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THE BLOWING-UP PHENOMENON FOR A REACTION DIFFUSION EQUATION WITH A LOCALIZED NON LINEAR SOURCE TERM AND DIRICHLET-NEUMANN BOUNDARY CONDITIONS

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Abstract

In this paper, we study a localized nonlinear reaction diffusion equation. We investigate interactions among the localized and local sources, nonlinear diffusion with the zero boundary value condition to establish the blow-up solution and estimate the numerical approximation for the following initial-boundary value problem:

$$\begin{cases} u_t(x, t) = u_{xx}(x, t) + f(u(x_0, t)), & (x, t) \in (-1, 1) \times (0, T), \\ u(-1, t) = 0, \quad u_x(1, t) = 0, & t \in (0, T), \\ u(x, 0) = u_0(x) \geq 0, & x \in [-1, 1], \end{cases}$$

where $f(s)$ is a positive, increasing, convex function for the nonnegative values of s, $\int_0^{+\infty} \frac{ds}{f(s)} < +\infty$, $x_0 = 1$, $u_0 \in C^1([-1, 1])$, $u_0(-1) = 0$, $u_0'(1) = 0$.

We find some conditions under which the solution of a discrete form of the above problem blows up in a finite time and a numerical method is proposed for estimating its numerical blow-up time. We also prove the convergence of the numerical blow-up time to the theoretical one. Finally, we give some numerical results to illustrate our analysis.

1. Introduction

Consider the following initial-boundary value problem:

$$u_t(x, t) = u_{xx}(x, t) + f(u(x_0, t)), \quad (x, t) \in (-1, 1) \times (0, T), \tag{1.1}$$

$$u(-1, t) = 0, \quad u_x(1, t) = 0, \quad t \in (0, T), \tag{1.2}$$

$$u(x, 0) = u_0(x) \geq 0, \quad x \in [-1, 1], \tag{1.3}$$

where $f(s)$ is a positive, increasing, convex function for the nonnegative values of s, $\int_0^{+\infty} \frac{ds}{f(s)} < +\infty$, $x_0 = 1$, $u_0 \in C^1([-1, 1])$, $u_0(-1) = 0$, $u_0'(1) = 0$, which models the temperature distribution of a large number of physical phenomena from physics, chemistry and biology. The particularity of the problem described in (1.1)-(1.3) is that it represents a model in physical phenomena where the reaction is driven by the temperature at a single site. This kind of phenomena is observed in biological and chemical diffusion processes in which the reaction takes place only at some local sites. This model is appropriate to describe:

(i) The influence of defect structures on a catalytic surface.

(ii) The temperature in a solid-fuel combustion scenario where the heat that is input into the system is localized, say as in a laser focused on one spot in the domain.

(iii) Chemical reaction-diffusion processes in which, due to effect of catalyst, the reaction takes place only at a single site.

(iv) A heat stationary source which can support an explosive reaction. A stationary source provides a continuous supply of heat to the same environment.

(v) The ignition of a combustible medium with damping, where either a heated wire or a pair of small electrodes supplies a large amount of energy to every confined area.

For more physical motivation, see [8].

(A_0) $u_0 : [-1, 1] \to [0, \infty)$ is a positive, nondecreasing c^1 function. $u_0(-1) = 0$, $u_0'(1) = 0$,

(A_f) $f : [0, \infty) \to [0, \infty)$ is a positive, increasing, convex function for the nonnegative values of s, c^1 function, $\int_0^{+\infty} \frac{ds}{f(s)} < +\infty$. Here $(0, T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in a finite time, namely,

$$\lim_{t\to T} \|u(., t)\|_\infty = +\infty,$$

where $\|u(., t)\|_\infty = \max_{-1\le x\le 1} |u(x, t)|$.

In this case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u.

This kind of the phenomena where the solutions of localized nonlinear heat equations which blow up in a finite time have been the subject of investigation of many authors (see [17], [44]-[46] and the references cited therein). In particular, the above problem has been studied and existence and uniqueness of a classical solution has been proved. Under some assumptions, it is also shown that the classical solution blows up in a finite time (see [17], [44], [45]).

In this paper, we are interesting in the numerical study of the above problem. Firstly, we show that under some assumptions, the solution of a discrete form of (1.1)-(1.3) blows up in a finite time and estimate its numerical blow-up time. We also show that the numerical blow-up time converges to the real one when the mesh size goes to zero. At the end of the paper, we have shown how one may treat the case of Dirichlet boundary conditions. One may find in [1]-[7], [12], [13], [16], [18]-[28], [31]-[33], [41] similar studies concerning other parabolic problems. Let us notice that many authors have used numerical methods to study the phenomenon of blow-up but there are only a few studies on the convergence of the numerical blow-up time for solutions which blow up in L^{∞} norm. For instance in [2], the authors have proved the convergence of numerical blow-up time for solutions which blow up in L^{p} norm with $1 < p < \infty$.

The rest of the paper is organized as follows. In the next section, we give some results which will be used later. In the third section, under some assumptions, we show that the solution of a discrete form of (1.1)-(1.3) blows up in a finite time and estimate its numerical blow-up time. In the fourth section, we show that, under some additional hypotheses, the numerical blow-up time goes to the real one when the mesh size goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. The Discrete Blow-up Solution

In this section, we study the phenomenon of blow-up using a full discrete explicit scheme of (1)-(3). We start by the construction of a scheme as follows. Let I be a positive integer, where $h = \frac{1}{I}$ is the mesh parameter and define the grid $x_i = ih$ or $x_{i+1} = x_i + h$ and $\Delta t_n = x_{i+1} - x_i$ the step size, φ_i is the discrete approximation of the initial data.

(A_{φ_i}) $\varphi_i : [0, I] \to [0, \infty)$ is a positive function. Approximate the problem (1.1)-(1.3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \dots, U_I^{(n)})^T$, and approximate the initial condition $u_0(x)$ by $U_i^{(0)} = \varphi_i$ of the following discrete equations:

$$\delta_t U_i^{(n)} = \delta^2 U_i^{(n)} + f(U_I^{(n)}), \quad 1 \le i \le I, \tag{2.1}$$

$$U_0^{(n)} = 0, \tag{2.2}$$

$$U_i^{(0)} = \varphi_i \ge 0, \quad 1 \le i \le I, \tag{2.3}$$

where $n \ge 0$, $\varphi_{i+1} \ge \varphi_i$, $0 \le i \le I - 1$,

$$\delta^2 U_i^{(n)} = \begin{cases} \dfrac{U_{i+1}^{(n)} - 2U_i^n + U_{i-1}^{(n)}}{h^2}, & 1 \le i \le I - 1, \\ \dfrac{2}{h^2}(U_{I-1}^{(n)} - U_I^{(n)}), & i = I, \end{cases}$$

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n},$$

then

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} - \frac{U_{i+1}^{(n)} - 2U_i^n + U_{i-1}^{(n)}}{h^2} = f(U_i^{(n)}), \quad 1 \leq i \leq I - 1,$$

$$U_0^{(n)} = 0,$$

$$U_i^{(0)} = \varphi_i \geq 0, \quad 1 \leq i \leq I,$$

$$\frac{2}{h^2}(U_{I-1}^{(n)} - U_I^{(n)}) + \frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = f\left(U_I^{(n)}\right), \quad i = I,$$

where

$$\Delta t_n = \min\left\{\frac{h^2}{3}, \frac{\tau}{f(\|U_h^{(n)}\|_\infty)}\right\}, \quad 0 < \tau < 1.$$

Let us notice that the restriction on the time step ensure the positivity of the discrete solution.

Definition 2.1. We say that the solution u of (1.1)-(1.3) blows up in a finite time, if there exist a finite time T such that $\|u(\cdot, t)\|_\infty < \infty$ for $t \in [0, T)$ but

$$\lim_{t \to T} \|u(\cdot, t)\|_\infty = \infty,$$

where $\|u(\cdot, t)\|_\infty = \sup_{x \in \Omega}|u(x, t)|$ and the time T is called the blow-up time of the solution u, when T is infinite, we say that the solution u exists globally.

