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THE RETENTIVITY OF TRANSITIVITY UNDER THE CONDITION OF UNIFORM CONVERGENCE

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Abstract

In this paper, (X, d) be a compact metric space without isolated points and $f_n : X \to X$ be a sequence of continuous functions which converges uniformly to a function *f*. Under the conditions of that, (f_n) is transitive for each integer $n \geq 1$, and

$$\lim_{n \to \infty} d_\infty(f_n^n, f^n) = 0,$$

where $d_\infty(.,.)$ is defined in Section 2, some necessary and sufficient conditions, or sufficient conditions for *f* to be ω-transitive, syndetically transitive, or exact are obtained. These results extend and improve the preservation of some chaotic properties under limiting functions.

1. Introduction

Topologically transitive is an eternal topic in the study of dynamical systems. The concept of transitive could trace back to Birkhoff (see [1, 2]). It is used to describe some chaotic properties of dynamical system together with other dynamic behaviours. A dynamical system may be defined as a deterministic mathematical model for evolving the state of a system forward in time, and which can be represented by a set of functions (rules, or equations) that specify how variables change over time. For instance, in [3], the authors discussed several population growth models (such as Malthus's growth model and Logistic growth model). They showed that the Malthus's model is non-chaotic but not real model of population growth. While, Logistic model, as a more real model of population growth, is often regarded as an intuitive description of how complex, chaotic behaviour can arise from very simple nonlinear dynamical equations.

For a dynamical system (X, f) where the pair (X, f) is given by a metric space X and a continuous function $f : X \to X$, as a motivation for the notion of topologically transitivity, one may think of a real physical system, its state is unknown in most situations, but the empirical observation (or experimental data) shows chaotic symptoms (for example, unpredictability, very small differences in starting values lead to very different behaviours). Consider a point $x \in X$, its trajectory is $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$, where f^n is the n-th iteration of f. The point $f^n(x)$ is the position of x after n units of time. The trajectory of x under f is called the orbit of x, denoted by $O(x, f)$. Instead of points one should study open subsets of the phase space and describe how they move in that space. If for instant the minimality of (X, f) is defined by requiring that every point $x \in X$ visit every open set V in X (i.e., $f^n(x) \in V$ for some $n \in \mathbb{N}$) then, instead, one may wish to study the following case. For every nonempty open subset U, V of X, $f^n(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$ or these n not exist. If the system (X, f) has this property, then it is called topologically transitive. Intuitively, a topologically transitive map f has points which eventually move under iteration from one arbitrarily small neighbourhood to any other. Consequently, the dynamical system can not be broken down or decomposed into two subsystems which do not interact under f, i.e., are invariant by $f(A) \subset X$ is invariant $(f(A) \subset A)$ (see [3]).

In 1989, Devaney [4] introduced the concept of Devaney's chaos, $f : X \to X$ is chaotic in the sense of Devaney if the following three conditions are satisfied:

(D1) f is topologically transitive;

(D2) the periodic points of f are dense in X;

(D3) f has sensitive dependence on initial conditions.

In this definition, sensitive dependence on initial conditions means that, there is a $\delta > 0$ such that, for each $x \in X$ and any open neighbourhood V of x, there is a $y \in V$ and a positive integer n such that

$d(f^n(x), f^n(y)) > \delta$. In 1992, Banks et al. [5] proved that topologically transitive and periodic density imply sensitive dependence on initial conditions. In 1994, Vellekoop and Berglund [6] proved that for continuous functions on intervals in R, transitivity implies chaos. Since the end of the 20th century, transitivity and sensitivity has been hotly discussed, see [7-13], and others. Especially, the convergence of chaotic functions has attracted the attention of many scholars. Some research have contributed with interesting application in physics and engineering. For example, the convergence of chaotic linear semigroups has been used for solving certain parabolic equations arising in problems associated to diffusion phenomena, such as the propagation of gas in the air or the temperature distribution on a surface (see [14, 15]). And the approximation of functions by chaotic polynomials has been used for solving certain stochastic differential equations arising in transport problems as well as in flow-structure interactions (see [16, 17]). Actually, it is interesting to find conditions assuring the preservation of chaotic property under limit operations. In [18], on the one hand, the authors show that for a sequence of continuous functions (f_n) which converges uniformly to a function f, f is not necessarily topologically transitive even if (f_n) is topologically transitive for each $n \geq 1$. On the other hand, the authors give a necessary and sufficient condition for the limit function f to be topologically transitive. In [19], the authors consider a compact metric space X, and a sequence of continuous functions $f_n : X \to X$ which converges uniformly to a function f, and give some necessary and sufficient conditions for topologically transitive (resp., topologically weak mixing, topologically mixing, syndetically transitive).

