Journal of Mathematical Sciences: Advances and Applications
Volume 60, 2020, Pages 39-50
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100122111

THE MATHEMATICAL STRUCTURE OF $\mathcal{L} = \{P \mid$ PERFECTLY NORMAL IS STRONGER THAN OR EQUAL TO P AND P IS STRONGER THAN OR EQUAL TO $T_0\}$ WITH THE PARTIAL ORDER $\leq$ ON $\mathcal{L}$ DEFINED BY $P \leq Q$ IFF P IS WEAKER THAN OR EQUAL TO Q, AND OTHERS

CHARLES DORSETT

Department of Mathematics
Texas A&M University-Commerce
Texas 75429
USA
e-mail: charles.dorsett@tamuc.edu

Abstract

In classical topology the separation axioms T_0, T_1, T_2, Urysohn, T_3, completely Hausdorff, $T_{3\frac{1}{2}}$, and T_4 were introduced and investigated. Later the perfectly normal and perfectly Hausdorff separation axioms were added. In this paper,

2020 Mathematics Subject Classification: 54D10, 54D15, 54H12.
Keywords and phrases: no between topological properties, lattice structures in topology.
Received December 29, 2019

the mathematical structure of $\mathcal{L} = \{P \mid$ perfectly normal is stronger than or equal to P and P is stronger than or equal to $T_0\}$, with the partial order $\leq$ on $\mathcal{L}$ defined by $P \leq Q$ iff P is weaker than or equal to Q, and others are investigated and shown to be complete lattices.

1. Introduction

In classical topology the separation axioms T_0, T_1, T_2, Urysohn, T_3, completely Hausdorff, $T_{3\frac{1}{2}}$, and T_4 were introduced and investigated. Later the perfectly normal [1] and perfectly Hausdorff [2] separation axioms were added. Definitions of the given separation axioms are available in Willard's book [4] and others, along with the following relationships between the above properties given in decreasing strengths: perfectly normal in one branch implies perfectly Hausdorff, which implies completely Hausdorff, which implies Urysohn; and on another branch from perfectly normal implies T_4, which implies $T_{3\frac{1}{2}}$, which in one branch implies completely Hausdorff, and on another branch implies T_3, which implies Urysohn, which implies T_2, which implies T_1, which implies T_0, with none of the implications reversible. The topological properties T_0 and perfectly normal can be used to define the set $\mathcal{L} = \{P \mid$ perfectly normal is stronger than or equal to P and P is stronger than or equal to $T_0\}$ and on $\mathcal{L}$ the relation $\leq$ defined by $P \leq Q$ iff P is weaker than or equal to Q is a partial ordering, i.e., on $\mathcal{L}$, $\leq$ satisfies the reflexive, antisymmetric, and transitive properties. Also, as given above, $\mathcal{L}$ has smallest element T_0 and biggest element perfectly normal. Since for the subset $\mathcal{B} = \{\text{completely Hausdorff}, T_3\}$ of $\mathcal{L}$, completely Hausdorff $\not\leq T_3$, $T_3 \not\leq$ completely Hausdorff, and completely Hausdorff $\neq T_3$, then $\leq$ is not a linear order on $\mathcal{L}$. Since the subset $\mathcal{B}$ of $\mathcal{L}$ has no smallest element, then $\leq$ is not a well-ordering of $\mathcal{L}$. Below $\mathcal{L}$ with the partial order $\leq$ is further investigated and the results used to give additional, similar results.

2. The Mathematical Structure of $(\mathcal{L}, <)$

With the known relationships of the separation axioms above, a natural question to ask concerning $(\mathcal{L}, \leq)$ is whether it is a lattice and, if so, is it a complete lattice.

Definition 2.1. A partially order set $\mathcal{S}$ is a lattice iff each two element set $\{a, b\}$ in $\mathcal{S}$ has a least upper bound (lub) and a greatest lower bound (glb). If every nonempty subset of $\mathcal{S}$ has a lub and a glb, $\mathcal{S}$ is a complete lattice [4].

