, then for any continuous function $f : V \longrightarrow \mathbb{R}$

$$\int_V f d\mu_V = \int f(\pi_V(x))d\mu(x)$$

whenever these integrals exist. For s an integer with $1 \le m \le s < n$, we denote the s-energy of a measure μ by

$$I_s(\mu) = \iint |x-y|^{-s} d\mu(x)d\mu(y). \tag{2.1}$$

Frostman [24] showed that the Hausdorff dimension of a Borel subset E of $\mathbb{R}^n$ is the supremum of the positive reals s for which there exists a Borel probability measure μ charging E and for which the s-energy of μ is finite. This characterization is used by Kaufman [29] and Mattila [31] to prove their results on the preservation of the Hausdorff dimension. The conditions : $I_s(\mu)$ is finite, implies that $\dim_H(\mu) \geq s$, on the other hand if $\mu(B(x, r)) \leq r^s$ for all x and all sufficiently small r, then μ has a finite s-energy. Notice Mattila [31] proved that if $I_m(\mu)$ is finite, then for almost every m-dimensional subspaces V, measure μ_V is absolutely continuous with respect to Lebesgue measure $\mathcal{L}_V^m$ on V identified with $\mathbb{R}^m$, where $\mathcal{L}_V^m(E) = \mathcal{L}^m(E \cap V)$ for $E \subset \mathbb{R}^m$, and $\mu_V \in L^2(V)$.

Throughout this paper, we suppose that μ and ν be two compactly supported Borel probability measures with $\operatorname{supp}\mu = \operatorname{supp}\nu = K$. Our main results are the following: in the next few results, we describe how if the couple of measures (μ, ν) having a finite s-energy for some $s > m$, then we may deduce a little more about the usual structure of the projections of measures.

Theorem 2.1. *Let* $1 \leq m \leq s < n$, *suppose that* $I_s(\mu) < \infty$ *and* $I_s(\nu) < \infty$. *Then for almost every* m*-dimensional subspace* V *and all* $t, q \geq 1$, *we have*

(1) *If* $m < s < 2m$, *then*

$$\begin{aligned} m(1-(q+t)) &\leq b_{\mu_V,\nu_V}(q, t) \leq B_{\mu_V,\nu_V}(q, t) \leq \Lambda_{\mu_V,\nu_V}(q, t) \\ &\leq \max\left\{-\frac{s(q+t)}{2},\, m(2-(q+t)),\, -\frac{st}{2}+m(1-q),\, -\frac{sq}{2}+m(1-t)\right\}. \end{aligned}$$

(2) *If* $s \geq 2m$, *then*

$$\begin{aligned} m(1-(q+t)) \leq b_{\mu_V,\nu_V}(q, t) &\leq B_{\mu_V,\nu_V}(q, t) \leq \Lambda_{\mu_V,\nu_V}(q, t) \\ &\leq m(2-(q+t)). \end{aligned}$$

Definition 2.1. We say that two Borel measures μ and ν are equivalent and we write $\nu \sharp \mu$ if there exists $c > 1$ such that for a Borel set A in $\mathbb{R}^n$,

$$c^{-1}\mu_{\lfloor \operatorname{supp} \nu}(A) \leq \nu(A) \leq c\ \mu_{\lfloor \operatorname{supp} \nu}(A).$$

Theorem 2.2. *Let* $1 \leq m \leq s < n$, *we assume that* $I_s(\mu) < \infty$ *and* $\nu \sharp \mathcal{L}^n$. *Then for almost every* m*-dimensional subspace* V *and all* $(q, t) \in [1, +\infty[\times \mathbb{R}$, *we have*

(1) *If* $m < s < 2m$, *then*

(a) *For* $q \geq \dfrac{2m}{2m - s}$

$$m(1 - (q + t)) \leq b_{\mu_V, \nu_V}(q, t) \leq B_{\mu_V, \nu_V}(q, t)$$

$$\leq \Lambda_{\mu_V, \nu_V}(q, t) \leq -\left(\frac{sq}{2} + mt\right).$$

(b) *For* $1 \leq q \leq \dfrac{2m}{2m - s}$

$$b_{\mu_V, \nu_V}(q, t) = B_{\mu_V, \nu_V}(q, t) = \Lambda_{\mu_V, \nu_V}(q, t) = m(1 - (q + t)).$$

(2) *If* $s \geq 2m$, *then*

$$b_{\mu_V, \nu_V}(q, t) = B_{\mu_V, \nu_V}(q, t) = \Lambda_{\mu_V, \nu_V}(q, t) = m(1 - (q + t)).$$

The symmetrical results are true as well.