The current work discuss some other definitions related to transitivity and present some necessary and sufficient conditions or sufficient condition under the conditions of uniform convergence and $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0$.

2. Preliminary

In this paper, let (X, d) be a metric space and $f : X \to X$ be a continuous function, $x \in X$. A point $y \in X$ is called ω-limit point if there is a subsequence $\{f^{n_k}(x)\}_{k=0}^{\infty}$ of $O(x, f) = \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ such that $\{f^{n_k}(x)\}_{k=0}^{\infty}$ converges to y. Let $\omega(x, f)$ denote the set of all the ω-limit points of x. A subset S of nonnegative integers is said to be syndetic if there is an $N \in \mathbb{N}$ such that $[n, n + N] \cap S \neq \emptyset$ for each nonnegative integer n. Let A be a subset of X, a point $x \in A$ is called an isolated point if there is a neighbourhood U of x such that $U \cap A = \{x\}$.

Definition 2.1. Let (X, d) be a metric space and $f : X \to X$ be a continuous function. Then f is said to be

(1) topologically transitive (briefly, transitive) if for any pair of nonempty open subsets U and V of X, there is a $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$;

(2) Z-transitive if for any pair of nonempty open subsets U and V of X, there is an integer n such that $f^n(U) \cap V \neq \emptyset$;

(3) totally transitive if f^n is transitive for each $n \geq 1$;

(4) topologically weak mixing (briefly, wmixing) if $f \times f$ is transitive on $X \times X$;

(5) topologically mixing (briefly, mixing) if for any pair of nonempty open subsets U and V of X, there is an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for every $n \geq N$;

(6) exact if there is an $m \in \mathbb{N}$ such that $f^n(U) = X$ for every nonempty open subset U of X;

(7) ω-transitive if there is an $x \in X$ such that $\omega(x, f) = X$;

(8) syndetically transitive if $\{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$ is syndetic for any pair of nonempty open subsets U and V of X.

It is obvious to observe that topologically transitive implies Z-transitive; Z-transitive implies totally transitive; both ω-transitive and syndetically transitive imply topologically transitive. And in [20], the author shows, on closed interval of $\mathbb{R}$,

$$\text{Exact} \Rightarrow \text{Mixing} \Rightarrow \text{Weak mixing} \Rightarrow \text{Topologically transitive.}$$

3. Main Results

For giving the proofs of main results, the following lemmas are needed.

Lemma 3.1 ([19]). *Let (X, d) be a metric space, $f : X \to X$ be a continuous function and f be a surjection. If the sequence $\{x_n\}_{n=0}^{\infty}$ is dense in X, then so does $\{f(x_n)\}_{n=0}^{\infty}$.*

Remark. As a corollary of this lemma, one can obtain that, if the orbit $\{x_n\}_{n=0}^{\infty}$ is dense in X, then for every integer $k \geq 1$, $\{f^k(x_n)\}_{n=0}^{\infty}$ is also dense in X.

Lemma 3.2 ([18]). *Let (X, d) be a compact metric space without isolated points. If $f : X \to X$ is a continuous function and f is a surjection, then f is transitive if and only if the orbit $O(x, f) = \{f^n(x) : n = 0, 1, 2, \dots\}$ of some $x \in X$ is dense in X.*

Remark. If f is surjection, i.e., $f(X) = X$. For every $x \in X$, there is an x_0 such that $f(x_0) = x$. One has

$$\begin{aligned}\{f^n(x) : n = 0, 1, 2, \dots\} &= \{x, f(x), f^2(x), \dots\} \\ &= \{f(x_0), f^2(x_0), f^3(x_0), \dots\}\end{aligned}$$

rewrite $O(x, f)$ by $\{f(x_n)\}_{n=0}^{\infty}$, where $\{x_n\}_{n=0}^{\infty} = \{f^n(x_0) : n = 0, 1, 2, \dots\}$ is the orbit of x_0. Then we can rewrite Lemma 3.2 to Lemma 3.3.

