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APPLICATION OF THE LAPLACE-ADOMIAN METHOD AND THE SBA METHOD TO SOLVING THE PARTIAL DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we use firstly the Laplace-Adomian method to construct the solution of a kind of Fischer's equation and secondly the SBA method to construct the solution of an integral-differential equation.

1. Introduction

Many problems in natural and engineering sciences are modelled by partial differential equations (PDEs) and integral equations. Most of these equations are nonlinear. Several techniques have been used to find

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the analytical solutions of such equations, like the perturbation method, the homotopy perturbation method, the Adomian decomposition method, the Laplace-Adomian method, the SBA method and others [3], [4], [19], [20]. Here we use the Laplace-Adomian method and the SBA method to investigate a kind of Fischer's equation and an Fredholm integro-differential equation.

1.1. About Laplace-Adomian method and the SBA method

About Laplace-Adomian method, we can see [3]-[9]. Using the simple Laplace transformations, it is difficult to find the solution of nonlinear differential equation. But, the combination of the Laplace transformations and the Adomian decomposition method allows us to avoid this difficulty.

1.1.1. The Laplace-Adomian decomposition method (LADM)

Suppose that we need to solve the following equation:

$$F(u(x, t)) = h(x, t) \tag{1}$$

with the following initial condition:

$$u(x, 0) = f(x) \tag{2}$$

in a Banach space E, where $F : E \to E$ is a linear or a nonlinear operator, $h \in E$ and u is the unknown function. The principle of LADM decomposition which has been the subject of several studies [6], [7], [19] is based on the decomposition of the operator F in the following form:

$$F = L + R + N, \tag{3}$$

where $L + R$ is linear, N is nonlinear, L is invertible with L^{-1} as inverse.

Using that decomposition, Equation (1) is equivalent to

$$Lu + Ru + Nu = h(x, t). \tag{4}$$

We denote $L = \frac{\partial(.)}{\partial t}$ and $L^{-1} = \int_0^t (.)ds.$

Let's note the Laplace transform by $\mathcal{L}(Lu(x, t)) = \int_0^\infty u(x, t)e^{-st}dt.$

From (4), we have

$$\mathcal{L}(Lu(x, t)) + \mathcal{L}(Ru(x, t)) + \mathcal{L}(Nu(x, t)) = \mathcal{L}(h(x, t)) \tag{5}$$

$\Leftrightarrow$

$$s\mathcal{L}(u(x, t)) - u(x, 0) + \mathcal{L}(Ru(x, t)) + \mathcal{L}(Nu(x, t)) = \mathcal{L}(h(x, t)) \tag{6}$$

$\Leftrightarrow$

$$s\mathcal{L}(u(x, t)) = u(x, 0) + \mathcal{L}(h(x, t)) - \mathcal{L}(Ru(x, t)) - \mathcal{L}(Nu(x, t)). \tag{7}$$

Thus

$$\mathcal{L}(u(x, t)) = \frac{1}{s}u(x, 0) + \frac{1}{s}\mathcal{L}(h(x, t)) - \frac{1}{s}\mathcal{L}(Ru(x, t)) - \frac{1}{s}\mathcal{L}(Nu(x, t)) \tag{8}$$

$\Leftrightarrow$

$$\mathcal{L}(u(x, t)) = \frac{1}{s}f(x) + \frac{1}{s}\mathcal{L}(h(x, t)) - \frac{1}{s}\mathcal{L}(Ru(x, t)) - \frac{1}{s}\mathcal{L}(Nu(x, t)). \tag{9}$$

We look for the solution of (1) in the following form:

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \tag{10}$$

and suppose that

$$N(u(x, t)) = \sum_{n=0}^{\infty} A_n(x, t), \tag{11}$$

where A_n are special polynomials $u_0, u_1, u_2, \cdots, u_n$ called Adomian polynomials and defined by [5]-[8]:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{+\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \; n = 0, 1, 2, \dots, \tag{12}$$

where λ is a parameter used by "convenience".

