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# GENERALIZED CLUSTER TILTING OF *n*-ABELIAN CATEGORIES

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## Abstract

In this paper, we study the $(m, n)$-cluster tilting subcategories of $n$-abelian categories as a generalization of $m$-cluster tilting subcategories of abelian categories and prove that the $(m, n)$-cluster tilting subcategories of certain $n$-abelian categories are $mn$-abelian categories.

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## 1. Introduction

In recent years, $n$-cluster tilting theory of representation theory of finite dimensional algebras, abelian categories, triangulated categories, derived categories comes into focus [1, 2, 3, 4, 7]. Motivated by these, Jasso introduced the $n$-abelian categories and $n$-exact categories [5] as a generalization of the classical abelian categories and exact categories, he proved that $n$-cluster tilting subcategories of abelian categories are $n$-abelian categories. As a generalization of homological theory of abelian categories, homological properties of $n$-abelian categories were introduced in [8, 9] via higher (co)homology of $n$-(co)resolutions under right (left) exact functors for $n$-exact sequences.

In this paper, we study the $(m, n)$-cluster tilting subcategories of $n$-abelian categories via $n$-homological theory of abelian categories as a generalization of $m$-cluster tilting subcategories of abelian categories.

This paper is organized as follows. In Section 2, we recall some notions and notations of (co)homology properties of $n$-abelian categories. In Section 3, we introduce the $(m, n)$-cluster tilting subcategories of certain $n$-abelian categories, and show that the $(m, n)$-cluster-tilting subcategories of $n$-abelian categories are $mn$-abelian categories, and study the relationship between $nm$-exact sequences and $m$-fold $n$-exact sequences.

## 2. Definitions and Preliminaries

### 2.1. $n$-Abelian categories

Let $n$ be a positive integer and $\mathcal{C}$ be an additive category. We denote the category of cochain complexes of $\mathcal{C}$ by $\mathsf{Ch}(\mathcal{C})$ and the homotopy category of $\mathcal{C}$ by $\mathsf{H}(\mathcal{C})$. Also, we denote by $\mathsf{Ch}^n(\mathcal{C})$ the full subcategory of $\mathsf{Ch}(\mathcal{C})$ given by all complexes

$$X^0 \overset{d^0}{\to} X^1 \overset{d^1}{\to} \cdots \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} X^{n+1},$$

which are concentrated in degrees $0, 1, \ldots, n+1$. We write $\mathcal{C}(X, Y)$ for the morphisms in $\mathcal{C}$ from $X$ to $Y$, if $X, Y \in \operatorname{ob}\mathcal{C}$.

Let $d^0 : X^0 \to X^1$ be a morphism in $\mathcal{C}$. An *n-cokernel of* $d^0$ is a sequence of morphisms

$$(d^1, \dots, d^n) : X^1 \overset{d^1}{\to} X^2 \overset{d^2}{\to} X^3 \to \cdots \overset{d^n}{\to} X^{n+1}$$

such that for all $1 \leq k \leq n-1$ the morphism $d^k$ is a weak cokernel of $d^{k-1}$, and $d^n$ is moreover a cokernel of $d^{n-1}$. In this case, we say the sequence

$$(d^0, d^1, \dots, d^n) : X^0 \overset{d^0}{\to} X^1 \overset{d^1}{\to} \cdots \overset{d^n}{\to} X^{n+1} \tag{1}$$

is *right n-exact*. The concepts of *n-kernel* of a morphism and *left n-exact* are defined dually. If $n \geq 2$, the $n$-cokernels and $n$-kernels are not unique in general, but their are unique up to isomorphism in $\mathsf{H}(\mathcal{C})$ [5]. (1) is called an *n-exact sequence* if it is both right $n$-exact and left $n$-exact. A sequence

$$X^0 \to X^1 \to \cdots \to X^{mn+1}$$

is called an $m$-fold $n$-exact sequence if it can be split into $m$ $n$-exact sequences $Y^{in} \to X^{in+1} \to \cdots \to X^{(i+1)n} \to Y^{(i+1)n}$ for $i = 0, 1, \cdots, m-1$ where $Y^0 = X^0$ and $Y^{mn} = X^{mn+1}$.

As a generalization of the notion of classical abelian categories, Jasso introduced the $n$-abelian categories in [5] as follows.

**Definition 2.1** ($n$-abelian category, Definition 3.1, [5])**.** An $n$-abelian category is an additive category $\mathcal{A}$ which satisfies the following axioms:

(A0) The category $\mathcal{A}$ is idempotent complete.