Lemma 3.3. *Let* (X, d) *be a compact metric space without isolated points. If* $f : X \to X$ *is a continuous function and f is a surjection, then f is transitive if and only if there is an orbit* $\{x_n\}_{n=0}^{\infty}$ *such that* $\{f(x_n)\}_{n=0}^{\infty}$ *is dense in X.*

Combine these lemmas, the following theorem is right.

Theorem 3.1. *Let* (X, d) *be a compact metric space without isolated points. If* $f : X \to X$ *is a continuous function and f is a surjection, then f is transitive if and only if f is totally transitive.*

Proof. It is sufficient to show the necessity. By Lemma 3.3, if f is transitive, then there is an orbit $\{x_n\}_{n=0}^{\infty}$ which is dense in X. By Lemma 3.1, one has that $\{f^k(x_n)\}_{n=0}^{\infty}$ is dense in X for every integer $k \geq 1$. Namely, for every integer $k \geq 1$, there is an orbit $\{x_n\}_{n=0}^{\infty}$ such that $\{f^k(x_n)\}_{n=0}^{\infty}$ is dense in X. This implies f^k is transitive, so f is totally transitive.

This completes the proof.

Theorem 3.2. *For any compact metric space, the following conclusions are hold*:

(a) *each topologically mixing function f is syndetically transitive*;

(b) *each* ω*-transitive function f is syndetically transitive.*

Proof. (a) If f is topologically mixing, i.e., there is an $N_0 \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for any pair of nonempty subsets U and V of X and any $n > N_0$.

For a given $m \in \mathbb{N}$, if $m \geq N_0$, one has $f^m(U) \cap V \neq \emptyset$ for the above open sets U and V. It means that for any $M \in \mathbb{N}$,

$$\{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\} \cap [m, m + M] \neq \emptyset.$$

If $m < N_0$, there is always an $M \in \mathbb{N}$ such that $[m, m + M] \cap [N_0, +\infty] \cap \mathbb{N} \neq \emptyset$, it implies

$$\{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\} \cap [m, m + M] \neq \emptyset.$$

Thus f is syndetically transitive.

(b) If f is ω-transitive, i.e., there is some $x \in X$ such that for any $y \in X$, $y \in \omega(x, f)$. For any nonempty open subsets U of X, there is a y_0 and its neighbourhood $B(y_0, \varepsilon_0)$ such that $y_0 \in B(y_0, \varepsilon_0) \subset U$. For $y_0 \in X = \omega(x, f)$, there is a subsequence $\{f^{n_k}(x)\}_{k=0}^{\infty}$ of $O(x, f)$ which converges to y_0. That is to say, there is an $N \in \mathbb{N}$ such that $f^{n_k}(x) \in B(y_0, \varepsilon)$ for any $\varepsilon > 0$ and any $k \geq N$. For any pair of nonempty open subsets U and V of X, since f is ω-transitive, one has $\sup\{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\} = \infty$. And it follows that for any given $m \in \mathbb{N}$, there is an M such that

$$\{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\} \cap [m, m + M] \neq \emptyset.$$

Thus f is syndetically transitive.

This completes the proof.

Now, let f and g be continuous functions on X, and denote $d_\infty(f, g) = \sup_{x \in X} d(f(x), g(x))$. In ([18], [19]), necessary and sufficient conditions of transitive or topologically weak mixing have been proved by adding the condition (C1) (Propositions 3.1-3.3). Inspired by this, we prove the necessary and sufficient conditions of ω-transitive and syndetically transitive (Theorems 3.3-3.4). Moreover, sufficient conditions of exact is given (Theorem 3.5).