We can also use the following formulas:

$$\begin{cases} A = N(\varphi_0), \\ A_{n+1} = \dfrac{1}{n+1}\displaystyle\sum_{\mathrm{k}=0}^{n}(k+1)\varphi_{k+1}\,\dfrac{\partial A_n}{\partial \varphi_k}. \end{cases} \tag{13}$$

From (10) and (11), the Equation (9) becomes

$$\sum_{n=0}^{\infty}\mathcal{L}(u_n(x,t)) = \frac{1}{s}f(x) + \frac{1}{s}\mathcal{L}(h(x,t)) - \sum_{n=0}^{\infty}\left(\frac{1}{s}\mathcal{L}(Ru_n(x,t)) + \frac{1}{s}\mathcal{L}(A_n(x,t))\right). \tag{14}$$

According to the classic theory of the Adomian decomposition method [6]-[8], we construct the following Adomian algorithm:

$$\begin{cases} \mathcal{L}(u_0(x,t)) = \dfrac{1}{s}f(x) + \dfrac{1}{s}\mathcal{L}(h(x,t)), \\ \mathcal{L}(u_{n+1}(x,t)) = -\dfrac{1}{s}\mathcal{L}(Ru_n(x,t)) - \dfrac{1}{s}\mathcal{L}(A_n(x,t)),\ n \geq 0. \end{cases} \tag{15}$$

The inverse Laplace transform gives as

$$\begin{cases} u_0(x,t) = \mathcal{L}^{-1}\left(\dfrac{1}{s}f(x) + \dfrac{1}{s}\mathcal{L}(h(x,t))\right), \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left(-\dfrac{1}{s}\mathcal{L}(Ru_n(x,t)) - \dfrac{1}{s}\mathcal{L}(A_n(x,t)),\ n \geq 0\right). \end{cases} \tag{16}$$

1.1.2. SBA method

About SBA method, we can see [1], [2]. The SBA method is powerful than the Adomian decomposition method because the SBA method avoids us the calculation of the Adomian polynomials.

The SBA method is based on the Adomian decomposition method, the successive approximation method and the Picard principle. Suppose that we need to solve the following equation:

$$Au = f, \tag{17}$$

where $A : H \rightharpoonup H,$ is a linear or nonlinear operator and H is a Hilbert space. Let's suppose that we can decompose the operator A in the following form:

$$A = L + R + N, \tag{18}$$

where $L + R$ is the linear part and N is the nonlinear part. We suppose that L is invertible with L^{-1} as inverse, then the Equation (17) becomes

$$Lu + Ru + Nu = f \Leftrightarrow u = \theta + L^{-1}f - L^{-1}Ru - L^{-1}Nu, \tag{19}$$

where θ is such that $L\theta = 0.$ Equation (19) is the Adomian canonical form. Using the successive approximations, we get

$$\begin{cases} u^k = \theta^k + L^{-1}f^k - L^{-1}R(u^k) - L^{-1}N(u^{k-1});\ k \geq 1, \\ \text{with} \\ f^1 = f^2 = \cdots = f;\ \theta^1 = \theta^2 = \cdots = \theta. \end{cases} \tag{20}$$

We look for the solution of (17), for every $k \geq 1$ in the following form $u^k = \sum_{n=0}^{+\infty} u_n^k(x, t)$ and we suppose that series converges.

Therefore, $u = \lim_{k \to +\infty} u^k$ is the solution of the Equation (17).

(20) yields the following Adomian algorithm [2]-[12]:

$$\begin{cases} u_0^k = \theta^k + L^{-1}f^k - L^{-1}N(u^{k-1}) \\ u_{n+1}^k = -L^{-1}R(u_n^k),\ n \geq 0 \end{cases} k \geq 1. \tag{21}$$

According to the Picard principle, for every $k \geq 1,$ one must choose u^{k-1} as $N(u^{k-1}) = 0.$ Thus;

For $k = 1$, we have

$$\begin{cases} u_0^1 = \theta + L^{-1} f, \\ u_{n+1}^1 = -L^{-1} R(u_n^1), \; n \geq 0, \end{cases} \tag{22}$$

we suppose that the condition $N(u) = 0$ is satisfied then $u^1 = \sum_{n=0}^{+\infty} u_n^1(x, t)$.

For $k = 2$, we have

$$\begin{cases} u_0^2 = \theta + L^{-1} f, \\ u_{n+1}^2 = -L^{-1} R(u_n^2), \; n \geq 0. \end{cases} \tag{23}$$

Here we must verify that $N(u^1) = 0$ then $u^2 = \sum_{n=0}^{+\infty} u_n^2(x, t)$.

We make the same procedure for $k \geq 2$ and the solution of Equation (17) is $u = \lim_{\mathrm{k} \to +\infty} u^k$.