(A1) Every morphism in $\mathcal{A}$ has an $n$-kernel and an $n$-cokernel.

(A2) For every monomorphism $f^0 : X^0 \to X^1$ in $\mathcal{A}$ there exists an $n$-exact sequence:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

(A2$^{op}$) For every epimorphism $f^n : X^n \to X^{n+1}$ in $\mathcal{A}$ there exists an $n$-exact sequence:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

Note that 1-abelian categories are precisely abelian categories in the usual sense.

Let $\mathcal{A}$ be an abelian category and $\mathcal{D}$ be a generating-cogenerating full subcategory of $\mathcal{A}$. $\mathcal{D}$ is called an *n-cluster-tilting subcategory of* $\mathcal{A}$ if $\mathcal{D}$ is functorially finite in $\mathcal{A}$ and

$$\mathcal{D} = \{X \in \mathcal{A} | \forall i \in \{1, \ldots, n-1\} \mathrm{Ext}^i_{\mathcal{A}}(X, \mathcal{D}) = 0\}$$

$$= \{X \in \mathcal{A} | \forall i \in \{1, \ldots, n-1\} \mathrm{Ext}^i_{\mathcal{A}}(\mathcal{D}, X) = 0\}.$$

Note that $\mathcal{A}$ itself is the unique 1-cluster-tilting subcategory of $\mathcal{A}$.

**Lemma 2.2** (Theorem 3.16, [5])**.** *Let* $\mathcal{A}$ *be an abelian category and* $\mathcal{D}$ *be an n-cluster tilting subcategory of* $\mathcal{A}$*. Then,* $\mathcal{D}$ *is an n-abelian category.*

### 2.2. (co)Homology of *n*-abelian categories

In this subsection, we recall the right (resp., left) derived functors of covariant or contravariant left (resp., right) $n$-exact functors and study their basic properties.

Let $\mathcal{A}$ be an $n$-abelian category and $\mathcal{B}$ be an abelian category, and let $G : \mathcal{A} \to \mathcal{B}$ be a covariant additive functor. Let $X : X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$ in $\mathrm{Ch}^n(\mathcal{A})$ be an $n$-exact sequence. We say that $G$ is

(i) *left $n$-exact* if $0 \to GX^0 \to GX^1 \to \cdots \to GX^n \to GX^{n+1}$ is an exact sequence of $\mathcal{B}$.

(ii) *right $n$-exact* if $GX^0 \to GX^1 \to \cdots \to GX^n \to GX^{n+1} \to 0$ is an exact sequence of $\mathcal{B}$.

(iii) *$n$-exact* if $0 \to GX^0 \to GX^1 \to \cdots \to GX^n \to GX^{n+1} \to 0$ is an exact sequence of $\mathcal{B}$.

The notions of covariant (contravariant) additive left (right) $n$-exact functors are defined dually. For example, the hom-functors $\mathcal{A}(M, -)$ (resp., $\mathcal{A}(-, M)$) is covariant (resp., contravariant) left $n$-exact by the definition of $n$-kernel (resp., $n$-cokernel).

We say that an $n$-abelian category $\mathcal{A}$ *has enough projectives* if for every object $M \in \mathcal{A}$, there exist projective objects $P_1, P_2, \ldots, P_n \in \mathcal{A}$ and an $n$-exact sequence $N \to P_n \to \cdots \to P_1 \to M$. The notion of *having enough injectives* is defined dually. Let $\mathcal{A}$ has enough projectives, $M \in \mathcal{A}$, there are $n$-exact sequences

$$\Omega_n M \xrightarrow{j_1} P_n \xrightarrow{d_n} \cdots \to P_1 \to M$$

$$\Omega_n^2 M \xrightarrow{j_2} P_{2n} \xrightarrow{d_{2n}} \cdots \to P_{n+1} \xrightarrow{\pi_1} \Omega_n M.$$

Connecting them, let $d_{in+1} = j_i \pi_i$, we call the sequence

$$\cdots \to P_{3n} \xrightarrow{d_{3n}} \cdots \to P_{2n+1} \xrightarrow{d_{2n+1}} P_{2n} \xrightarrow{d_{2n}} \cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \to P_1 \xrightarrow{d_1} M \tag{2}$$

a *projective n-resolution of* $M$, also denoted simply as $P_\bullet \xrightarrow{d_1} M$. We call $\Omega_n^k M$ the $k$-th $n$-syzygy of $M$ for $k \geq 0$. The notions of *injective n-resolution, k-th n-cosyzygy* $\Omega_n^{-k}M$ of $M$ are defined dually.