Proposition 3.1 ([18]). *Let* (X, d) *be a compact metric space without isolated points, and* $f_n : X \to X$ *be a sequence of continuous and transitive functions such that* (f_n) *converges uniformly to a function f. Additionally, suppose that*

(C1) $\lim\limits_{n \to \infty} d_\infty(f_n^n, f^n) = 0.$

Then f is transitive if and only if

(C2) $\{f_n^n(x)\}$ *is dense in X for some* $x \in X$.

Remark. According to condition (C1), for arbitrary $\varepsilon > 0$, there is an $N > 0$ such that $d(f_n^n(x), f^n(x)) < \varepsilon$, for any $n > N$ and any $x \in X$.

Proposition 3.2 ([19]). *Let* (X, d) *be a compact metric space without isolated points, and* $f_n : X \to X$ *be a sequence of continuous and transitive functions such that* (f_n) *converges uniformly to a function f. Additionally, suppose that*

(C1) $\lim\limits_{n \to \infty} d_\infty(f_n^n, f^n) = 0.$

Then f is topologically weak mixing if and only if

(C3) $\sup\{n \in \mathbb{N} : f_n^n(U_1) \cap V_1 \neq \emptyset, f_n^n(U_2) \cap V_2 \neq \emptyset\} = \infty$ *for any nonempty open subsets* $U_i, V_i \subset X (i = 1, 2)$.

Proposition 3.3 ([19]). *Let* (X, d) *be a compact metric space without isolated points, and* $f_n : X \to X$ *be a sequence of continuous and transitive functions such that* (f_n) *converges uniformly to a function f. Additionally, suppose that*

(C1) $\lim\limits_{n \to \infty} d_\infty(f_n^n, f^n) = 0.$

Then f is topologically weak mixing if and only if

(C4) $[m, +\infty] \subset \{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\}$ *for some* $m > 0$ *and any pair of nonempty open subsets U and V of X.*

Theorem 3.3. *Let* (X, d) *be a compact metric space without isolated points, and* $f_n : X \to X$ *be a sequence of continuous and transitive functions such that* (f_n) *converges uniformly to a function f. And*

(C1) $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0.$

Then f is ω*-transitive if and only if*

(C5) $\{n \in \mathbb{N} : f_n^n(\{x\}) \cap U \neq \emptyset\}$ *is infinite, for some* $x \in X$ *and every nonempty open subset U of X.*

Proof. If $\{n \in \mathbb{N} : f_n^n(\{x\}) \cap U \neq \emptyset\}$ is infinite for some $x \in X$ and every nonempty open subset U of X. For any $y \in X$, consider the open ε-spherical neighbourhood $B(y, \varepsilon)$ of y. One has that $\{n \in \mathbb{N} : f_n^n(\{x\}) \cap B(y, \varepsilon) \neq \emptyset\}$ is infinite. That is, there are infinitely many integers $k > 0$ and infinitely many $n_k \in \mathbb{N}$ such that $f^{n_k}(x) \in B(y, \varepsilon)$.

Due to the condition (C1), for the above $\varepsilon > 0$, there is an $N > 0$ such that $d(f_n^n(x), f^n(x)) < \varepsilon$ for any $n > N$ and any $x \in X$. So there are infinitely many $k > N$ and infinitely many $n_k > N$ such that $f^{n_k}(x) \in B(y, \varepsilon)$. That is to say, there is a subsequence $\{f^{n_k}(x)\}_{k=0}^{\infty}$ of $\{f^n(x)\}_{n=0}^{\infty}$ converges to y, i.e., $y \in \omega(x, f) = X$. This implies f is ω-transitive.

If f is ω-transitive, i.e., $\omega(x, f) = X$. For any open subset U of X, there is a ε_0-spherical neighbourhood $B(y, \varepsilon_0)$ of y such that $y \in B(y, \varepsilon_0) \subset U$. Since $y \in X = \omega(x, f)$, there is a sequence $\{f^{n_k}(x)\}_{k=0}^{\infty}$ which converges to y. For arbitrary $\varepsilon > 0$, there is an $N_0 > 0$ such that $f^{n_k}(x) \in B(y, \varepsilon)$ for any $k > N_0$.