Theorem 2.3. *Let* $1 \leq m \leq s < n$, *we suppose that* $\mu \sharp \mathcal{L}^n$ *and* $I_s(\nu) < \infty$. *Then for almost every* m*-dimensional subspace* V *and all* $(q, t) \in \mathbb{R} \times [1, +\infty[$, *we have*

(1) *If* $m < s < 2m$, *then*

(a) *For* $t \geq \dfrac{2m}{2m-s}$

$$m(1-(q+t)) \leq b_{\mu_V,\nu_V}(q,t) \leq B_{\mu_V,\nu_V}(q,t)$$

$$\leq \Lambda_{\mu_V,\nu_V}(q,t) \leq -\left(\frac{st}{2}+mq\right).$$

(b) *For* $1 \leq t \leq \dfrac{2m}{2m-s}$

$$b_{\mu_V,\nu_V}(q,t) = B_{\mu_V,\nu_V}(q,t) = \Lambda_{\mu_V,\nu_V}(q,t) = m(1-(q+t)).$$

(2) *If* $s \geq 2m$, *then*

$$b_{\mu_V,\nu_V}(q,t) = B_{\mu_V,\nu_V}(q,t) = \Lambda_{\mu_V,\nu_V}(q,t) = m(1-(q+t)).$$

Remark 2.1.

(1) We assume that μ and ν be two Borel probability measures having the same compact support with $\mu\sharp\mathcal{L}^n$ and $\nu\sharp\mathcal{L}^n$. Then for all m-dimensional subspace V and all $(t,q) \in \mathbb{R}^2$, we have

$$b_{\mu_V,\nu_V}(q,t) = B_{\mu_V,\nu_V}(q,t) = \Lambda_{\mu_V,\nu_V}(q,t) = m(1-(q+t)). \tag{2.2}$$

(2) The results developed by O'Neil in [34] are obtained as a special case of the mutual multifractal theorems by setting $q = 0$ or $t = 0$.

Example 2.1. The hypothesis $\operatorname{supp}\mu = \operatorname{supp}\nu$ is sufficient to obtain the conclusion (2.2). In fact, considering a special case where $(q,t) = (0,0)$, we can find two measures μ and ν such that $\operatorname{supp}\mu \subset \operatorname{supp}\nu$ and

$$b_{\mu_V,\nu_V}(q,t) < b_{\mu,\nu}(q,t) < m(1-(q+t)).$$

Let $\mathcal{C}$ be the usual Cantor subset of $[0, 1]$. Also, let $\mathcal{H}^s$ be the normalized $s = \frac{\log 2}{\log 3}$-dimensional Hausdorff measure on $\mathcal{C}$ and write

$$\mu = \mathcal{H}^s \times \delta_0,$$

where δ_0 denotes the Dirac measure concentrated at 0 and

$$K = \operatorname{supp} \mu = \mathcal{C} \times \{0\}.$$

Taking ν to be the normalized 2-dimensional Lebesgue measure on a ball with center at $(0, 0)$ and radius equal to 2, it is clear that $\operatorname{supp} \mu \subset \operatorname{supp} \nu$. We note that for any 1-dimensional subspace V of $\mathbb{R}^2$, we have

$$b_{\mu, \nu}(0, 0) = \dim_H(\operatorname{supp} \mu) = \dim_H(K) = s,$$

and

$$b_{\mu_V, \nu_V}(0, 0) = \dim_H(\operatorname{supp} \mu_V) = \dim_H(K_V),$$

where K_V denotes the projection of K onto V. For $V = \{0\} \times \mathbb{R}$ we have $K_V = \{(0, 0)\}$, whence

$$b_{\mu_V, \nu_V}(0, 0) = \dim_H(K_V) = 0 < s = \dim_H(K) = b_{\mu, \nu}(0, 0) < 1.$$

3. Proof of Main Results

Before proving the main results we need some preliminary results. We begin by investigating the multifractal spectrum of a measure which is absolutely continuous with respect to Lebesgue measure.