3. Properties of the Discrete Scheme

In this section, we give some important results which will be used later. The following Lemmas 3.1 to 3.4 are a form of the maximum principal for discrete equations.

Lemma 3.1. *Let* $U_h^{(n)}$ *be the solution of* (2.1)-(2.3). *Then we have* $U_{i+1}^{(n)} \geq U_i^{(n)}$, $0 \leq i \leq I-1$.

Proof. Let $Z_i^{(n)} = U_{i+1}^{(n)} - U_i^{(n)}$, $0 \leq i \leq I-1$. Obviously $Z_0^{(n)} \geq 0$ because $\varphi_{i+1} \geq \varphi_i$.

A routine computation reveals that $Z_i^{(n+1)} = U_{i+1}^{(n+1)} - U_i^{(n+1)}$ then

$$\frac{Z_i^{(n+1)} - Z_i^{(n)}}{\Delta t_n} = \frac{Z_{i+1}^{(n)} - 2Z_i^{(n)} + Z_{i-1}^{(n)}}{h^2},\ 1 \leq i \leq I-2,$$

$$\frac{Z_{I-1}^{(n+1)} - Z_{I-1}^{(n)}}{\Delta t_n} = \frac{-3Z_{I-1}^{(n)} + Z_{I-2}^{(n)}}{h^2},\ i = I-1,$$

which implies that

$$Z_i^{(n+1)} = \frac{\Delta t_n}{h^2} Z_{i+1}^{(n)} + (1 - \frac{2\Delta t_n}{h^2})Z_i^{(n)} + \frac{\Delta t_n}{h^2} Z_{i-1}^{(n)}, \quad 1 \leq i \leq I-2,$$

$$Z_{I-1}^{(n+1)} = \frac{\Delta t_n}{h^2} Z_{I-2}^{(n)} + (1 - \frac{3\Delta t_n}{h^2})Z_{I-1}^{(n)}.$$

Since $Z_i^{(0)} \geq 0, 1 \leq i \leq I-1$, we deduce by induction that $Z_i^{(n)} \geq 0$, $0 \leq i \leq I-1$ and the proof is complete. □

Lemma 3.2. *Let* $a^{(n)}$ *be a nonnegative sequence and let* $V_h^{(n)}$ *be a sequence such that*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} - a^{(n)} V_i^{(n)} \geq 0, \quad 1 \leq i \leq I, \quad n \geq 0,$$

$$V_0^{(n)} \geq 0, \quad n \geq 0,$$

$$V_i^{(0)} \geq 0, \quad 0 \leq i \leq I.$$

Then $V_i^{(n)} \geq 0$ *for* $n \geq 0, 0 \leq i \leq I$, *if* $\Delta t_n \leq \frac{h^2}{3}$.

Proof. A routine calculation gives

$$\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} - \frac{V_{i+1}^{(n)} - 2V_i^{(n)} + V_{i-1}^{(n)}}{h^2} - a^{(n)} V_i^{(n)} \geq 0, \quad 1 \leq i \leq I - 1,$$

which implies that

$$V_i^{(n+1)} \geq \frac{\Delta t_n}{h^2} V_{i+1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2})V_i^{(n)} + \frac{\Delta t_n}{h^2} V_{i-1}^{(n)} + \Delta t_n a^{(n)} V_I^{(n)}, 1 \leq i \leq I - 1,$$

$$V_I^{(n+1)} \geq \frac{2\Delta t_n}{h^2} V_{I-1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2})V_I^{(n)} + \Delta t_n a^{(n)} V_I^{(n)}.$$

Since $\Delta t_n \leq \frac{h^2}{3}$, we see that $1 - 2\frac{\Delta t_n}{h^2}$ is nonnegative. Due to the fact that $V_h^{(0)} \geq 0$, we deduce by induction that $V_h^{(n)} \geq 0$ for $n \geq 0$, which ends the proof. □

A direct consequence of the above result is the following comparison lemma. Its proof is straightforward.

Lemma 3.3. *Suppose that* $a^{(n)}$ *and* $b^{(n)}$ *are two sequences such that* $a^{(n)}$ *is nonnegative. Let* $V_h^{(n)}$ *and* $W_h^{(n)}$ *two sequences such that*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} - a^{(n)} V_i^{(n)} + b^{(n)} \leq \delta_t W_i^{(n)} - \delta^2 W_i^{(n)} - a^{(n)} W_i^{(n)} + b^{(n)},$$

$$1 \leq i \leq I,\ n \geq 0,$$

$$V_0^{(n)} \leq W_0^{(n)}, \quad n \geq 0,$$

$$V_i^{(0)} \leq W_i^{(0)}, \quad 0 \leq i \leq I.$$

Then $V_i^{(n)} \leq W_i^{(n)}$ *for* $n \geq 0, 0 \leq i \leq I,$ *if* $\Delta t_n \leq \frac{h^2}{3}$.

Now, let us give the property of operator δ_t.

Lemma 3.4. *Let* $f : [0, \infty) \to [0, \infty)$ *be a* C^1 *convex function and* $U_h^{(n)} \in \mathbb{R}$ *be a sequence such that* $U_h^{(n)} \geq 0$. *Then we have*

$$\delta_t f(U_h^{(n)}) \geq f'(U_h^{(n)})\delta_t U_h^{(n)}, \quad n \geq 0,$$

$$\delta^2 f(U_h^{(n)}) \geq f'(U_h^{(n)})\delta^2 U_h^{(n)}.$$

Proof. We apply Taylor's expansion to obtain

$$f(U_h^{(n+1)}) = f(U_h^{(n)}) + (U_h^{(n+1)} - (U_h^{(n)}))f'(U_h^{(n)}) + \frac{(U_h^{(n+1)} - (U_h^{(n)}))^2}{2} f''(\theta),$$

where θ is an intermediate value between $U_h^{(n+1)}$ and $U_h^{(n)}$;

$$f(U_h^{(n-1)}) = f(U_h^{(n)}) + (U_h^{(n-1)} - (U_h^{(n)}))f'(U_h^{(n)}) + \frac{(U_h^{(n-1)} - (U_h^{(n)}))^2}{2} f''(\tau),$$

where τ is an intermediate value between $U_h^{(n-1)}$ and $U_h^{(n)}$.

These two equations above imply that

$$f(U_h^{(n+1)}) - 2f(U_h^{(n)}) + f(U_h^{(n-1)}) = (U_h^{(n+1)} - 2U_h^{(n)} + U_h^{(n-1)})f'(U_h^{(n)})$$

$$+ \frac{(U_h^{(n+1)} - U_h^{(n)})^2}{2} f''(\theta) + \frac{(U_h^{(n-1)} - U_h^{(n)})^2}{2} f''(\tau)$$

$$\frac{f(U_h^{(n+1)}) - 2f(U_h^{(n)}) + f(U_h^{(n-1)})}{h^2} = \frac{(U_h^{(n+1)} - 2U_h^{(n)} + U_h^{(n-1)})}{h^2} f'(U_h^{(n)})$$

$$+ \frac{(U_h^{(n+1)} - U_h^{(n)})^2}{2h^2} f''(\theta) + \frac{(U_h^{(n-1)} - U_h^{(n)})^2}{2h^2} f''(\tau).$$

The first equation and the last one imply that

$$\delta^2 f(U_h^{(n)}) = f'(U_h^{(n)})\delta^2 U_h^{(n)} + \frac{(U_h^{(n+1)} - U_h^{(n)})^2}{2h^2} f''(\theta) + \frac{(U_h^{(n-1)} - U_h^{(n)})^2}{2h^2} f''(\tau),$$

and

$$\delta_t f(U_h^{(n)}) = f'(U_h^{(n)})\delta_t U_h^{(n)} + \frac{1}{2}\Delta t_n \delta_t (U_h^{(n)})^2 f''(\theta).$$

Use the fact that $U_h^{(n)} \geq 0$ for $n \geq 0$ and using the convexcity of f we obtain the desire result.