Due to the condition (C1), for the above $\varepsilon > 0$, there is an $N_1 > 0$ such that $d(f_n^n(x), f^n(x)) < \varepsilon$ for any $n > N_1$ and any $x \in X$. Thus there are infinitely many $n_k > \max\{N_0, N_1\}$ such that $f_{n_k}^{n_k}(x) \in B(y, \varepsilon_0) \subset U$. This implies $\{n \in \mathbb{N} : f_n^n(\{x\}) \cap U \neq \emptyset\}$ is infinite.

This completes the proof.

The following theorem has been proved (see [19] for more details), for the integrity of this paper, we present the theorem and give a new proof here.

Theorem 3.4 ([19])**.** *Let (X, d) be a compact metric space without isolated points, and $f_n : X \to X$ be a sequence of continuous and transitive functions such that (f_n) converges uniformly to a function f. And*

(C1) $\lim_{n\to\infty} d_\infty(f_n^n, f^n) = 0.$

Then f is syndetically transitive if and only if

(C6) $\{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\}$ *is syndetic for any pair of nonempty open subsets U and V of X.*

Proof. If $\{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\}$ is syndetic for some any pair of nonempty open subsets U and V of X. Let $\{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\} = \{n_k \in \mathbb{N} : n_k < n_{k+1}\}$. Take $n_k \in \{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\}$, then there is an $x_{n_k} \in U$ such that $f_{n_k}^{n_k}(x_{n_k}) \in V$. By the condition (C1), for a given $m \in \mathbb{N}$ and for any $\varepsilon > 0$, there is an $N > m$ such that $d(f_n^n(x), f^n(x)) < \varepsilon$, for any $n > N$ and any $x \in X$. For the above $N > 0$, due to the syndeticity of the set $\{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\}$, there is an $N_0 \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\} \cap [N, N + N_0] \neq \emptyset.$$

Thus there is an $n_{k_0} \in \{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\} \cap [N, N + N_0]$ such that $f_{n_{k_0}}^{n_{k_0}}(x_{n_{k_0}}) \in V$ and $x_{n_{k_0}} \in U$. Since $n_{k_0} \geq N$, it follows that $f^{n_{k_0}}(x_{n_{k_0}}) \in V$. For the above $m \in \mathbb{N}$, there is an $M \geq N + N_0 - m$ such that $n_{k_0} \in [N, N + N_0] \subset [m, m + M]$. So

$$n_{k_0} \in \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\} \cap [m, m + M].$$

This implies $\{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$ is syndetic.

Similar to above argument, one can deduce that if f is syndetically transitive, then $\{n \in \mathbb{N} : f_n^n(U) \cap V \neq \emptyset\}$ is syndetic for any pair of nonempty open subsets U and V of X.

This completes the proof.

Theorem 3.5. *Let (X, d) be a compact metric space without isolated points, and $f_n : X \to X$ be a sequence of continuous and transitive functions such that (f_n) converges uniformly to a function f. Additionally, suppose that*

(C1) $\lim_{n \to \infty} d_\infty(f_n^n, f^n) = 0.$

(C7) $\sup\{n \in \mathbb{N} : f_n^n(U) = X\} = \infty$, *for any open subset U of X.*

Then f is exact.

Proof. Since $\sup\{n \in \mathbb{N} : f_n^n(U) = X\} = \infty$, for any given $N \in \mathbb{N}$, there exists an $N_0 > N$ such that $N_0 \in \{n \in \mathbb{N} : f_n^n(U) = X\}$ for any open subset U of X.

Due to the condition (C1), for arbitrary $\varepsilon > 0$, there is an $N_1 > 0$ such that $d(f_n^n(x), f^n(x)) < \varepsilon$, for any $n > N_1$ and any $x \in X$. For the above N_1, there is an N_2 such that $N_2 \in \{n \in \mathbb{N} : f_n^n(U) = X\}$, i.e., $f_{N_2}^{N_2}(U) = X$. Due to the condition (C1), one has $f^{N_2}(U) = X$ for any open subset U of X. This implies f is exact.

This completes the proof.

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