2. Application of the Laplace-Adomian Method to Solve a Kind of Fischer's Equation

We consider the following initial value problem [17]:

$$\begin{cases} \dfrac{\partial u(x, t)}{\partial t} - \alpha \dfrac{\partial^2 u(x, t)}{\partial x^2} = \varepsilon u(x, t) - \lambda u^{q+1}(x, t), \quad \forall \; t > 0, \\ u(x, 0) = \beta, \end{cases} \tag{24}$$

where $\lambda, \alpha, \varepsilon \in \mathbb{R}$ and $q \in \mathbb{R}^*$.

Let's put

$$Nu(x, t) = u^{q+1}(x, t). \tag{25}$$

From (24), we have

$$\frac{\partial u(x,\, t)}{\partial t} = u(x,\, t) + \alpha \frac{\partial^2 u(x,\, t)}{\partial x^2} - \lambda u^{q+1}(x,\, t). \tag{26}$$

Let's note the Laplace transform by

$$U(x,\, s) = \mathcal{L}(u(x,\, t)) = \int_0^{\infty} u(x,\, t) e^{-st} dt. \tag{27}$$

From (27), we obtain

$$\mathcal{L}\left(\frac{\partial u(x,\, t)}{\partial t} \right) = \int_0^{\infty} e^{-st} \frac{\partial u}{\partial t} dt = sU(x,\, s) - u(x,\, 0). \tag{28}$$

Applying the Laplace transformations on (26), we have

$$\mathcal{L}(u(x,\, t)) = \frac{\beta}{s - \varepsilon} + \frac{\alpha}{s - \varepsilon} \mathcal{L}\left(\frac{\partial^2 u(x,\, t)}{\partial x^2} \right) - \frac{\lambda}{s - \varepsilon} \mathcal{L}(Nu(x,\, t)). \tag{29}$$

Using the Laplace inverse transformation, we obtain

$$u(x,\, t) = \beta e^{\varepsilon t} + \alpha \mathcal{L}^{-1}\left(\frac{1}{s - \varepsilon} \mathcal{L}\left(\frac{\partial^2 u(x,\, t)}{\partial x^2} \right) \right) - \lambda \mathcal{L}^{-1}\left(\frac{1}{s - \varepsilon} \mathcal{L}(Nu(x,\, t)) \right). \tag{30}$$

According to the Adomian decomposition method, we look for the solution of (24) in the following form:

$$u(x,\, t) = \sum_{n=0}^{+\infty} u_n(x,\, t), \tag{31}$$

and we suppose that

$$Nu(x,\, t) = \sum_{n=0}^{+\infty} A_n(x,\, t). \tag{32}$$

From [13], we get

$$\begin{cases} A_0(x,\, t) = N(u_0(x,\, t)), \\ A_{n+1}(x,\, t) = \dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n} (k+1) u_{k+1} \frac{\partial A_n}{\partial u_k}, \; n \in \mathbb{N}. \end{cases} \tag{33}$$

Substituting (31) and (32) into (30), we obtain the following Adomian algorithm:

$$\begin{cases} u_0(x, t) = \beta e^{\varepsilon t}, \\ u_{n+1}(x, t) = +\alpha \mathcal{L}^{-1}\left(\frac{1}{s-\varepsilon} \mathcal{L}(\frac{\partial^2 u(x, t)}{\partial x^2}) \right) - \lambda \mathcal{L}^{-1}(\frac{1}{s-\varepsilon} \mathcal{L}(A_n(x, t))). \end{cases} \tag{34}$$

From (33), we obtain

$$\begin{cases} A_0 = u_0^{q+1}, \\ A_1 = (q+1)(\frac{u_1}{u_0}) u_0^{q+1}, \\ A_2 = \frac{(q+1)(2q+1)}{2!} (\frac{u_1}{u_0})^2 u_0^{q+1}, \\ A_3 = \frac{(q+1)(2q+1)(3q+1)}{3!} (\frac{u_1}{u_0})^3 u_0^{q+1}, \\ \qquad \cdots \\ A_n = \frac{\prod_{k=0}^{n} (kq+1)}{n!} (\frac{u_1}{u_0})^n u_0^{q+1}. \end{cases} \tag{35}$$