4. Blow-up Time in the Discrete Solution

In this section, under some assumptions, we show that the discrete solution blows up in a finite time and its numerical blow-up time converges to the real one when the mesh size tends to zero.

Definition 4.1. We say that the solution $U_h^{(n)}$ of (2.1)-(2.3) blows up in a finite time, if $\|U_h^{(n)}\|_\infty = +\infty$, and $T_h^{\Delta t} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta t_i < +\infty$, the number $T_h^{\Delta t}$ is called the numerical blow-up time of the solution $U_h^{(n)}$.

The following theorem shows that the discrete solution blows up under some conditions.

Theorem 4.1. *Suppose that there exists a positive constant* $A \in (0, 1]$ *such that the initial data at* (2.3) *satisfies*:

$$\delta^2\varphi_i + f(\varphi_i) \geq 0,\ 0 \leq i \leq I-1,\ \delta^2\varphi_I + f(\varphi_I) \geq Af(\varphi_I), \tag{4.1}$$

$$\delta^2\varphi_I + f(\varphi_I) \geq Af(\varphi_I), \tag{4.2}$$

with φ_i *is an approximation of the initial data. Then the solution* $U_h^{(n)}$ *of* (2.1)-(2.3) *blows up in a finite time and its numerical blow-up time* $T_h^{\Delta t}$ *is estimated as follows*:

$$T_h^{\Delta t} \leq \frac{\tau}{f(\|\varphi_h\|_\infty)} + \frac{\tau}{\tau'}\int_{\|\varphi_h\|_\infty}^{+\infty} \frac{ds}{f(s)},$$

where $\tau' = \min\{\frac{h^2}{3} f(\|\varphi_h\|_\infty),\ \tau\}$.

Proof. Introduce the vector $J_h^{(n)}$ such that

$$J_i^{(n)} = \delta_t U_i^{(n)},\ 0 \le i \le I-1,\ J_I^{(n)} = \delta_t U_I^{(n)} - Af(U_I^{(n)}).$$

A straightforward computation yields

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t(\delta_t U_i^{(n)} - \delta^2 U_i^{(n)}),\ 0 \le i \le I-1,\ n \ge 0,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} = \delta_t(\delta_t U_I^{(n)} - \delta^2 U_I^{(n)}) - A\delta_t f(U_I^{(n)}) + A\delta^2 f(U_I^{(n)}).$$

Using (2.4), we arrive at

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t f(U_i^{(n)}), \quad 1 \le i \le I-1,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} = (1-A)\delta_t f(U_I^{(n)}) + A\delta^2 f(U_I^{(n)}).$$

It follows from Lemma 3.4 that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \ge f'(U_i^{(n)})\delta_t U_i^{(n)}, \quad 1 \le i \le I-1,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} \ge (1-A)f'(U_I^{(n)})\delta_t U_I^{(n)} + Af'(U_I^{(n)})\delta^2 U_I^{(n)}.$$

Taking into account (2.4), we deduce that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \ge f'(U_i^{(n)})\delta_t U_i^{(n)} - Af'(U_i^{(n)})f(U_i^{(n)}), \tag{4.3}$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} \ge f'(U_I^{(n)})\delta_t U_I^{(n)} - Af'(U_I^{(n)})f(U_I^{(n)}), \tag{4.4}$$

which implies that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \ge f'(U_i^{(n)})J_i^{(n)}, \quad 1 \le i \le I. \tag{4.5}$$

Obviously, we have $J_0^{(n)} = 0$. From (4.1), we obtain $J_i^{(0)} \ge 0$. It follows from Lemma 3.2 that $J_i^{(n)} \ge 0,\ 0 \le i \le I$. Hence, we have

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} \ge Af(U_I^{(n)}). \tag{4.6}$$

Consequently, we get

$$U_I^{(n+1)} \geq U_I^{(n)} + A\Delta t_n f(U_I^{(n)}).$$

Since $U_I^{(n)} = \|U_h^{(n)}\|_\infty$, we arrive at

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty + A\Delta t_n f(\|U_h^{(n)}\|_\infty). \tag{4.7}$$

It not difficult to see that

$$\Delta t_n f(\|U_h^{(n)}\|_\infty) = \min\{\frac{h^2}{3} f(\|U_h^{(n)}\|_\infty), \tau\}.$$

From (4.7), we get $\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty$ and by induction we obtain $\|U_h^{(n)}\|_\infty \geq \|U_h^{(0)}\|_\infty = \|\varphi_h\|_\infty$. It follows that

$$\Delta t_n f(\|U_h^{(n)}\|_\infty) \geq \min\{\frac{h^2}{3} f(\|\varphi_h\|_\infty), \tau\} = \tau'.$$

Consequently, we have

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty + \tau'. \tag{4.8}$$

Using a recursion argument, we discover that

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(0)}\|_\infty + n\tau' = \|\varphi_h\|_\infty + \tau'. \tag{4.9}$$

Hence, we see that $\|U_h^{(n)}\|_\infty$ goes to infinity as n approaches infinity. Now let us estimate the numerical blow-up time. From the restriction on the time step, we get

$$\sum_{n=0}^{\infty} \Delta t_n \leq \sum_{n=0}^{\infty} \frac{\tau}{f(\|U_h^{(n)}\|_\infty)}.$$

Due to (4.8), we arrive at

$$\sum_{n=0}^{\infty} \Delta t_n \leq \sum_{n=0}^{\infty} \frac{\tau}{f(\|\varphi_h\|_\infty + n\tau')}.$$

We observe that

$$\int_0^{+\infty} \frac{ds}{f(\|\varphi_h\|_\infty + s\tau')} = \sum_{n=0}^{\infty} \int_n^{n+1} \frac{ds}{f(\|\varphi_h\|_\infty + s\tau')}$$

$$\geq \sum_{n=0}^{\infty} \frac{1}{f(\|\varphi_h\|_\infty + (n+1)\tau')}.$$

Since

$$\int_0^{+\infty} \frac{ds}{f(\|\varphi_h\|_\infty + s\tau')} = \frac{1}{\tau'} \int_{\|\varphi_h\|_\infty}^{+\infty} \frac{ds}{f(s)},$$

we deduce that

$$\sum_{n=0}^{\infty} \Delta t_n \leq \frac{\tau}{f(\|\varphi_h\|_\infty)} + \frac{\tau}{\tau'} \int_{\|\varphi_h\|_\infty}^{+\infty} \frac{ds}{f(s)}.$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. $\square$

Remark 4.1. From (4.8),

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty + \tau',$$

we get by induction that

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(q)}\|_\infty + \tau'(n-q).$$

Hence

$$T_h^{\Delta t} - T_q = \sum_{n=q}^{\infty} \Delta t_n \leq \sum_{n=q}^{\infty} \frac{\tau}{f(\|U_h^{(n)}\|_\infty)} \leq \sum_{n=q}^{\infty} \frac{\tau}{f(\|U_h^{(q)}\|_\infty + (n-q)\tau')},$$

where $t_q = \sum_{n=q}^{\infty} \Delta t_n$. We observe that

$$\int_0^{+\infty} \frac{ds}{f(\|U_h^{(q)}\|_\infty + s\tau')} = \sum_{n=0}^{\infty} \int_n^{n+1} \frac{ds}{f(\|U_h^{(q)}\|_\infty + s\tau')}$$

$$\geq \sum_{n=0}^{\infty} \frac{1}{f(\|U_h^{(q)}\|_\infty + (n+1)\tau')}.$$

Since

$$\int_0^{+\infty} \frac{ds}{f(\|U_h^{(q)}\|_\infty + s\tau')} = \frac{1}{\tau'} \int_{\|U_h^{(q)}\|_\infty}^{+\infty} \frac{ds}{f(s)},$$

we get

$$T_h^{\Delta t} - t_q \leq \frac{\tau}{f(\|U_h^{(q)}\|_\infty)} + \frac{\tau}{\tau'} \int_{\|U_h^{(q)}\|_\infty}^{+\infty} \frac{ds}{f(s)}.$$

Since $\tau' = \min\{\frac{h^2}{3} f(\|\varphi_h\|_\infty), \tau\}$, if we take $\tau = h^2$, we get

$$\frac{\tau}{\tau'} = \min\{\frac{1}{3} f(\|\varphi_h\|_\infty), 1\},$$

which implies that there exists a positive constant B such that $\frac{\tau}{\tau'} \leq B$.