Let $F : \mathcal{A} \to \mathcal{B}$ be a contravariant left $n$-exact functor. The *right n-derived functors* $\mathsf{nR}^i F$ for $i \geq 0$ as follows, for any $M \in \mathcal{A}$, choose a projective $n$-resolution $P_\bullet \to M$ as (2) and define

$$\mathsf{nR}^i F(M) := H_{in+1}(FP_\bullet) := \mathsf{Ker} F d_{in+2} \,/\, \mathsf{Im} G d_{in+1} \text{ for } i = 0, 1, \cdots.$$

Note that $\mathsf{nR}^0 F(M) \simeq FM$. $\mathsf{nR}^i F(-)$ is an additive functor from $\mathcal{A}$ to $\mathcal{B}$ and $\mathsf{nR}^i F(P) = 0$ for all projective object $P$ for any $i > 0$. The notions of *right (left) n-derived functors of covariant or contravariant left (right) n-exact functors* are defined dually. Specially, for contravariant (resp., covariant) left $n$-exact functor $\mathcal{A}(-, B)$ (resp., $\mathcal{A}(A, -)$), we define the right $n$-derived functors

$$\mathsf{nExt}^i_{\mathcal{A}}(-, B) = \mathsf{nR}^i \mathcal{A}(-, B) \quad \text{resp., } \mathsf{nExt}^i_{\mathcal{A}}(A, -) = \mathsf{nR}^i \mathcal{A}(A, -).$$

In particular, $\mathsf{nExt}^0_{\mathcal{A}}(-, B) = \mathcal{A}(-, B)$, $\mathsf{nExt}^0_{\mathcal{A}}(A, -) = \mathcal{A}(A, -)$.

There is an isomorphism $\mathsf{nE}^m(A, B) \cong \mathsf{nExt}^m_{\mathcal{A}}(A, B)$, here $\mathsf{nE}^m(A, B)$ is the equivalence classes of $m$-fold $n$-extensions of $A$ by $B$, it is an abelian group under $n$-Baer sum [8]. So, we can define $\mathsf{nExt}^m_{\mathcal{A}}(A, B)$ even without of projective objects and injective objects.

**Lemma 2.3** ([8], Proposition 4.3). *Let* $\mathcal{A}$ *be an n-abelian category,* $A, B \in \mathcal{A}$, *we have*

(i) $nExt^i_{\mathcal{A}}(A, -)(B) \simeq nExt^i_{\mathcal{A}}(-, B)(A) = nExt^i_{\mathcal{A}}(A, B)$.

(ii) *If* $\mathcal{A}$ *is an n-cluster tilting subcategory of a projectively generated injectivity cogenerated abelian category* $\mathcal{D}$*. Then* $nExt^{m}_{\mathcal{A}}(A, B) \simeq Ext^{mn}_{\mathcal{D}}(A, B)$, $Ext^{mn+i}_{\mathcal{D}}(A, B) = 0\ \forall A, B \in \mathcal{A}, m \geq 0, 1 \leq i \leq n-1$.

(iii) $A$ *is a projective object if and only if* $\mathcal{A}(A, -)$ *is an exact functor if and only if* $nExt^{i}_{\mathcal{A}}(A, B) = 0$ *for all* $i \neq 0$ *and all* $B$ *if and only if* $nExt^{1}_{\mathcal{A}}(A, B) = 0$ *for all* $B$.

If an $n$-abelian category $\mathcal{A}$ is injectively cogenerated, then by the results of Jasso and Kvamme [5, 6], it follows that $\mathcal{A}$ is equivalent to an $n$-cluster tilting subcategory in the dual of the category of finitely presented covariant functors over the full subcategory of injective objects of $\mathcal{A}$ which is an injectively cogenerated abelian category.

**Lemma 2.4.** *An injectively cogenerated additive category* $\mathcal{C}$ *is an n-abelian category if and only if there exists an injectively-cogenerated abelian categories* $\mathcal{A}$ *such that* $\mathcal{C}$ *can be embedded to* $\mathcal{A}$ *as an n-cluster tilting subcategory*.