Definition 2.2. The lub of a subset $\mathcal{B}$ of a partially ordered set $\mathcal{S}$ is the smallest element of the set $\{a \in \mathcal{S} \mid b \leq a \text{ for each } b \in \mathcal{B}\}$, if it exists, and the glb of a subset $\mathcal{B}$ of a partially ordered set $\mathcal{S}$ is the largest element of the set $\{a \in \mathcal{S} \mid a \leq b \text{ for each } b \in \mathcal{B}\}$, if it exists [4].

To determine if $\mathcal{L}$, with the given partial order, is a lattice, the elements of $\mathcal{L}$ would have to be fully known and each considered to see if the requirements for a lattice are satisfied. Clearly $\mathcal{L}$ has 10 elements, but there could be more depending on whether there are elements between any of the 10 given topological properties. If there are more, they would have to be fully known and addressed to resolve the given question.

Until recently, the question of whether there are topological properties between two consecutive topological properties in the ordered listing of 10 properties above was unaddressed. In the paper [3], it was shown that there are no topological properties between any two consecutive topological properties given in the ordered listing above, giving the resolution of the question above. To arrive at the conclusion there are no topological properties between each two consecutive topological properties P and Q in the ordered listing above, where P implies Q, it was shown that P is a minimal strengthening of Q and thus there are no topological properties between P and Q.

Theorem 2.1. *$\mathcal{L}$ contains only the* 10 *topological properties given in the ordered listing of the* 10 *topological properties above.*

Proof. Since there are no topological properties between any two consecutive topological properties in the ordered listing above, then $\mathcal{L}$ contains only the ten listed topological properties.

Theorem 2.2. $(\mathcal{L}, \leq)$ *is a complete lattice.*

Proof. Let $\mathcal{B}$ be a nonempty subset of $\mathcal{L}$. If $\mathcal{B}$ is a singleton set, then the single element is both the lub and the glb of the set. If $\mathcal{B}$ contains only elements in the set $\{T_0, T_1, T_2, \text{Urysohn}\}$, then $\mathcal{B}$ has a lub and a glb. Consider the cases where $\mathcal{B}$ contains no element in $\{T_0, T_1, T_2, \text{Urysohn}\}$.

$\mathcal{B} = \{$completely Hausdorff, $T_3\}$ has lub $T_{3\frac{1}{2}}$ and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, $T_{3\frac{1}{2}}\}$ has lub $T_{3\frac{1}{2}}$ and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, $T_4\}$ has lub T_4 and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, perfectly Hausdorff$\}$ has lub perfectly Hausdorff and glb completely Hausdorff. $\mathcal{B} = \{T_3, T_{3\frac{1}{2}}\}$ has lub $T_{3\frac{1}{2}}$ and glb T_3. $\mathcal{B} = \{T_3, T_4\}$ has lub T_4 and glb T_3. $\mathcal{B} = \{T_3$, perfectly normal$\}$ has lub perfectly normal and glb T_3. $\mathcal{B} = \{T_3$, perfectly Hausdorff$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{T_{3\frac{1}{2}}, T_4\}$ has lub T_4 and glb $T_{3\frac{1}{2}}$. $\mathcal{B} = \{T_{3\frac{1}{2}}$, perfectly normal$\}$ has lub perfectly normal and glb $T_{3\frac{1}{2}}$. $\mathcal{B} = \{T_{3\frac{1}{2}}$, perfectly Hausdorff$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{T_4$, perfectly Hausdorff$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{T_4$, perfectly normal$\}$ has lub perfectly normal and glb T_4. $\mathcal{B} = \{$perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb perfectly Hausdorff.