From (34), we obtain

$$\begin{cases} u_0(x, t) = \beta e^{\varepsilon t}, \\ u_1(x, t) = \left(-\frac{\lambda \beta^q}{\varepsilon q} (e^{q\varepsilon t} - 1) \right) \beta e^{\varepsilon t}, \\ u_2(x, t) = \frac{q+1}{2!} \left(-\frac{\lambda \beta^q}{\varepsilon q} (e^{q\varepsilon t} - 1) \right)^2 \beta e^{\varepsilon t}, \\ u_3(x, t) = \frac{(q+1)(2q+1)}{3!} \left(-\frac{\lambda \beta^q}{\varepsilon q} (e^{q\varepsilon t} - 1) \right)^3 \beta e^{\varepsilon t}, \\ u_4(x, t) = \frac{(q+1)(2q+1)(3q+1)}{4!} \left(-\frac{\lambda \beta^q}{\varepsilon q} (e^{q\varepsilon t} - 1) \right)^4 \beta e^{\varepsilon t}, \\ \qquad \cdots\cdots \\ u_n(x, t) = \frac{\prod_{k=0}^{n-1} (\mathrm{k}q+1)}{n!} \left(-\frac{\lambda \beta^q}{\varepsilon q} (e^{q\varepsilon t} - 1) \right)^n \beta e^{\varepsilon t}, \; n \geq 1. \end{cases} \tag{36}$$

So,

$$u(x,\ t) = \left[1 + \sum_{n=1}^{+\infty} \frac{\prod_{k=0}^{n-1}(kq+1)}{n!}\left(-\frac{\lambda\beta^q}{\varepsilon q}\left(e^{q\varepsilon t}-1\right)\right)^n\right]\beta e^{\varepsilon t}. \tag{37}$$

Lemma. *According to the D'Alembert criteria, the following series*

$$\sum_{n=1}^{+\infty} \frac{\prod_{k=0}^{n-1}(kq+1)}{n!}\left(-\frac{\lambda\beta^q}{\varepsilon q}\left(e^{q\varepsilon t}-1\right)\right)^n \tag{38}$$

is absolutely convergent for all $t \in \left[\frac{1}{q\varepsilon}\ln\left(1 - \frac{q\varepsilon}{\lambda(|q|+1)\beta^q}\right), \frac{1}{q\varepsilon}\ln\left(1 + \frac{q\varepsilon}{\lambda(|q|+1)\beta^q}\right)\right]$ *and the remain term* R_n *verifies the following relation:*

$$0 \le R_n \le \frac{\left|-\frac{\lambda\beta^q}{\varepsilon}\left(e^{q\varepsilon t}-1\right)\right|}{1 - \left|-\frac{\lambda\beta^q}{\varepsilon}\left(e^{q\varepsilon t}-1\right)\right|}. \tag{39}$$

Proof. Let's denote

$$a_n = \frac{\prod_{k=0}^{n-1}(kq+1)}{n!}, \tag{40}$$

and

$$b_n = \left(-\frac{\lambda\beta^q}{\varepsilon q}\left(e^{q\varepsilon t}-1\right)\right)^n. \tag{41}$$

We remark that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq \left|\frac{nq+1}{n+1}\right| \leq \left|\frac{nq}{n+1}\right| + \left|\frac{1}{n+1}\right| \leq |q| + 1, \tag{42}$$

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t} - 1)\right|. \tag{43}$$

From (42) and (43), we have

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\frac{b_{n+1}}{b_n}\right| = \left|\frac{a_{n+1}}{a_n}\right|\left|\frac{b_{n+1}}{b_n}\right| \leq 1 &\Leftrightarrow (|q|+1)\left|-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t} - 1)\right| \leq 1 \\
&\Leftrightarrow \left|-\frac{\lambda\beta^q(|q|+1)}{\varepsilon q}(e^{q\varepsilon t} - 1)\right| \leq 1 \\
&\Leftrightarrow -1 \leq -\frac{\lambda\beta^q(|q|+1)}{\varepsilon q}(e^{q\varepsilon t} - 1) \leq 1 \\
&\Leftrightarrow -\frac{\varepsilon q}{\lambda(|q|+1)\beta^q} \leq e^{q\varepsilon t} - 1 \leq \frac{\varepsilon q}{\lambda(|q|+1)\beta^q} \\
&\Leftrightarrow 1 - \frac{\varepsilon q}{\lambda(|q|+1)\beta^q} \leq e^{q\varepsilon t} \leq 1 + \frac{\varepsilon q}{\lambda(|q|+1)\beta^q}.
\end{aligned} \tag{44}$$