Proposition 3.1. *Suppose that* μ *and* ν *are absolutely continuous with respect to Lebesgue measure on K. Then for* $q, t \geq 0$

$$b_{\mu, \nu}(q, t) \geq n(1 - (q + t)).$$

Proof. Suppose that $f \geq 0$ is such that $\mu = f\mathcal{L}_K^n$. Then, as $\mu(K) > 0$ we can find a Borel set $A \subset K$ of positive Lebesgue measure and $\gamma_1 > 0$ such that for all $x \in A$, $f(x) \geq \gamma_1$. Similarly, we suppose the same for g, i.e., there exist a Borel set $B \subset K$ of positive Lebesgue measure and $\gamma_2 > 0$ such that for all $x \in B$, $g(x) \geq \gamma_2$, where $g \geq 0$ such that $\nu = g\mathcal{L}_K^n$. Let $(B_i = B(x_i, r_i))_i$ be a centered δ-covering of K. For $s < n(1-(q+t))$, we obtain

$$\begin{aligned}
\sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s &\geq 2^s \sum_i \left(\int_{B_i} f d\mathcal{L}^n\right)^q \left(\int_{B_i} g d\mathcal{L}^n\right)^t r_i^s \\
&\geq 2^s \gamma_1^q \gamma_2^t \sum_i \mathcal{L}^n(A \cap B_i)^q \mathcal{L}^n(B \cap B_i)^t r_i^s \\
&\geq 2^s \alpha(n)^{-\frac{s}{n}} \gamma_1^q \gamma_2^t \sum_i \sup\left\{\mathcal{L}_A^n(B_i)^{q+t+\frac{s}{n}}, \mathcal{L}_B^n(B_i)^{q+t+\frac{s}{n}}\right\} \\
&\geq 2^s \alpha(n)^{-\frac{s}{n}} \gamma_1^q \gamma_2^t \sup\left\{\mathcal{L}^n(A)^{q+t+\frac{s}{n}}, \mathcal{L}^n(B)^{q+t+\frac{s}{n}}\right\} \\
&> 0,
\end{aligned}$$

where $\alpha(n)$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Hence $\mathcal{H}_{\mu,\nu}^{q,t,s}(K) > 0$ which implies that $b_{\mu,\nu}(q, t) \geq s$ and the inequality follows.

Definition 3.1. For a measure μ on $\mathbb{R}^n$ we say, for $p \geq 1$, that $\mu \in L^p(\mathbb{R}^n)$ if there is a function $f \in L^p(\mathbb{R}^n)$ such that f is the Radon-Nikodym derivative of μ with respect to $\mathcal{L}^n$ for μ-a.e. x.

Proposition 3.2. *Fix* $p \in [1, +\infty[$. *Suppose that* $\mu, \nu \in L^p(\mathbb{R}^n)$. *Then for* $q, t \geq 1$,

$$\Lambda_{\mu,\nu}(q, t) \leq \max\left\{n\frac{(q+t)(1-p)}{p},\ n(2-(q+t)),\right.$$
$$\left. n\left(\frac{t(1-p)}{p}+1-q\right),\ n\left(\frac{q(1-p)}{p}+1-t\right)\right\}.$$

Proof. Since $\mu, \nu \in L^p(\mathbb{R}^n)$, there exists a compactly supported function f (resp., g) on K such that f (resp., g) is the Radon-Nikodym derivative of μ (resp., ν) with respect to $\mathcal{L}^n$ for μ (resp., ν)-a.e. x.

We distinguish four different cases.

• **Case 1:** For $t, q \geq p \geq 1$.