$\mathcal{B} = \{$completely Hausdorff, T_3, $T_{3\frac{1}{2}}\}$ has lub $T_{3\frac{1}{2}}$ and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, T_3, $T_4\}$ has lub T_4 and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, T_3, perfectly normal$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, T_3, perfectly Hausdorff$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, $T_{3\frac{1}{2}}$, $T_4\}$ has lub T_4 and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, $T_{3\frac{1}{2}}$, perfectly normal$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, $T_{3\frac{1}{2}}$, perfectly Hausdorff$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, T_4, perfectly normal$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, T_4, perfectly Hausdorff$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{T_3, T_{3\frac{1}{2}}, T_4\}$ has lub T_4 and glb T_3. $\mathcal{B} = \{T_3, T_{3\frac{1}{2}}$, perfectly normal$\}$ has lub perfectly normal and glb T_3. $\mathcal{B} = \{T_3, T_{3\frac{1}{2}}$, perfectly Hausdorff$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{T_3, T_4$, perfectly normal$\}$ has lub perfectly normal and glb T_3. $\mathcal{B} = \{T_3, T_4$, perfectly Hausdorff$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{T_3$, perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and lub Urysohn. $\mathcal{B} = \{T_{3\frac{1}{2}}, T_4$, perfectly normal$\}$ has lub perfectly normal and glb $T_{3\frac{1}{2}}$. $\mathcal{B} = \{T_{3\frac{1}{2}}, T_4,$ perfectly Hausdorff$\}$ has lub perfectly normal and lub completely Hausdorff. $\mathcal{B} = \{T_{3\frac{1}{2}},$ perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and lub completely Hausdorff. $\mathcal{B} = \{T_4,$ perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and lub completely Hausdorff.

$\mathcal{B} = \{$completely Hausdorff, T_3, $T_{3\frac{1}{2}}$, $T_4\}$ has lub T_4 and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, T_3, $T_{3\frac{1}{2}}$, perfectly normal$\}$ has lub $T_{3\frac{1}{2}}$ and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, T_3, T_4, perfectly Hausdorff$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, T_3, T_4, perfectly normal$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, T_3, T_4, perfectly Hausdorff$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, T_3, perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, $T_{3\frac{1}{2}}$, T_4, perfectly normal$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, $T_{3\frac{1}{2}}$, T_4, perfectly Hausdorff$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, $T_{3\frac{1}{2}}$, perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{$completely Hausdorff, T_4, perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{T_3, T_{3\frac{1}{2}}, T_4,$ perfectly normal$\}$ has lub perfectly normal and glb T_3. $\mathcal{B} = \{T_3, T_{3\frac{1}{2}}, T_4,$ perfectly Hausdorff$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{T_3, T_{3\frac{1}{2}},$ perfectly Hausdorff, Perfectly normal$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{T_3, T_4,$ perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{T_{3\frac{1}{2}}, T_4,$ perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb completely Hausdorff.

$\mathcal{B} = \{$completely Hausdorff, $T_3, T_{3\frac{1}{2}}, T_4,$ perfectly normal$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, $T_3, T_{3\frac{1}{2}},$ $T_4,$ perfectly Hausdorff$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, $T_3, T_{3\frac{1}{2}},$ perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, $T_3, T_4,$ perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb Urysohn. $\mathcal{B} = \{$completely Hausdorff, $T_{3\frac{1}{2}}, T_4,$ perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb completely Hausdorff. $\mathcal{B} = \{T_3, T_{3\frac{1}{2}}, T_4,$ perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb Urysohn.

$\mathcal{B} = \{$completely Hausdorff, $T_3, T_{3\frac{1}{2}}, T_4,$ perfectly Hausdorff, perfectly normal$\}$ has lub perfectly normal and glb Urysohn.

If $\mathcal{B}$ is a subset of $\mathcal{L}$ such that both $\mathcal{C} = (\mathcal{L} \cap \{T_0, T_1, T_2, \text{Urysohn}\}) \neq \phi$ and $\mathcal{D} = (\mathcal{L} \cap \{$completely Hausdorff, $T_3, T_{3\frac{1}{2}}, T_4,$ perfectly Hausdorff, perfectly normal$\})$ exist, then the lub of $\mathcal{B}$ is the lub of $\mathcal{D}$ above and the glb of $\mathcal{B}$ is the least element in $\mathcal{C}$.

Therefore $(\mathcal{L}, \leq)$ is a completely lattice.

3. No Between Topological Properties and a Topological Lattice

Theorem 3.1. *Let Q and P be consecutive topological properties in the ordering listing of topological properties from above, with Q weaker than P, and let W be a topological property such that each of $(W$ and $Q)$ and $(W$ and $P)$ exist. Then there are no topological properties between the two topological properties.*

Proof. If $(W$ and $Q)$ and $(W$ and $P)$ are equivalent, then there are no topological properties between $(W$ and $Q)$ and $(W$ and $P)$. Otherwise, applying the minimal strengthening of Q to give P to $(W$ and $Q)$ gives $(W$ and $P)$. Thus there are no topological properties between $(W$ and $Q)$ and $(W$ and $P)$.