5. Convergence of the Blow-up Time

In this section, under some conditions, we show that the discrete solution blows up in a finite time and that its numerical blow-up time goes to the analytic one when the mesh size goes to zero.

In order to prove the convergence of the discrete blow-up time, we need to show that the discrete scheme converges for each fixed time interval $[0, T]$. We denote by $u_h(t_n) = (u(x_0, t_n), \ldots, u(x_I, t_n))^T$ and state the result on the convergence of our scheme by the following.

Theorem 5.1. *Suppose that the problem* (1.1)-(1.3) *has a solution* $u \in C^{4,2}([-1, 1] \times [0, T])$, *and that* $U_h^{(n)}$ *approximates the solution* u *of* (1.1)-(1.3) *with* $U_h^{(0)} = \varphi_h$. *Assume that the initial data at* (2.3) *verifies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0. \tag{5.1}$$

Then the problem (2.1)-(2.3) *has a solution* $U_h^{(n)}$ *for h sufficiently small*, $0 \leq n \leq J$ *and we have the following estimate*:

$$\max_{0 \leq n \leq J} \|U_h^{(n)} - u_h(t_n)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2 + \Delta t_n) \quad \text{as} \quad h \to 0,$$

where J is such that $\sum_{n=0}^{J-1} \Delta t_n \leq T$ *and* $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

Proof. For each h, the problem (2.1)-(2.3) has a solution $U_h^{(n)}$, we want to proof that $U_h^{(n)}$ approaches to u_h as $h \to 0$.

Let $N \leq J$ be the greatest value of n such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < 1 \quad \text{for} \quad n < N. \tag{5.2}$$

We know that $N \geq 1$ because of (5.1). Due to the fact that $u \in C^{4,2}([-1, 1] \times [0, T])$, there exists a positive constant K such that $\|u\| \leq K$.

Applying the triangle inequality, we obtain

$$\|U_h^{(n)}\|_\infty \leq \|u_h(t_n)\|_\infty + \|U_h^{(n)} - u_h(t_n)\|_\infty \leq 1 + K. \tag{5.3}$$

Since $u \in C^{4,2}([-1, 1] \times [0, T])$, taking the derivative in x on both sides of (1.1) and due to the fact that u_x, u_{xt} vanish at $x = 1$. We observe that u_{xxx} also vanishes at $x = 1$. Using Taylor's expansion, we find that

$$\delta_t u(x_i, t_n) - \delta^2 u(x_i, t_n) - f(u(x_I, t_n)) = -\frac{h^2}{12} u_{xxxx}(\tilde{x}_i, t_n) + \frac{\Delta t_n}{2} u_{tt}(x_i, \tilde{t}_n),$$

$$\text{for } 1 \leq i \leq I.$$

To establish the above equality for $i = I$, we have used the fact that u_{xxx} vanishes at $x = 1$. Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization. From the mean value theorem, we get

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} = f'(\varsigma_i^{(n)})e_i^{(n)} + \frac{h^2}{12}u_{xxxx}(\widetilde{x}_i, t_n) - \frac{\Delta t_n}{2}u_{tt}(x_i, \widetilde{t}_n), 1 \le i \le I,$$

where $\varsigma_i^{(n)}$ is an intermediate value between $u(x_i, t_n)$ and $U_i^{(n)}$. Since $u_{xxxx}(x, t)$, $u_{tt}(x, t)$ are bounded, there exists a positive constant M such that

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} \le f'(\varsigma_i^{(n)})e_i^{(n)} + M\Delta t_n + Mh^2, \quad 0 \le i \le I. \tag{5.4}$$

Let $P = 1 + K$ and introduce the vector $V_h^{(n)}$ defined as follows:

$$V_i^{(n)} = e^{(P+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Mh^2 + M\Delta t_n), \quad 0 \le i \le I.$$

A straightforward computation gives

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} \ge f'(\varsigma_i^{(n)})V_i^{(n)} + M\Delta t_n + Mh^2, \quad 1 \le i \le I,$$

$$V_0^{(n)} \ge e_0^{(n)},$$

$$V_i^{(0)} \ge e_i^{(0)}, \quad 0 \le i \le I.$$

We observe that $f'(\varsigma_i^{(n)})$ is bounded from above by $f(P)$. It follows from comparison Lemma 3.3 that $V_h^{(n)} \ge e_h^{(n)}$. By the same way, we also prove that $V_h^{(n)} \ge -e_h^{(n)}$ which implies that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty \le e^{(P+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Mh^2 + M\Delta t_n).$$

Let us show that $N = J$.

Suppose that $N < J$, if we replace n by N in the above inequality and use (5.2), we find that

$$1 \le \|U_h^{(N)} - u_h(t_N)\|_\infty \le e^{(P+1)t_N}(\|\varphi_h - u_h(0)\|_\infty + Mh^2 + M\Delta t_n).$$

Since term on the right hand side of the second inequality goes to zero as h tends to zero, we deduce that $1 \leq 0$, which is a contradiction and the proof is complete. □

Now, we are in a position to prove the main theorem of this section.

Theorem 5.2. *Suppose that the problem* (1.1)-(1.3) *has a solution* u *which blows up in a finite time* T_0 *and* $u \in C^{4,2}([-1, 1] \times [0, T_0))$. *Assume that the initial data at* (2.3) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0. \tag{5.5}$$

Under the assumption of Theorem 4.1, *the problem* (2.1)-(2.3) *has a solution* $U_h^{(n)}$ *which blows up in a finite time* $T_h^{\Delta t}$ *and the following relation holds*:

$$\lim_{h \to 0} T_h^{\Delta t} = T_0.$$

Proof. We know from Remark 4.1 that $\frac{\tau}{\tau'}$ is bounded. Letting $\varepsilon > 0$, there exists a constant $R > 0$ such that

$$\frac{\tau}{f(x)} + \frac{\tau}{\tau'} \int_x^{+\infty} \frac{ds}{f(s)} < \frac{\varepsilon}{2} \quad \text{for} \quad x \in [R, +\infty). \tag{5.6}$$

Since u blows up at the time T_0, there exists $T_1 \in (T_0 - \frac{\varepsilon}{2}, T_0)$ such that

$$\|u(., t)\|_\infty \geq 2R \quad \text{for} \quad t \in [T_1, T_0).$$

Let $T_2 = \frac{T_1 + T_2}{2}$ and q be a positive integer such that $t_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_1, T_2]$ for h small enough.