If $\lambda(|q|+1)\beta^q > \varepsilon q$, we get

$$\frac{1}{q\varepsilon}\ln\left(1 - \frac{\varepsilon q}{\lambda(|q|+1)\beta^q}\right) \leq t \leq \frac{1}{q\varepsilon}\ln\left(1 + \frac{\varepsilon q}{\lambda(|q|+1)\beta^q}\right). \tag{45}$$

Proposition. *Suppose that* $(q, m) \in \mathbb{R}^* \times \mathbb{N}^*$, $(\beta, \lambda, \varepsilon) \in \mathbb{R}^3$, $x \in \mathbb{R}^m$, $\alpha \in \mathbb{R}^m$, $\delta \in \mathbb{R}^m$, $\Omega = \left[\frac{1}{q\varepsilon}\ln\left(1 - \frac{q\varepsilon}{\lambda(|q|+1)\beta^q}\right), \frac{1}{q\varepsilon}\ln\left(1 + \frac{q\varepsilon}{\lambda(|q|+1)\beta^q}\right)\right]$,

then,

the solution of the following problem:

$$\frac{\partial u(x,\ t)}{\partial t} - \nabla^k u(x,\ t).\ \alpha + \nabla u(x,\ t).\ \delta = \varepsilon u(x,\ t) - \lambda u^{q+1}(x,\ t),\ k \geq 2, \quad (46)$$

$$u(x,\ 0) = \beta \quad (47)$$

on the set $\mathbb{R}^m \times \Omega$ *is*

$$u(x,\ t) = \left[1 + \sum_{n=1}^{+\infty} \frac{\prod_{k=0}^{n-1}(kq+1)}{n!}\left(-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t} - 1)\right)^n\right]\beta e^{\varepsilon t}, \quad (48)$$

where $u.v$ *is the scalar product of* u *and* v. *Otherwise, this series solution is convergent on* $\mathbb{R}^m \times \Omega$.

Proof. If $t = 0$, from (48), we obtain (46).

If $t \neq 0$, from (48), we get

$$\frac{\partial u(x,\ t)}{\partial t} = u_0(x,\ t)\frac{d}{dt}\left[1 + \sum_{n=1}^{+\infty} \frac{\prod_{k=0}^{n-1}(kq+1)}{n!}\left(-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t} - 1)\right)^n\right]$$

$$+ \left[1 + \sum_{n=1}^{+\infty} \frac{\prod_{k=0}^{n-1}(kq+1)}{n!}\left(-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t} - 1)\right)^n\right]\frac{\partial u_0(x,\ t)}{\partial t}. \quad (49)$$

We firstly remark that

$$\frac{\partial u_0(x,\ t)}{\partial t} = \varepsilon u_0(x,\ t) \text{ and } \left[1 + \sum_{n=1}^{+\infty} \frac{\prod_{k=0}^{n-1}(kq+1)}{n!}\left(-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t} - 1)\right)^n\right] = \frac{u(x,\ t)}{u_0(x,\ t)},$$

so

$$\left[1+\sum_{n=1}^{+\infty}\frac{\prod_{k=0}^{n-1}(kq+1)}{n!}\left(-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t}-1)\right)^n\right]\frac{\partial u_0(x,t)}{\partial t}=\varepsilon u(x,t), \tag{50}$$

on the other hand,

$$u_0(x,t)\frac{d}{dt}\left[1+\sum_{n=1}^{+\infty}\frac{\prod_{k=0}^{n-1}(kq+1)}{n!}\left(-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t}-1)\right)^n\right]=-\lambda u_0^{q+1}(x,t)\sum_{n=1}^{+\infty}\frac{\prod_{k=0}^{n-1}(kq+1)}{(n-1)!}$$

$$\times\left(-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t}-1)\right)^{n-1}$$

$$=-\lambda u_0^{q+1}(x,t)\sum_{N=0}^{+\infty}\frac{\prod_{k=0}^{N}(kq+1)}{N!}\left(-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t}-1)\right)^N. \tag{51}$$