Theorem 3.2. *Let Q, M, and P be three consecutive topological properties in the ordered listing of topological properties above, with P implying M, which implies Q, and let W be a topological property such that each of $(W$ and $P)$, $(W$ and $M)$, and $(W$ and $Q)$ exist and are distinct. Then $(W$ and $M)$ is "isolated" in the sense that it is the only topological property between $(W$ and $Q)$ and $(W$ and $P)$.*

Proof. By hypothesis $(W$ and $M)$ is a topological property between $(W$ and $Q)$ and $(W$ and $P)$, and by Theorem 3.1 there are no topological properties between $(W$ and $Q)$ and $(W$ and $M)$ or between $(W$ and $M)$ and $(W$ and $P)$. Thus $(W$ and $M)$ is the only topological property between $(W$ and $Q)$ and $(W$ and $P)$.

Theorem 3.3. *Let Q be a topological property in the ordered listing above weaker than perfectly normal with only one immediate predecessor P, let W be a topological property such that both $(W$ and $Q)$ and $(W$ and $P)$ exist and are distinct, and let $\mathcal{A}_Q = \{Z \mid Z$ is a topological property stronger than $(W$ and $Q)\}$. Then $\mathcal{A}_Q$ has least element $(W$ and $P)$.*

Proof. Since $(W$ and $P)$. is stronger than $(W$ and $Q)$, then $(W$ and $P)$ is in $\mathcal{A}_Q$ and since there are no topological properties between $(W$ and $Q)$ and $(W$ and $P)$, then $(W$ and $P)$ is the least element of $\mathcal{A}_Q$.

Let W be a topological property such that for distinct separation axioms P and Q in the ordered listing of separation axiom above, both $(W$ and $P)$ and $(W$ and $Q)$ exist and are distinct, let $\mathcal{LW} = \{Z \mid Z$ is a topological property weaker than or equal to $(W$ and perfectly normal$)$ and stronger than or equal to $(W$ and $T_0\}$, and let $\leq^*$ be the partial order on $\mathcal{LW}$ defined by $(W$ and $Q) \leq^* (W$ and $P)$ iff $(W$ and $Q)$ is weaker than or equal to $(W$ and $P)$. Is $(\mathcal{LW}, \leq^*)$ a complete lattice?

Definition 3.1. Let $(\mathcal{M}, \leq)$ and $(\mathcal{O}, \leq^*)$ be partially ordered sets. Then $(\mathcal{M}, \leq)$ and $(\mathcal{O}, \leq^*)$ are exact order related iff there exists a one-to-one function f from $(\mathcal{M}, \leq)$ onto $(\mathcal{O}, \leq^*)$ that preserves order, i.e., if $a \leq b$ in $\mathcal{M}$, then $f(a) \leq^* f(b)$ in $\mathcal{O}$.

Theorem 3.4. *Let $(\mathcal{M}, \leq)$ and $(\mathcal{O}, \leq^*)$ be partially ordered sets that are exact order related, where $(\mathcal{M}, \leq)$ is a complete lattice. Then $(\mathcal{O}, \leq^*)$ is a complete lattice.*

Proof. Let $f : (\mathcal{M}, \leq\ ) \to (\mathcal{O}, \leq^*)$ preserve order between the partially ordered sets. Let $\mathcal{B}$ be a nonempty subset of $\mathcal{O}$. Then $f^{-1}(\mathcal{B})$ is a nonempty subset of $\mathcal{M}$ and has lub u and glb v. Thus $\{m \mid m \in \mathcal{M}$ and $c \leq m$ for each $c \in f^{-1}(\mathcal{B})\}$ exists and $\{o \mid o \in \mathcal{O}$ and $b \leq^* o$ for each $b \in \mathcal{B}\}$ exists. Since u is the smallest element in $\{m \mid m \in \mathcal{M}$ and $c \leq m$ for each $c \in f^{-1}(\mathcal{B})\}$, then $f(u) \in \{o \mid o \in \mathcal{O}$ and $b \leq^* o$ for each $b \in \mathcal{B}\}$. Let $z \in \{o \mid o \in \mathcal{O}$ and $b \leq^* o$ for each $b \in \mathcal{B}\}$. Then $f^{-1}(z) \in \{m \mid m \in \mathcal{M}$ and $c \leq m$ for each $c \in f^{-1}(\mathcal{B})\}$ and $u \leq f^{-1}(z)$. Hence $f(u) \leq^* z$ in $\mathcal{O}$ and $f(u)$ is the lub of $\mathcal{B}$. Similarly, $f(v)$ is the glb of $\mathcal{B}$. Hence $(\mathcal{O}, \leq^*)$ is a complete lattice.