We have $\sup_{t\in[0, T_2]} \|u(., t)\|_\infty < +\infty$. It follows from Theorem 4.1 that the problem (2.1)-(2.3) has a solution $U_h^{(n)}$ which obeys

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < R \quad \text{for} \quad n \le q,$$

which implies that

$$\|U_h^{(q)}\|_\infty \ge \|u_h(t_q)\|_\infty - \|U_h^{(q)} - u_h(t_q)\|_\infty \ge R.$$

From Theorem 4.1, $U_h^{(n)}$ blows up at the time $T_h^{\Delta t}$. It follows from Remark 4.1 and (5.6) that

$$|T_h^{\Delta t} - t_q| \le \frac{\tau}{f(\|U_h^{(q)}\|_\infty)} + \frac{\tau}{\tau'}\int_{\|U_h^{(q)}\|_\infty}^{+\infty} \frac{ds}{f(s)} \le \frac{\varepsilon}{2},$$

because $\|U_h^{(q)}\|_\infty \ge R$. We deduce that

$$|T_0 - T_h^{\Delta t}| \le |T_0 - t_q| + |t_q - T_h^{\Delta t}| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon,$$

and the proof is complete. □

Remark 5.1. Consider the following initial-boundary value problem:

$$u_t(x, t) = u_{xx}(x, t) + f(u(0, t)), \quad (x, t) \in (-1, 1) \times (0, T), \tag{5.7}$$

$$u(-1, t) = 0, \quad u(1, t) = 0, \quad t \in (0, T), \tag{5.8}$$

$$u(x, 0) = u_0(x) \ge 0, \quad x \in [-1, 1], \tag{5.9}$$

where $u_0(x)$ is a positive and symmetric function in $[-1, 1]$ and $u_0'(x) \ge 0$ for $x \in (-1, 0)$. Since $u_0(x)$ is symmetric in $[-1, 1]$, from the maximum principle u is also symmetric in $[-1, 1]$. We observe that $ux(0, t) = 0$ because $u(x, t) = u(-x, t)$.

Consider now the solution v of the following initial-boundary value problem below:

$$z_t(x, t) = u_{xx}(x, t) + f(u(0, t)), \quad (x, t) \in (-1, 0) \times (0, T), \tag{5.10}$$

$$z(-1, t) = 0, \quad z_x(1, t) = 0, \quad t \in (0, T), \tag{5.11}$$

$$z(x, 0) = u_0(x) \geq 0, \quad x \in [-1, 0]. \tag{5.12}$$

Since u is symmetric, we have $\max_{-1 \leq x \leq 1}|u(x, t)| = \max_{-1 \leq x \leq 0}|u(x, t)| = \max_{-1 \leq x \leq 0}|z(x, t)|$. Hence, to get an approximation of the blow-up time of the solution u, it suffices to obtain the one of the classical solution z which has been the subject of investigation of the present paper.

6. Numerical Experiments

In this section, we present some numerical approximations to the blow-up time of the problem (1.1)-(1.3). We approximate the solution u of the problem (1.1)-(1.3) by the solution $U_h^{(n)}$ of the following explicit scheme:

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n)} + f(U_I^{(n)}), \quad 1 \leq i \leq I,$$

$$U_0^{(n)} = 0,$$

$$U_i^{(0)} = \varphi_i \geq 0, \quad 0 \leq i \leq I,$$

where

$$\Delta t_n = \min\left\{\frac{h^2}{3}; \frac{\tau}{f(\|U_h^{(n)}\|_\infty)}\right\}.$$

We can also approximate the solution u of the problem (1.1)-(1.3) by the solution $U_h^{(n)}$ of the following implicit scheme:

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} + f(U_I^{(n)}), \quad 1 \le i \le I,$$

$$U_0^{(n)} = 0,$$

$$U_i^{(0)} = \varphi_i \ge 0, \quad 0 \le i \le I,$$

where

$$\Delta t_n = \frac{\tau}{f(\|U_h^{(n)}\|_\infty)},$$

$\tau = h^2$. In both cases, we take $\varphi_i = \varepsilon \sin(\frac{i\pi h}{2})$, $1 \le i \le I$. The explicit scheme may be written as follows:

$$U_i^{(n+1)} = U_i^{(n)} + \Delta t_n \delta^2 U_i^{(n+1)} + \Delta t_n f(U_i^{(n)}),$$

$$U_i^{(n+1)} = U_i^{(n)} + \Delta t_n \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \Delta t_n f(U_i^{(n)}),$$

$$U_i^{(n+1)} = (1 - \frac{2\Delta t_n}{h^2})U_i^{(n)} + \frac{\Delta t_n}{h^2}\left(U_{i+1}^{(n)} + U_{i-1}^{(n)}\right) + \Delta t_n f(U_i^{(n)}),$$

$$U_i^{(n+1)} = \frac{\Delta t_n}{h^2}(U_{i+1}^{(n)}) + (1 - \frac{2\Delta t_n}{h^2})U_i^{(n)} + \frac{\Delta t_n}{h^2}(U_{i-1}^{(n)}) + \Delta t_n f(U_i^{(n)}).$$

For $i = 1$,

$$U_1^{(n+1)} = \frac{\Delta t_n}{h^2}(U_2^{(n)}) + (1 - \frac{2\Delta t_n}{h^2})U_1^{(n)} + \frac{\Delta t_n}{h^2}(U_0^{(n)}) + \Delta t_n f(U_1^{(n)}).$$

For $i = 1$,

$$U_1^{(n+1)} = (1 - \frac{2\Delta t_n}{h^2})U_1^{(n)} + \frac{\Delta t_n}{h^2}(U_2^{(n)}) + \Delta t_n f(U_1^{(n)}).$$

For $i = 2$,

$$U_2^{(n+1)} = \frac{\Delta t_n}{h^2}(U_3^{(n)}) + (1 - \frac{2\Delta t_n}{h^2})U_2^{(n)} + \frac{\Delta t_n}{h^2}(U_1^{(n)}) + \Delta t_n f(U_2^{(n)}).$$

Let us notice that the restriction on the time step $\Delta t_n \leq \frac{h^2}{3}$ ensure the nonnegativity of the discrete solution.

According to the implicit scheme, it may be written in the following form:

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + f(U_i^{(n)}),$$

$$U_0^{(n+1)} = 0; \quad U_I^{(n+1)} = 0,$$

$$U_i^{(n+1)} = \frac{\Delta t_n}{h^2}U_i^{(n+1)} - \frac{2\Delta t_n}{h^2}U_i^{(n+1)} + \frac{\Delta t_n}{h^2}U_i^{(n+1)} + U_i^{(n)} + \Delta t_n f(U_i^{(n)})$$

$$(1 + \frac{2\Delta t_n}{h^2})U_i^{(n+1)} - \frac{\Delta t_n}{h^2}U_{i+1}^{(n+1)} - \frac{\Delta t_n}{h^2}U_{i-1}^{(n+1)} = U_i^{(n)} + \Delta t_n f(U_i^{(n)}),$$

or

$$-\frac{\Delta t_n}{h^2}U_{i-1}^{(n+1)} + (1 + \frac{2\Delta t_n}{h^2})U_i^{(n+1)} - \frac{\Delta t_n}{h^2}U_{i+1}^{(n+1)} = U_i^{(n)} + \Delta t_n f(U_i^{(n)}).$$

For $i = 1$,

$$(1 + \frac{2\Delta t_n}{h^2})U_1^{(n+1)} - \frac{\Delta t_n}{h^2}U_2^{(n+1)} = U_1^{(n)} + \Delta t_n f(U_1^{(n)}).$$

For $i = 2$,

$$-\frac{\Delta t_n}{h^2}U_1^{(n+1)} + (1 + \frac{2\Delta t_n}{h^2})U_2^{(n+1)} - \frac{\Delta t_n}{h^2}U_3^{(n+1)} = U_2^{(n)} + \Delta t_n f(U_2^{(n)}).$$

…

For $i = I - 1$,

$$-\frac{\Delta t_n}{h^2} U_2^{(n+1)} + (1 + \frac{2\Delta t_n}{h^2})U_{I-1}^{(n+1)} = U_1^{(n)} + \Delta t_n f(U_1^{(n)}),$$

lead us to the linear system below

$$A_h^{(n)} U_h^{(n+1)} = F_h^{(n)},$$

where $A_h^{(n)}$ is an $I \times I$ tridiagonal matrix defined as follows:

$$A_h^{(n)} = \begin{pmatrix} 1 + 2\frac{\Delta t_n}{h^2} & -\frac{\Delta t_n}{h^2} & 0 & \cdots & 0 \\ -\frac{\Delta t_n}{h^2} & 1 + 2\frac{\Delta t_n}{h^2} & -\frac{\Delta t_n}{h^2} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -\frac{\Delta t_n}{h^2} \\ 0 & \cdots & 0 & -\frac{\Delta t_n}{h^2} & 1 + 2\frac{\Delta t_n}{h^2} \end{pmatrix}$$

implies that

$$A_h^{(n)} = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ c_0 & a_0 & b_0 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & b_0 \\ 0 & \cdots & 0 & c_0 & a_0 \end{pmatrix},$$

with

$$a_0 = 1 + 2\frac{\Delta t_n}{h^2},$$

$$b_0 = -\frac{\Delta t_n}{h^2}, \quad i = 1, \dots, I - 2,$$

$$c_0 = -\frac{\Delta t_n}{h^2}, \quad i = 1, \dots, I - 1,$$

$\left(F^{(n)}\right)_i - U_i^{(n)} + \Delta t_n f(U_i^{(n)})$ and $A_h^{(n)}$ a three-diagonal matrix verifying the following properties:

$\left(A_h^{(n)}\right)_{i,i} = (1 + \frac{2\Delta t_n}{h^2}) > 0$, $0 \le i \le I$ and $\left(A_h^{(n)}\right)_{i-1,i} = -\frac{\Delta t_n}{h^2} = \left(A_h^{(n)}\right)_{i,i+1} \le 0$, $2 \le i \le I - 2$ so that $\left(A_h^{(n)}\right)_{i,i} \ge \sum_{i \ne j} |\left(A_h^{(n)}\right)_{i,j}|$.

It follows that $U_h^{(n)}$ exists for $n \ge 0$. In addition, since $U_h^{(0)}$ is nonnegative, $U_h^{(n)}$ is also nonnegative for $n \ge 0$.

We need the following definition.

Definition 6.1. The discrete solution $U_h^{(n)}$ of the explicit or of the implicit scheme blows up in a finite time if $\lim_{n \to +\infty} \|U_h^{(n)}\|_\infty = +\infty$ and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the numerical blow-up time of the solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical blow-up time, values of n, the CPU time and the orders of the approximations corresponding to the meshes of 16, 32, 64, 128, 256, 512.

We take for the numerical blow-up time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |T_{n+1} - T_n| \le 10^{-16}.$$

The initial condition is $\varphi_i = \varepsilon \sin(\frac{i\pi h}{2})$, $1 \le i \le I$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h}) / (T_{2h} - T_h))}{\log(2)}.$$

For the numerical values, we take, $U_i^{(0)} = \varphi_i = \varepsilon \sin(\frac{i\pi h}{2})$.

Numerical experiments for $f(u) = \beta e^{U_I^{(n)}}$.

First case: $\beta = 5$; $\varphi_i = \sin(\frac{i\pi h}{2})$.

Table 1. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

16	0.200507	1688	2	–
32	0.206603	6496	6	–
64	0.211148	24932	21	1.50
128	0.211205	33708	45	2.00

Table 2. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPUt	s
16	0.200514	1689	2	–
32	0.206628	6502	6	–
64	0.211201	24939	21	1.50
128	0.211215	33708	45	2.00

Second case: $\beta = 10$, $\varepsilon = 0$.

Table 3. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

16	0.111327	843	1	–
32	0.110838	3228	5	–
64	0.111079	12350	24	1.57
128	0.111088	13956	33	1.58

Table 4. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPUt	s
16	0.111327	843	1	–
32	0.110838	3228	8	–
64	0.111149	12402	34	1.57
128	0.111188	13989	38	1.58

Third case: $\beta = 20$; $\varepsilon = 0$.

Table 5. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

16	0.052386	418	1	–
32	0.051354	1596	5	–
64	0.051378	6096	24	2.00
128	0.051106	23249	120	1.50

Table 6. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPUt	s
16	0.052389	454	1	–
32	0.051355	1598	6	–
64	0.051377	6194	26	2.00
128	0.051115	23255	124	1.50

Fourth case: $\beta = 100$; $\varepsilon = 0$.

Table 7. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

16	0.011277	84	1	–
32	0.010322	319	3	–
64	0.010080	1216	23	1.70
128	0.001025	3708	30	2.00

Table 8. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPUt	s
16	0.011277	84	1	–
32	0.010322	319	3	–
64	0.010080	1216	23	1.70
128	0.010020	4633	30	2.00

Remark 6.1. In the case where the initial data is null, $\phi = 0$, and the reaction term increases as a function of β it is not hard to see that the blow-up time of the solution equals $\frac{1}{\beta}$. We observe from Tables 1-6 that the numerical blow-up time tends to $\frac{1}{\beta}$ for $\beta = 10$, $\beta = 20$, and $\beta = 100$.

When $\varphi_i = \sin(\frac{i\pi h}{2})$ with $\beta = 5$ it is not hard to see that the blow-up time of the solution equals $\frac{1}{\beta}$. See the Tables 7-8.

In the following, we also give some plots to illustrate our analysis. In Figures 1 to 6, we can appreciate that the discrete solution blows up globally. Let us notice that, theoretically, we know that the continuous solution blows up globally under the assumptions given in the introduction of the present paper (see [17], [46])

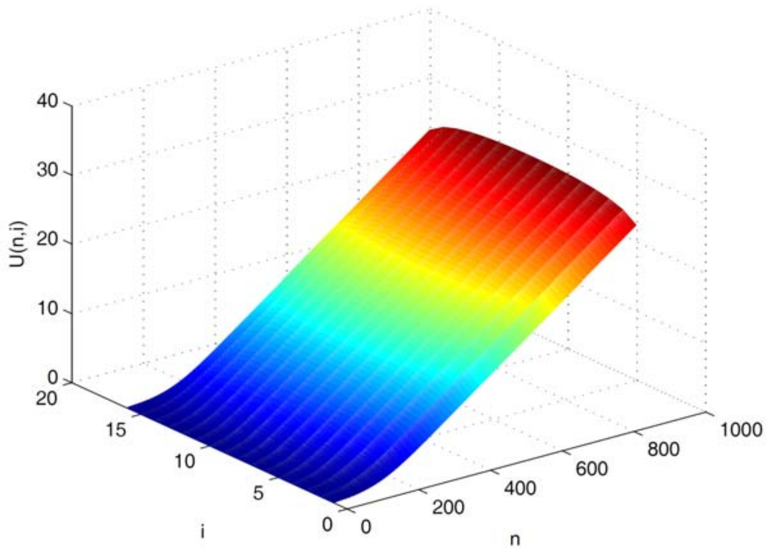

Figure 1. Evolution of the discrete solution, source $f(u) = \beta e^{u}$, $\beta = 10$, $\varepsilon = 0$, $I = 16$ (implicit scheme).

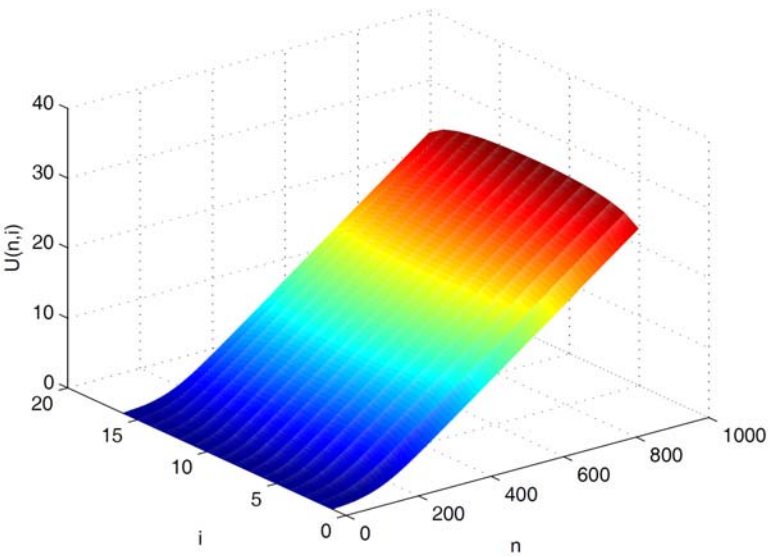

Figure 2. Evolution of the discrete solution, source $f(u) = \beta e^{u}$, $\beta = 10$, $\varepsilon = 0$, $I = 16$ (explicit scheme).