Secondly:

$$\sum_{N=0}^{+\infty}\frac{\prod_{k=0}^{N}(kq+1)}{N!}\left(-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t}-1)\right)^N=\sum_{N=0}^{+\infty}\frac{A_N}{u_0^{q+1}(x,t)}, \tag{52}$$

so

$$u_0(x, t)\frac{d}{dt}\left[1 + \sum_{n=1}^{+\infty}\frac{\prod_{k=0}^{n-1}(kq+1)}{n!}\left(-\frac{\lambda\beta^q}{\varepsilon q}(e^{q\varepsilon t}-1)\right)^n\right] = -\lambda u_0^{q+1}(x, t)\sum_{n=0}^{+\infty}\frac{A_N}{u_0^{q+1}(x, t)}$$

$$= -\lambda\sum_{N=0}^{+\infty}A_N$$

$$= -\lambda Nu(x, t)$$

$$= -\lambda u^{q+1}(x, t). \tag{53}$$

Thirdly, $u(x, t)$ depends only on t variable, so

$$\nabla^k u(x, t).\ \alpha + \nabla u(x, t).\ \delta = 0. \tag{54}$$

Finally,

$$\frac{\partial u(x, t)}{\partial t} - \nabla^k u(x, t).\ \alpha + \nabla u(x, t).\ \delta = \varepsilon u(x, t) - \lambda u^{q+1}(x, t). \tag{55}$$

Remark. If $q = 1$, $\varepsilon = 1$, $\lambda = 1$, $\alpha = 1$, $\delta = 1$ and $m = 1$, we recover the case examined in [17], and we have

$$u(x, t) = \frac{\beta e^t}{1 - \beta + \beta e^t}. \tag{56}$$

3. Application of the SBA Method to Solve a Fredholm Integro-Differential Equation

Let's consider the following Cauchy problem [18]:

$$\begin{cases} \dfrac{\partial u(x,\, t)}{\partial t} = u(x,\, t) + \displaystyle\int_0^1 tu^{\alpha}(x,\, t)dt - \dfrac{f^{\alpha}(x)}{\alpha^2}((\alpha-1)e^{\alpha}+1),\ (x,\, t) \in \mathbb{R} \times \mathbb{R}_{+}^{*}, \\ u(x,\, 0) = f(x). \end{cases} \tag{57}$$

We are going to construct the solution of this problem using the SBA method, where $f \in \mathbb{C}^{\infty}(\mathbb{R}),\ \alpha \in \mathbb{R}^{*}$.

Let's note $N(u(x,\, t)) = \int_0^1 tu^{\alpha}(x,\, t)dt - \frac{f^{\alpha}(x)}{\alpha^2}((\alpha-1)e^{\alpha}+1)$. From (57), we have

$$u(t,\, x) = u(x,\, 0) + \int_0^t u(s,\, x)ds + \int_0^t \left[\int_0^t tu^{\alpha}(t,\, x)dt - \frac{f^{\alpha}(x)}{\alpha^2}((\alpha-1)e^{\alpha}+1)\right]ds. \tag{58}$$

(58) is equivalent to

$$u(x,\, t) = u(x,\, 0) + \int_0^t u(s,\, x)ds + \int_0^t N(u(x,\, s))ds. \tag{59}$$

According to the SBA method, we suppose that the solution of (57) has the following form:

$$u(x,\, t) = \lim_{k \to +\infty} u^{k}(x,\, t), \tag{60}$$

where

$$u^{k}(x,\, t) = \sum_{n=0}^{+\infty} u_n^{k}(x, t);\ k \geq 1, \tag{61}$$

and for every $k \geq 1,$ we get $u_n^k(x, t)$ for $n \geq 0$ through the following SBA algorithm:

$$\begin{cases} u_0^k(x, t) = u^k(x, 0) + \int_0^t Nu^{k-1}(x, s)ds; \; k \geq 1, \\ u_{n+1}^k(x, t) = \int_0^t u(s, x)ds; \; n \geq 0. \end{cases} \tag{62}$$

Here, $u^k(x, 0) = u(x, 0) = f(x).$

The SBA principle needs that, for $k = 1,$ we must choose $u^0(x, t)$ like $Nu^0(x, t) = 0$ and for $k > 1,$ we must verify that $Nu^{k-1}(x, t) = 0.$

Thus; for $k = 1,$ we have the following SBA algorithm:

$$\begin{cases} u_0^1(x, t) = f(x) + \int_0^t Nu^0(x, s)ds, \\ u_{n+1}^1(x, t) = \int_0^t u_n^1(x, s)ds; \; n \geq 0. \end{cases} \tag{63}$$

Here, taking $u^0(x, t) = e^t f(x),$ we obtain $Nu^0(x, t) = 0,$ and from (63) we have the following algorithm:

$$\begin{cases} u_0^1(x, t) = f(x), \\ u_{n+1}^1(x, t) = \int_0^t u_n^1(x, s)ds; \; n \geq 0. \end{cases} \tag{64}$$