Using the Lemma 2.4, we can generalize the "Long $n$-exact sequence Theorem 4.5" of [8] as following:

**Lemma 2.5.** *Let* $\mathcal{A}$ *be an injectively cogenerated n-abelian category.* $X : X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^n} X^{n+1}$ *an n-exact sequence of* $\mathcal{A}$, *for any object* $A \in \mathcal{A}$, *we have exact sequences*

$$0 \to \mathcal{A}(A, X^0) \to \cdots \to \mathcal{A}(A, X^{n+1}) \xrightarrow{\partial_n} nExt^{1}_{\mathcal{A}}(A, X^0) \to \cdots \to nExt^{1}_{\mathcal{A}}(A, X^{n+1}) \xrightarrow{\partial_n^1} \cdots \xrightarrow{\partial_n^{i-1}} nExt^{i}_{\mathcal{A}}(A, X^0) \to \cdots \to nExt^{i}_{\mathcal{A}}(A, X^{n+1}) \xrightarrow{\partial_n^i} \cdots.$$

$$0 \to \mathcal{A}(X^{n+1}, A) \to \cdots \to \mathcal{A}(X^0, A) \xrightarrow{\partial_n} nExt^{1}_{\mathcal{A}}(X^{n+1}, A) \to \cdots \to nExt^{1}_{\mathcal{A}}(X^0, A) \xrightarrow{\partial_n^1} \cdots \xrightarrow{\partial_n^{i-1}} nExt^{i}_{\mathcal{A}}(X^{n+1}, A) \to \cdots \to nExt^{i}_{\mathcal{A}}(X^0, A) \xrightarrow{\partial_n^i} \cdots.$$

### 3. Cluster Tilting Subcategories of *n*-Abelian Categories

By Lemma 2.2, any $n$-cluster tilting subcategory of abelian category is an $n$-abelian category. It is natural to define $(m, n)$-cluster tilting subcategories of $n$-abelian categories.

**Definition 3.1.** Let $\mathcal{A}$ be an $n$-abelian category and $\mathcal{D}$ be a generating-cogenerating full subcategory of $\mathcal{A}$. $\mathcal{D}$ is called an $(m, n)$-*cluster tilting subcategory of* $\mathcal{A}$ if $\mathcal{D}$ is functorially finite in $\mathcal{A}$ and

$$\mathcal{D} = \{X \in \mathcal{A} | \forall i \in \{1, \dots, m-1\} \ \mathrm{nExt}^i_{\mathcal{A}}(X, \mathcal{D}) = 0\}$$

$$= \{X \in \mathcal{A} | \forall i \in \{1, \dots, m-1\} \ \mathrm{nExt}^i_{\mathcal{A}}(\mathcal{D}, X) = 0\}.$$

Note that $\mathcal{A}$ itself is the unique $(1, n)$-cluster-tilting subcategory of $\mathcal{A}$.

Let $\mathcal{A}$ be an abelian category, $\mathcal{D}$ be an $n$-cluster tilting subcategory of $\mathcal{A}$, and $\mathcal{M}$ be an $(m, n)$-cluster tilting subcategory of $\mathcal{D}$. Then, if $m = n = 1$, $\mathcal{A} = \mathcal{D} = \mathcal{M}$. If $n = 1, m > 1, \mathcal{M}$ is an $m$-cluster tilting subcategory of $\mathcal{A}$. If $n > 1, m = 1, \mathcal{M}$ is an $n$-cluster tilting subcategory of $\mathcal{A}$.

Our next aim is to show that the $(m, n)$-cluster tilting subcategories of small $n$-abelian categories are $mn$-abelian categories under certain conditions as a generalization of Lemma 2.2. First, we generalize ([5], Propositions 3.17, 3.18).

**Lemma 3.2.** *Let* $\mathcal{A}$ *be an injectively cogenerated* $n$*-abelian category*, $\mathcal{M}$ *be an* $(m, n)$*-cluster tilting subcategory of* $\mathcal{A}$. *Then, for all* $A \in \mathcal{A}$, *there exist* $n$*-exact sequences*,

$$A \xrightarrow{f^0} M^1 \to \cdots \to M^n \xrightarrow{g^1} D^1$$

$$D^1 \xrightarrow{f^1} M^{n+1} \to \cdots \to M^{2n} \xrightarrow{g^2} D^2$$

$$\cdots$$

$$D^{m-2} \xrightarrow{f^{m-2}} M^{(m-2)n+1} \to \cdots \to M^{(m-1)n} \to M^{(m-1)n+1}$$

*satisfying the following properties*:

(i) $M^i \in \mathcal{M}$;

(ii) $f^i$ *are left* $\mathcal{M}$*-approximations*;

(iii) *For all* $M \in \mathcal{M}$, *the induced sequence of abelian groups*

$$0 \to \mathcal{A}(M^{(m-1)n+1}, M) \to \mathcal{A}(M^{(m-1)n}, M) \to \cdots \to \mathcal{A}(M^1, M) \to \mathcal{A}(A, M) \to 0$$

*is exact*.