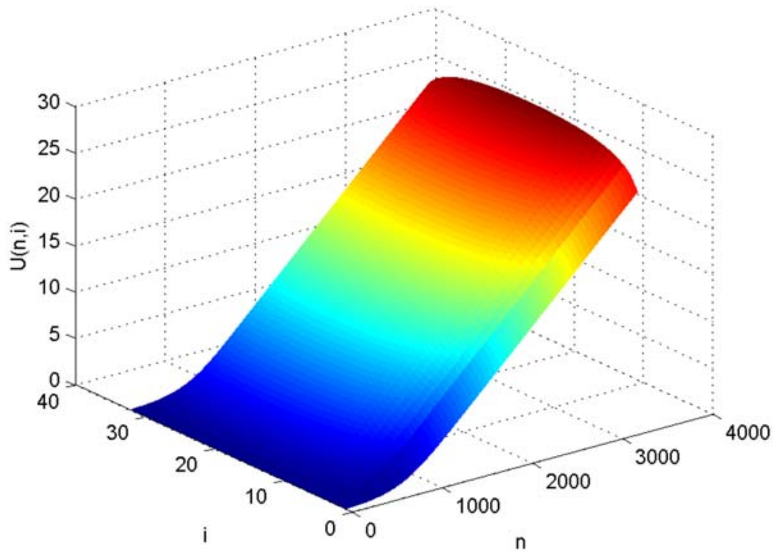

Figure 3. Evolution of the discrete solution, source $f(u) = \beta e^{u}$, $\beta = 10$, $\varepsilon = 0$, $I = 32$ (implicit scheme).

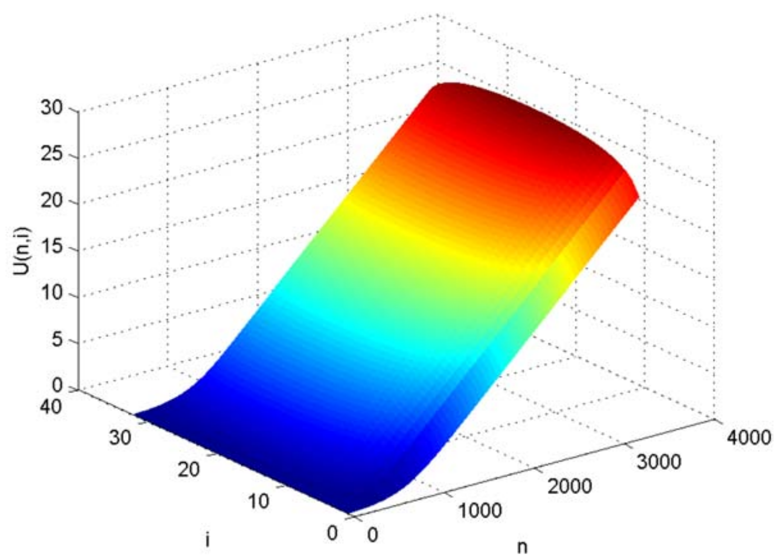

Figure 4. Evolution of the discrete solution, source $f(u) = \beta e^{u}$, $\beta = 10$, $\varepsilon = 0$, $I = 32$ (explicit scheme).

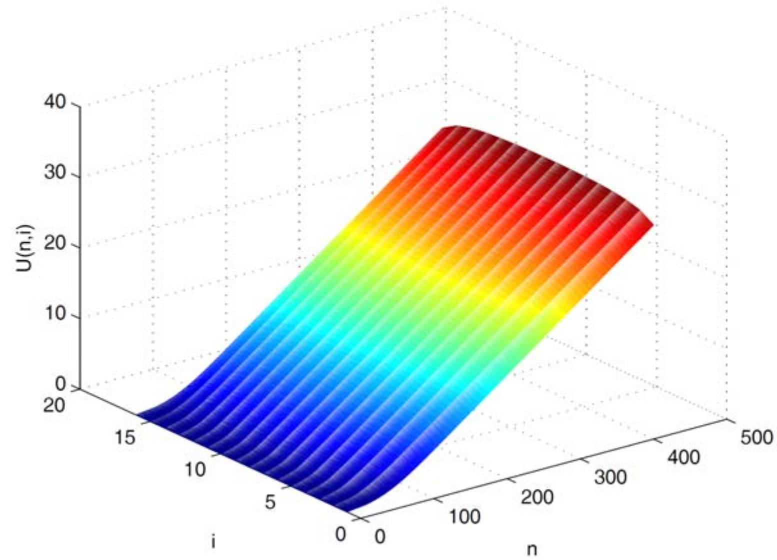

Figure 5. Evolution of the discrete solution, source $f(u) = \beta e^{u}$, $\beta = 20$, $\varepsilon = 0$ (implicit scheme).

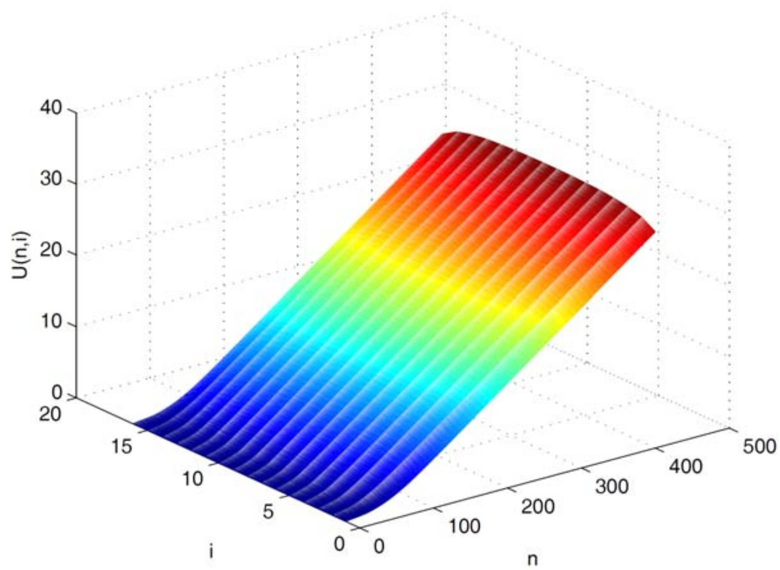

Figure 6. Evolution of the discrete solution, source $f(u) = \beta e^{u}$, $\beta = 20$, $\varepsilon = 0$ (explicit scheme).

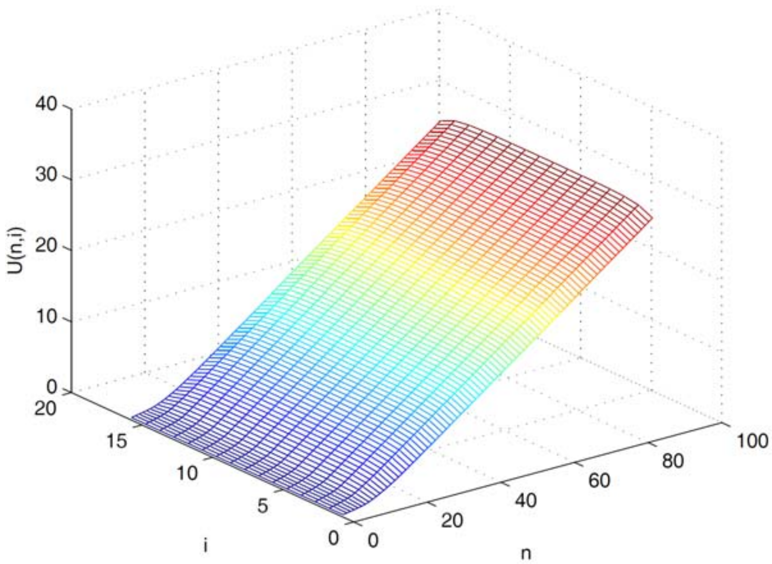

Figure 7. Evolution of the discrete solution, source $f(u) = \beta e^{u}$, $\beta = 100$, $\varepsilon = 0$, $I = 16$ (implicit scheme).