Let $s > -n(q+t)\frac{p-1}{p}$, $\delta > 0$ and $(B_i = B(x_i, r_i))_i$ be a centered δ-packing of K. By using Hölder's inequality, we obtain

$$\sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s$$

$$\leq 2^s \sum_i \left(\int_{B_i} f d\mathcal{L}^n\right)^q \left(\int_{B_i} g d\mathcal{L}^n\right)^t r_i^s$$

$$\leq 2^s \alpha(n)^{-\frac{s}{n}} \sum_i \left(\int_{B_i} f d\mathcal{L}^n\right)^q \left(\int_{B_i} g d\mathcal{L}^n\right)^t \mathcal{L}^n(B_i)^{\frac{s}{n}}$$

$$\leq 2^s \alpha(n)^{-\frac{s}{n}} \sum_i \left[\left(\int_{B_i} g^p d\mathcal{L}^n\right)^{\frac{q}{p}} \left(\int_{B_i} g^p d\mathcal{L}^n\right)^{\frac{t}{p}} \times \mathcal{L}^n(B_i)^{\frac{s}{n}} \mathcal{L}^n(B_i)^{\left(\frac{q(p-1)}{P}\right)} \mathcal{L}^n(B_i)^{\left(\frac{t(p-1)}{P}\right)}\right]$$

$$\leq 2^s \alpha(n)^{-\frac{s}{n}} \sum_i \left(\int_{B_i} f^p d\mathcal{L}^n\right)^{\frac{q}{p}} \left(\int_{B_i} g^p d\mathcal{L}^n\right)^{\frac{t}{p}} \mathcal{L}^n(B_i)^{\frac{s}{n}+(q+t)\frac{p-1}{p}}$$

$$\le 2^s \alpha(n)^{-\frac{s}{n}} \sum_i \left(\int_{B_i} f^p d\mathcal{L}^n \right)^{\frac{q}{p}} \left(\int_{B_i} g^p d\mathcal{L}^n \right)^{\frac{t}{p}}$$

$$\le 2^s \alpha(n)^{-\frac{s}{n}} \sum_i \left(\int_{B_i} f^p d\mathcal{L}^n \right)^{\frac{q}{p}} \sum_i \left(\int_{B_i} g^p d\mathcal{L}^n \right)^{\frac{t}{p}}$$

$$\le 2^s \alpha(n)^{-\frac{s}{n}} \left(\sum_i \int_{B_i} f^p d\mathcal{L}^n \right)^{\frac{q}{p}} \left(\sum_i \int_{B_i} g^p d\mathcal{L}^n \right)^{\frac{t}{p}}$$

$$= 2^s \alpha(n)^{-\frac{s}{n}} \left(\int f^p d\mathcal{L}^n \right)^{\frac{q}{p}} \left(\int g^p d\mathcal{L}^n \right)^{\frac{t}{p}} < \infty,$$

where $\alpha(n)$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

• **Case 2:** For $p \ge t \ge 1$ and $p \ge q \ge 1$.

Suppose that $s > -n(q + t - 2)$. Let $\delta > 0$ and $(B_i = B(x_i, r_i))_i$ be a centered δ-packing of K. Then we find, on using Hölder's inequality that

$$\sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s$$

$$\le 2^s \sum_i \left(\int_{B_i \cap K} f d\mathcal{L}^n \right)^q \left(\int_{B_i \cap K} g d\mathcal{L}^n \right)^t r_i^s$$

$$\le 2^s \alpha(n)^{-\frac{s}{n}} \sum_i \left(\int_{B_i \cap K} f^q d\mathcal{L}^n \right) \left(\int_{B_i \cap K} g^t d\mathcal{L}^n \right) \mathcal{L}_K^n (B_i)^{q+t-2+\frac{s}{n}}$$

$$\le 2^s \alpha(n)^{-\frac{s}{n}} \sum_i \left(\int_{B_i \cap K} f^q d\mathcal{L}^n \right) \sum_i \left(\int_{B_i \cap K} g^t d\mathcal{L}^n \right)$$

$$\le 2^s \alpha(n)^{-\frac{s}{n}} \left(\int_K f^q d\mathcal{L}^n \right) \left(\int_K g^t d\mathcal{L}^n \right) < \infty,$$

where $\alpha(n)$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

• **Case 3:** For $t \geq p \geq q \geq 1$.