From (64), we obtain

$$\begin{cases} u_1^1(x,\, t) = tf(x), \\ u_2^1(x,\, t) = \dfrac{t^2}{2!}\, f(x), \\ u_3^1(x,\, t) = \dfrac{t^3}{3!}\, f(x), \\ \dots \\ u_n^1(x,\, t) = \dfrac{t^n}{n!}\, f(x) \dfrac{t^n}{n!}\, f(x). \end{cases} \tag{65}$$

From (65), we obtain

$$u^1(x,\, t) = \sum_{n=0}^{+\infty} u_n^1(x,\, t) = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\, f(x) = f(x) \sum_{n=0}^{+\infty} \frac{t^n}{n!} = e^t f(x). \tag{66}$$

We note that $u^1(x,\, t) = u^0(x,\, t) = e^t f(x)$.

So; for $k = 2$, we have $Nu^1(x,\, t) = Nu^0(x,\, t) = 0$, therefore we have the same algorithm like for $k = 1$, then

$$\begin{cases} u_0^2(x,\, t) = f(x), \\ u_{n+1}^2(x,\, t) = \displaystyle\int_0^t u_n^1(x,\, s) ds;\ n \geq 0, \end{cases} \tag{67}$$

and we obtain

$$u^2(x,\, t) = e^t f(x). \tag{68}$$

Using the same procedure, for $k \geq 3$, we have $u^k(x,\, t) = e^t f(x)$.

So,

$$u(x,\, t) = \lim_{k \to +\infty} u^k(x,\, t) = \lim_{k \to +\infty} e^t f(x) = e^t f(x). \tag{69}$$

Proposition. *The solution of the following Cauchy problem*:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = u(x,t) + \displaystyle\int_0^1 tu^{\alpha}(x,t)dt - \dfrac{f^{\alpha}(x)}{\alpha^2}((\alpha-1)e^{\alpha}+1),\ (x,t) \in \mathbb{R}^m \times \mathbb{R}_+^*, \\ u(x,0) = f(x), \end{cases} \tag{70}$$

where $f \in \mathbb{C}^{\infty}(\mathbb{R}^m)$, $m \in \mathbb{N}^*$ *and* $\alpha \in \mathbb{R}^*$, *is*

$$u(x_1, x_2, \ldots, x_m, t) = f(x_1, x_2, \ldots, x_m, t)e^t. \tag{71}$$

Proof. If $t = 0$, we have $u(x_1, x_2, \ldots, x_m, 0) = f(x_1, x_2, \ldots, x_m)e^0$ $= f(x_1, x_2, \ldots, x_m)$.

If $t \neq 0$, we have

$$\begin{cases} \displaystyle\int_0^1 tu^{\alpha}(x,t)dt &= \displaystyle\int_0^1 tf^{\alpha}(x)e^{\alpha t}dt \\ &= f^{\alpha}(x)\displaystyle\int_0^1 te^{\alpha t}dt \\ &= f^{\alpha}(x)\left(\dfrac{e^{\alpha}}{\alpha} - \dfrac{1}{\alpha}\displaystyle\int_0^1 te^{\alpha t}dt\right) \\ &= f^{\alpha}(x)\left[\dfrac{e^{\alpha}}{\alpha} - \dfrac{1}{\alpha^2}(e^{\alpha}-1)\right] \\ &= f^{\alpha}(x)\dfrac{(\alpha-1)e^{\alpha}+1}{\alpha^2}, \end{cases} \tag{72}$$

and

$$\frac{\partial u(x_1, x_2, \ldots, x_m, t)}{\partial t} = f(x_1, x_2, \ldots, x_m, t)e^t = u(x_1, x_2, \ldots, x_m, t). \tag{73}$$

So

$$\frac{\partial u(x,t)}{\partial t} = u(x,t) + \int_0^1 tu^{\alpha}(x,t)dt - \frac{f^{\alpha}(x)}{\alpha^2}((\alpha-1)e^{\alpha}+1). \tag{74}$$

4. Conclusion

Through these two examples, we showed on the one hand the superiority of the Laplace-Adomian method in relation to the Laplace transformations and on the other hand the efficiency of the SBA method in the resolution of the ordinary and partial differential equations, and integro-differential equations.

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