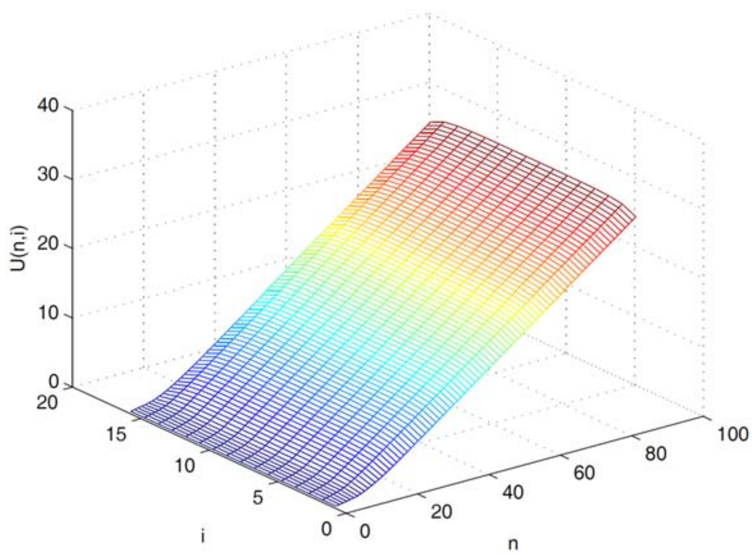

Figure 8. Evolution of the discrete solution, source $f(u) = \beta e^{u}$, $\beta = 100$, $\varepsilon = 0$, $I = 16$ (explicit scheme).

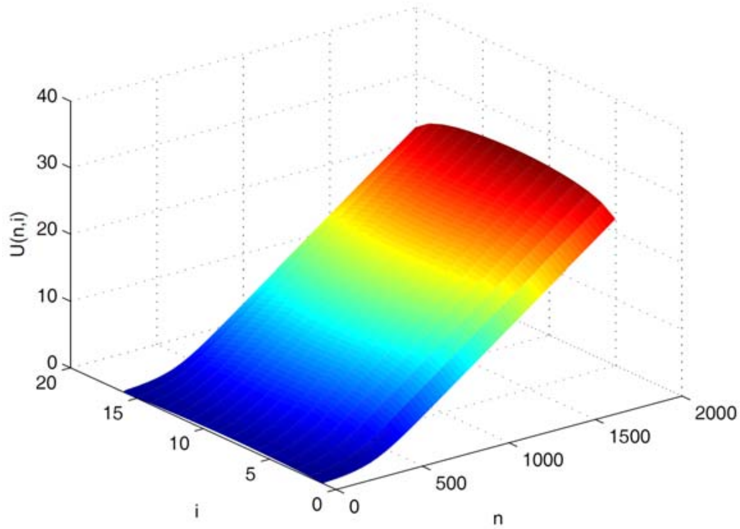

Figure 9. Evolution of the discrete solution, source $f(u) = \beta e^{u}$, $\beta = 5$, $\varphi_i = \sin(\frac{i\pi h}{2})$, $I = 16$ (implicit scheme).

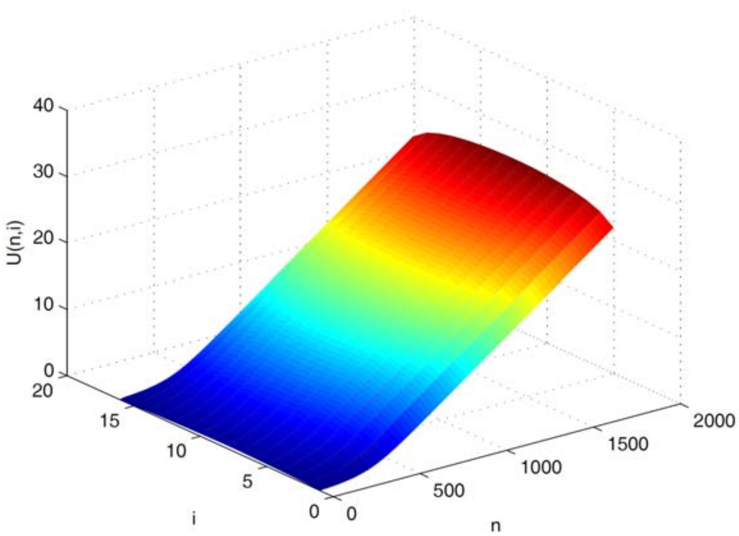

Figure 10. Evolution of the discrete solution, source $f(u) = \beta e^{u}$, $\beta = 5$, $\varphi_i = \sin(\frac{i\pi h}{2})$, $I = 16$ (explicit scheme).

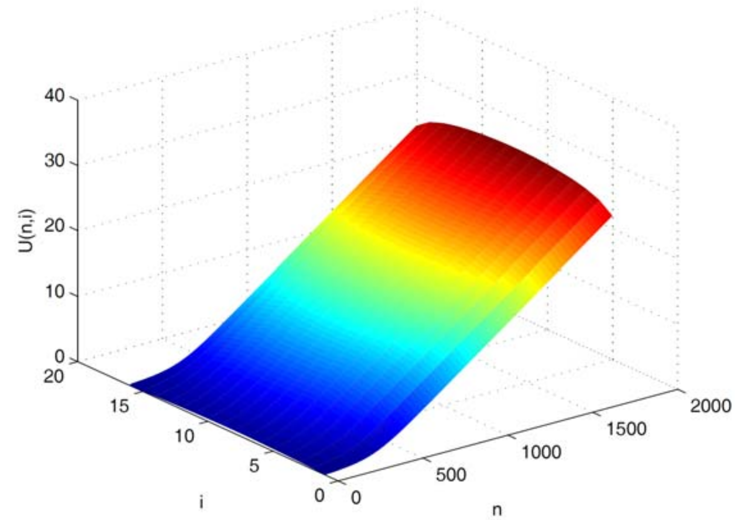

Figure 11. Evolution of the discrete solution, source $f(u) = \beta e^{u}, \beta = 5$, $\varphi_i = \sin(\frac{i\pi h}{2})$, $I = 32$ (implicit scheme).

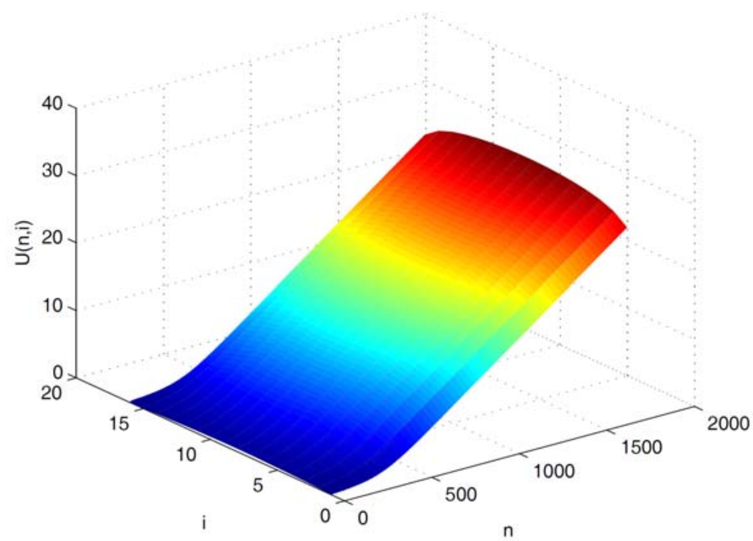

Figure 12. Evolution of the discrete solution, source $f(u) = \beta e^{u}, \beta = 5$, $\varphi_i = \sin(\frac{i\pi h}{2})$, $I = 32$ (explicit scheme).

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