Fix $s > -n(\frac{t(p-1)}{p} + q - 1)$, $\delta > 0$ and $(B_i = B(x_i, r_i))_i$ be a centered δ-packing of K. Hölder's inequality gives

$$\sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s$$

$$\leq 2^s \sum_i \left(\int_{B_i \cap K} f d\mathcal{L}^n \right)^q \left(\int_{B_i \cap K} g d\mathcal{L}^n \right)^t r_i^s$$

$$\leq 2^s \alpha(n)^{-\frac{s}{n}} \sum_i \left[\left(\int_{B_i \cap K} f^q d\mathcal{L}^n \right) \left(\int_{B_i \cap K} g^p d\mathcal{L}^n \right)^{\frac{t}{p}} \times \mathcal{L}^n (B_i \cap K)^{(\frac{t(p-1)}{P} + q - 1 + \frac{s}{n})} \right]$$

$$\leq 2^s \alpha(n)^{-\frac{s}{n}} \sum_i \left(\int_{B_i \cap K} f^q d\mathcal{L}^n \right) \left(\int_{B_i \cap K} g^p d\mathcal{L}^n \right)^{\frac{t}{p}}$$

$$\leq 2^s \alpha(n)^{-\frac{s}{n}} \sum_i \left(\int_{B_i \cap K} f^q d\mathcal{L}^n \right) \sum_i \left(\int_{B_i \cap K} g^p d\mathcal{L}^n \right)^{\frac{t}{p}}$$

$$\leq 2^s \alpha(n)^{-\frac{s}{n}} \left(\sum_i \int_{B_i \cap K} f^q d\mathcal{L}^n \right) \left(\sum_i \int_{B_i \cap K} g^p d\mathcal{L}^n \right)^{\frac{t}{p}}$$

$$\leq 2^s \alpha(n)^{-\frac{s}{n}} \left(\int_K f^q d\mathcal{L}^n \right) \left(\int_K g^p d\mathcal{L}^n \right)^{\frac{t}{p}} < \infty,$$

where $\alpha(n)$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

• **Case 4:** For $q \geq p \geq t \geq 1$. The proof of Case 4 is identical to the proof of the above case and is therefore omitted.

Finally, we obtain $\overline{\mathcal{P}}_{\mu, \nu}^{q, t, s}(K) < \infty$ which implies that $\Lambda_{\mu, \nu}(q, t) \leq s$ and the result follows. □

We now describe conditions which ensure that a measure has projections which are in L^p for some $p > 1$.

Theorem 3.1 ([23])**.** *Let* μ *be a compactly supported Radon measure on* $\mathbb{R}^n$. *Let* $m \leq s < n$ *and suppose that* $I_s(\mu) < \infty$. *Then* μ_V *is absolutely continuous with respect to* $\mathcal{L}_V^m$, *with* $\mu_V \in L^2(V)$ *for* $\gamma_{n, m}$*-almost all* $V \in G_{n, m}$.

Moreover, for almost every m-dimensional subspace V,

(1) *If* $m < s < 2m$, *then* $\mu_V \in L^p(V)$ *for all* p *satisfying* $1 \leq p < \dfrac{2m}{2m - s}$.

(2) *If* $2m \leq s < n$, *then the Radon-Nikodym derivation of* μ_V *with respect to* $\mathcal{L}_V^m$ *is bounded and essentially continuous.*

Proof of Theorem 2.1. Since $I_s(\mu) < \infty$ and $I_s(\nu) < \infty$, it follows from Theorem 3.1 that, for almost every m-dimensional subspace V,

(1) If $m < s < 2m$, then $\mu_V, \nu_V \in L^p(V)$ for all p satisfying $1 \leq p < \dfrac{2m}{2m - s}$.

(2) If $2m \leq s < n$, then the Radon-Nikodym derivation of $\mu_V(\nu_V)$ with respect to $\mathcal{L}_V^m$ is bounded and essentially continuous.

Finally, the desired result follows directly from Propositions 3.1 and 3.2.

Proof of Theorems 2.2 and 2.3. All of the ideas required to prove Theorems 2.2 and 2.3 can be found in the proof of Theorem 2.1.

Acknowledgements

The author is greatly indebted to the referee for his/her carefully reading the first submitted version of this paper and giving elaborate comments and valuable suggestions on revision so that the presentation can be greatly improved.

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