**Proof.** This proof is an adaptation of the proof of ([5], Proposition 3.17). Note that $D^0 = A$.

The existences of these $n$-exact sequences follow from the functorially finiteness of $\mathcal{M}$. Indeed, for any $A \in \mathcal{A}$, there exists a left $\mathcal{M}$-approximation $f^0 : A \to M^1$. Since $\mathcal{A}$ is $n$-abelian, there exists a weak cokernel $k^1 : M^1 \to C^1$ in $\mathcal{A}$, then taking a left $\mathcal{M}$-approximation $t^1 : C^1 \to M^2$, this constructs a weak cokernel $t^1k^1$ of $f^0$. Inductively, we can construct a $n$-exact sequence $A \xrightarrow{f^0} M^1 \to \cdots \to M^n \xrightarrow{g^1} D^1$, where $g^1$ is a cokernel of $M^{n-1} \to M^n$ by ([5], Proposition 3.7). Inductively, we can construct the desired $n$-exact sequences.

Given that for all $k \in \{0, \dots, m-2\}$ the morphism $f^k$ is a left $\mathcal{M}$-approximation, it readily follows that the sequence

$$0 \to \mathcal{A}(M^{(m-1)n+1}, M) \to \cdots \to \mathcal{A}(M^1, M) \to \mathcal{A}(A, M) \to 0$$

is exact. It remains to show that $M^{(m-1)n+1} \in \mathcal{M}$.

We claim that for each $M \in \mathcal{M}$, $\mathsf{nExt}^i_{\mathcal{A}}(M^{(m-1)n+1}, M) = 0$ for $i \in \{1, 2, \cdots, m-1\}$. First, note that for all $M \in \mathcal{M}$ applying the contravariant functor $\mathcal{A}(-, M)$ to the $n$-exact sequence (3), we have isomorphisms

$$\mathsf{nExt}^i_{\mathcal{A}}(M^{(m-1)n+1}, M) \simeq \mathsf{nExt}^{i-1}_{\mathcal{A}}(D^{m-2}, M) \simeq \cdots \simeq \mathsf{nExt}^1_{\mathcal{A}}(D^{m-i}, M),$$

for $i \in \{1, 2, \cdots, m-1\}$ by long $n$-exact sequence theorem. Moreover, the morphism $\mathcal{A}(M^{(m-i-1)n+1}, M) \to \mathcal{A}(D^{m-i-1}, M)$ is an epimorphism for $f^{m-i-1}$ is a left $\mathcal{M}$-approximation of $\mathcal{A}$. Thus we have $\mathsf{nExt}^i_{\mathcal{A}}(M^{(m-1)n+1}, M) = 0$ as required. $\square$

**Lemma 3.3.** *Let $\mathcal{A}$ be an injectively cogenerated $n$-abelian category. Let $B \in \mathcal{A}$, and $\mathcal{M}$ be a subcategory of $\mathcal{A}$ such that $\mathsf{nExt}^k_{\mathcal{A}}(\mathcal{M}, B) = 0$ for all $k \in \{1, \ldots, m-1\}$. Consider a composition of $n$-exact sequences*

$$A_1 \xrightarrow{f_1} M_n \to \cdots \to M_1 \xrightarrow{g_0} A$$

$$A_2 \xrightarrow{f_2} M_{2n} \to \cdots \to M_{n+1} \xrightarrow{g_1} A_1$$

$$\vdots$$

$$A_m \xrightarrow{f_m} M_{mn} \to \cdots \to M_{(m-1)n+1} \xrightarrow{g_{m-1}} A_{m-1}$$

*in $\mathcal{A}$ such that $M_k \in \mathcal{M}$ for all $k \in \{1, 2, \ldots, mn\}$. Then, for each $k \in \{1, \ldots, m-1\}$ there is an isomorphism between $\mathsf{nExt}^k_{\mathcal{A}}(A, B)$ and the cohomology of the induced complex*

$$\mathcal{A}(M_1, B) \to \mathcal{A}(M_2, B) \to \cdots \to \mathcal{A}(M_{mn}, B) \to \mathcal{A}(A_m, B) \tag{3}$$