Theorem 3.5. *Let* $\mathcal{LW} = \{Z \mid Z$ *is a topological property weaker than or equal to* $(W$ *and perfectly normal*$)$ *and stronger that or equal to* $(W$ *and* $T_0\}$, *and let* $\leq^*$ *be the partial order on* $\mathcal{LW}$ *defined by* $(W$ *and* $Q) \leq^* (W$ *and* $P)$ *iff* $(W$ *and* $Q)$ *is weaker than or equal to* $(W$ *and* $P)$, *as above. Then* $\mathcal{LW}$ *contains exactly* 10 *elements and* $(\mathcal{LW}, \leq^*)$ *is a complete lattice.*

Proof. Since perfectly normal in one branch implies perfectly Hausdorff, which implies completely Hausdorff, which implies Urysohn; and on another branch from perfectly normal implies T_4, which implies $T_{3\frac{1}{2}}$, which in one branch implies completely Hausdorff, and on another branch implies T_3, which implies Urysohn, which implies T_2, which implies T_1, which implies T_0, with none of the implications reversible, and W is a topological property such that for distinct elements P and Q in the ordered listing of separation axioms above, both $(W$ and $P)$ and $(W$ and $Q)$ exist and are distinct, then $(W$ and perfectly normal) in one branch implies (W and perfectly Hausdorff), which implies (W and completely Hausdorff), which implies (W and Urysohn); and on another branch from (W and perfectly normal) implies (W and T_4), which implies (W and $T_{3\frac{1}{2}}$), which in one branch implies (W and completely Hausdorff), and on another branch implies (W and T_3), which implies (W and Urysohn), which implies (W and T_2), which implies (W and T_1), which implies (W and T_0), with none of the implications reversible. Thus $\mathcal{LW}$ contains at least the 10 topological properties listed in the ordered listing of topological properties given immediately above. Since, by Theorem 3.1 above, for consecutive topological properties (W and P) and (W and Q) in the ordered listing above there are no topological properties between the two topological properties, then $\mathcal{LW}$ contains only the 10 elements.

Let $f : (\mathcal{L}, \leq) \to (\mathcal{LW}, \leq^*)$ defined by $f(P) = (W \text{ and } P)$. Since f is a one-to-one function from $(\mathcal{L}, \leq)$ onto $(\mathcal{LW}, \leq^*)$ that preserves order and $(\mathcal{L}, \leq)$ is a complete lattice, then, by Theorem 3.5 $(\mathcal{LW}, \leq^*)$ is a complete lattice.

Theorem 3.6. *Let Q be a separation axiom in the ordered listing of separation axioms above weaker than perfectly normal, let W be a topological property such that for distinct separation axioms P and Y in the ordered listing above (W and P) and (W and P) and (W and Y) are distinct, and let $\mathcal{A}_Q = \{Z \mid Z$ is a topological property stronger than (W and Q). Then $\mathcal{A}_Q$ has a glb.*

Proof. Let $\mathcal{B} = (\mathcal{A}_Q \cap \mathcal{L})$. Then $\mathcal{B} \neq \phi$ and has a glb in $\mathcal{L}$, which is the glb of $\mathcal{A}_Q$.

Letting $W = L$ gives the last result in the paper.

Corollary 3.1. *Let Q be a separation axiom in the ordered listing of separation axioms above weaker than perfectly normal and let $\mathcal{A}_Q = \{Z \mid Z$ is a topological property stronger than $Q\}$. Then $\mathcal{A}_Q$ has a glb.*

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