*at $\mathcal{A}(M_{kn+1}, B)$.*

**Proof.** Note that $A_0 = A$. First, let us show that for each $k \in \{1, \ldots, m-1\}$ there exist isomorphisms

$$\mathsf{nExt}^k_{\mathcal{A}}(A_0, B) \simeq \mathsf{nExt}^{k-1}_{\mathcal{A}}(A_1, B) \simeq \cdots \simeq \mathsf{nExt}^1_{\mathcal{A}}(A_{k-1}, B).$$

The case $k = 1$ is obvious. If $2 \leq k \leq m-1$, then for each $2 \leq \ell \leq k$ applying the functor $\mathcal{A}(-, B)$ to the exact sequence $0 \to A_{k-\ell+1} \to M_{(k-\ell+1)n} \to \cdots \to M_{(k-\ell)n+1} \to A_{k-\ell} \to 0$ yields an exact sequence

$$0 = \mathsf{nExt}^{\ell-1}_{\mathcal{A}}(M_{(k-\ell+1)n}, B) \to \mathsf{nExt}^{\ell-1}_{\mathcal{A}}(A_{k-\ell+1}, B) \to \mathsf{nExt}^{\ell}_{\mathcal{A}}(A_{k-\ell}, B) \to \mathsf{nExt}^{\ell}_{\mathcal{A}}(M_{(k-\ell)n+1}, B) = 0.$$

The claim follows.

Second, let us show that $\mathsf{nExt}^1_{\mathcal{A}}(A_{k-1}, B)$ is isomorphic to the cohomology of the complex (3) at $\mathcal{A}(M_{kn+1}, B)$. The conclusion follows from the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{A}(M_{kn}, B) & & \longrightarrow & & \mathcal{A}(M_{kn+1}, B) & \longrightarrow & \mathcal{A}(M_{kn+2}, B) \\
 & \searrow & & \nearrow & & & \\
 & & \mathcal{A}(A_k, B) & & & & \\
 & \nearrow & & \searrow & & & \\
0 & & & & \mathsf{nExt}^1_{\mathcal{A}}(A_{k-1}, B) & & \\
 & & & & & \searrow & \\
 & & & & & & \mathsf{nExt}^1_{\mathcal{A}}(M_{(k-1)n+1}, B) = 0
\end{array}$$

□

**Theorem 3.4.** *Let $\mathcal{A}$ be an injectively cogenerated n-abelian category and $\mathcal{M}$ be an $(m, n)$-cluster tilting subcategory of $\mathcal{A}$. Then, $\mathcal{M}$ is an mn-abelian category.*

**Proof.** We shall show that $\mathcal{M}$ satisfies the axioms of $mn$-abelian category.

(A0) Since the $n$-abelian category $\mathcal{A}$ is idempotent complete, it follows immediately from the definition of $(m, n)$-cluster tilting subcategory that $\mathcal{M}$ also is idempotent complete.

(A1) Let $d^0 : X^0 \to X^1$ be a morphism in $\mathcal{M}$. Let $X^1 \to \cdots \to X^{n+1}$ be an $n$-cokernel of $d^0$, applying Lemma 3.2 to $X^{n+1}$ gives the desired $mn$-cokernel of $d^0$. By duality, $d^0$ has an $mn$-kernel.

(A2) and $(\mathrm{A}2^{op})$ Let $f^0 : X^0 \to X^1$ be a monomorphism in $\mathcal{A}$ such that $X^0, X^1 \in \mathcal{M}$ and let $(f^k : X^k \to X^{k+1} | 1 \le k \le mn)$ be an $mn$-cokernel of $f^0$ in $\mathcal{M}$ obtained as in the previous paragraph. Applying the dual of Lemma 3.3 to $(f^k : X^k \to X^{k+1} | 0 \le k \le mn)$, we obtain that for all $Y \in \mathcal{M}$ and for all $k \in \{1, \ldots, m-1\}$ the cohomology of the induced complex

$$\mathcal{A}(Y, X^1) \to \cdots \to \mathcal{A}(Y, X^{mn}) \to \mathcal{A}(Y, X^{mn+1})$$

at $\mathcal{A}(Y, X^{(k+1)n+1})$ is isomorphic to $\mathsf{nExt}^k_{\mathcal{A}}(Y, X^0)$ which vanishes since $\mathcal{M}$ is an $(m, n)$-cluster-tilting subcategory of $\mathcal{A}$, cohomology of the sequence vanisher at $\mathcal{A}(Y, X^j)$ for $j \neq (k+1)n+1$. This shows that $(f^0, \ldots, f^{mn-1})$ is an $mn$-kernel of $f^{mn}$ in $\mathcal{M}$. $\mathcal{M}$ also satisfies axiom $(\mathrm{A}2^{op})$ follows by duality. □

**Proposition 3.5.** *Let $\mathcal{A}$ be an injectively cogenerated n-abelian category and $\mathcal{M}$ be an $(m, n)$-cluster tilting subcategory of $\mathcal{A}$. Then the sequence of morphisms*

$$X : X^0 \to X^1 \to \cdots \to X^{mn+1}$$

*is an mn-exact sequence of $\mathcal{M}$ if and only if is an m-fold n-exact sequence of $\mathcal{A}$.*

**Proof.** ($\Rightarrow$). We split $d^{in} : X^{in} \to X^{in+1}$ to $X^{in} \overset{\pi^i}{\to} C^i \overset{j^i}{\to} X^{in+1}$ such that $\pi^i$ is a cokernel of $d^{in-1}$ for $i \in \{1, 2, \cdots, m-1\}$. Then we show that $C^{i-1} \overset{j^{i-1}}{\to} X^{(i-1)n+1} \to \cdots \to X^{in} \overset{\pi^i}{\to} C^i$ are $n$-exact sequence of $\mathcal{A}$, it is enough to prove $j^{i-1}$ are monomorphisms and $\mathsf{nExt}^j_{\mathcal{A}}(M, C^i) = 0$ for any $i \in \{1, 2, \cdots, m\}$, $j \in \{1, 2, \cdots, m-1-i\}$ by induction, where $C^0 = X^0$, $C^m = X^{mn+1}$, $j^0 = d^0$ and $\pi^m = d^{mn}$.

For $i = 1$, $j^0$ is a monomorphism. Applying $\mathcal{A}(M, -)$ on $X^0 \to X^1 \to \cdots \to X^n \overset{\pi^1}{\to} C^1$, by long $n$-exact sequence theorem, we have $\mathsf{nExt}^j_{\mathcal{A}}(M, C^1) = 0$ for $j \in \{1, 2, \cdots, m-2\}$ since $\mathsf{nExt}^j_{\mathcal{A}}(M, X^s) = 0$ for $s \in \{0, 1, \cdots, n\}$, $j \in \{1, 2, \cdots, m-1\}$. Suppose that $1 \le k \le m-1$ and that for all $\ell \le k-1$, $j^\ell$ are monomorphisms and $\mathsf{nExt}^j_{\mathcal{A}}(M, C^{\ell+1}) = 0$ for $j \in \{1, 2, \cdots, m-2-\ell\}$. For $j^k$, let $u : A \to C^k$ be a morphism in $\mathcal{A}$ such that $j^k u = 0$. Taking a right $\mathcal{M}$-approximation $v : M \to A$ ($v$ is epic since $\mathcal{M}$ is generating). Applying $\mathcal{A}(M, -)$ to $n$-exact sequence $C^{k-1} \to X^{(k-1)n+1} \to \cdots \to X^{kn} \to C^k$, by long $n$-exact sequence theorem, we have an exact sequence of groups

$$0 \to \mathcal{A}(M, C^{k-1}) \to \mathcal{A}(M, X^{(k-1)n+1}) \to \cdots \to \mathcal{A}(M, X^{kn}) \to \mathcal{A}(M, C^k) \to \mathsf{nExt}^1_{\mathcal{A}}(M, C^{k-1}) = 0. \quad (4)$$

Then there exists a morphism $w : M \to X^{kn}$ such that $u\nu = \pi^k w$, we have

$$d^{kn} w = j^k \pi^k w = j^k u \nu = 0.$$

Therefore, since $d^{kn-1}$ is a weak kernel of $d^{kn}$, there exists a morphism $s : M \to X^{kn-1}$ such that $d^{kn-1}s = w$, thus $u\nu = \pi^k d^{kn-1}s = 0$, $u = 0$, since $\nu$ is an epimorphism, this provides that $j^k$ is a monomorphism. It follows that

$$C^k \to X^{kn+1} \to \cdots \to X^{(k+1)n} \to C^{k+1} \tag{5}$$

is an $n$-exact sequence. Applying $\mathcal{A}(M, -)$ to (5), by long $n$-exact sequence theorem, we have exact sequence of groups for $j = 1, 2, \cdots, m-2$

$$0 = \mathrm{nExt}^j_{\mathcal{A}}(M, X^{(k+1)n}) \to \mathrm{nExt}^j_{\mathcal{A}}(M, C^{k+1}) \to \mathrm{nExt}^{j+1}_{\mathcal{A}}(M, C^k) \to \mathrm{nExt}^{j+1}_{\mathcal{A}}(M, X^{kn+1}) = 0.$$

but, $\mathrm{nExt}^{j+1}_{\mathcal{A}}(M, C^k) = 0$ for $j = 0, \cdots, m-2-k$, this finishes the induction steps.

($\Leftarrow$). We split $X$ to $m$ $n$-exact sequences

$$C^{i-1} \xrightarrow{j^{i-1}} X^{(i-1)n+1} \to \cdots \to X^{in} \xrightarrow{\pi^i} C^i,$$

where $C^0 = X^0$, $C^m = X^{mn+1}$, $j^0 = d^0$ and $\pi^m = d^{mn}$ for $i = 1, 2, \cdots, m$. Applying $\mathcal{A}(M, -)$ to these $n$-exact sequences, by long $n$-exact sequence theorem, it is easily prove that $\mathrm{nExt}^1_{\mathcal{A}}(\mathcal{M}, C^j) = 0$ for $j = 0, 1, \cdots, m-2$.

We only need to show that $d^{in}$ is a weak kernel of $d^{in+1}$ and $d^{in}$ is a weak cokernel of $d^{in-1}$ for $i = 1, 2, \cdots, m-1$. We only show that $d^{in}$ is a weak kernel of $d^{in+1}$. Let $u : M \to X^{in+1}$ be a morphism in $\mathcal{M}$ such that $d^{in+1}u = 0$, since $j^i$ is a kernel of $d^{in+1}$, there exists a morphism $\nu : M \to C^i$ such that $j^i\nu = u$. Applying $\mathcal{A}(M, -)$ to $n$-exact sequence $C^{i-1} \to X^{(i-1)n+1} \to \cdots \to X^{in} \to C^i$, by long $n$-exact sequence theorem, we have an exact sequence of groups like (4), thus, there exists a morphism $w : M \to X^{in}$ such that $\pi^i w = \nu$, so $d^{in}w = u$. □

**Theorem 3.6.** *Let $\mathcal{A}$ be a projectively generated abelian category, $\mathcal{D}$ be an n-cluster tilting subcategory of $\mathcal{A}$ which closed under n-th syzygy, and $\mathcal{M}$ be an additive full subcategory of $\mathcal{D}$. Then, if $\mathcal{M}$ is an mn-cluster tilting subcategory of $\mathcal{A}$, then $\mathcal{M}$ is an $(m, n)$-cluster tilting subcategory of $\mathcal{D}$.*

**Proof.** If $\mathcal{M}$ is an $mn$-cluster tilting subcategory of $\mathcal{A}$, then

$$\mathcal{M} = \{X \in \mathcal{A} | \forall i \in \{1, 2, \dots, mn - 1\} \operatorname{Ext}^i_{\mathcal{A}}(X, \mathcal{M}) = 0\}$$

$$= \{X \in \mathcal{D} | \forall i \in \{1, 2, \dots, mn - 1\} \operatorname{Ext}^i_{\mathcal{A}}(X, \mathcal{M}) = 0\}$$

but, $\mathcal{D}$ is an $n$-cluster tilting subcategory closed under $n$-th syzygy, so $\mathcal{D}$ is an $n$-abelian category which has enough projective objects, by Lemma 2.3, we have $\operatorname{Ext}^j_{\mathcal{A}}(X, \mathcal{M}) = 0$ for all $j \neq kn, k \in \mathbb{N}^*$, so

$$\{X \in \mathcal{D} | \forall i \in \{1, 2, \dots, mn - 1\} \operatorname{Ext}^i_{\mathcal{A}}(X, \mathcal{M}) = 0\}$$

$$= \{X \in \mathcal{D} | \forall i \in \{1, 2, \dots, m - 1\} \operatorname{Ext}^{in}_{\mathcal{A}}(X, \mathcal{M}) = 0\}$$

$$= \{X \in \mathcal{D} | \forall i \in \{1, \dots, m - 1\} \operatorname{nExt}^i_{\mathcal{D}}(X, \mathcal{M}) = 0\}.$$

Since $\mathcal{M}$ is generating and cogenerating functorial finite subcategory of $\mathcal{A}$, so is generating and cogenerating functorial finite subcategory of $\mathcal{D}$. So, $\mathcal{M}$ is an $(m, n)$-cluster tilting subcategory of $\mathcal{D}$. □

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