Journal of Mathematical Sciences: Advances and Applications
Volume 59, 2019, Pages 33-60
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100122093

POWER-EXPONENTIAL TRANSSERIES AS SOLUTIONS TO ODE

ALEXANDER BRUNO

Keldysh Institute of Applied Mathematics
Moscow 125047
Russia
e-mail: abruno@keldysh.ru

Abstract

A polynomial ordinary differential equation (ODE) of order n in a neighbourhood of zero or infinity of the independent variable is considered. In 2004, a method was proposed for computing its solutions in the form of power series and an exponential addition that involves another power series. The addition contains an arbitrary constant, exists only in a set E_1 consisting of sectors of the complex plane, and is found by solving an ODE of order $n-1$. It is possible a hierarchical sequence of exponential additions, each is determined by an ODE of progressively lower order $n-i$ and each exists in its own set E_i. In this case, one has to check that the intersection of the existence sets $E_1 \cap \ldots \cap E_i$ is nonempty. Each exponential addition extends to its own exponential expansion involving a countable set of power series. Finally, the solution is expanded into a transseries involving a countable set of power series, some of which are summable. The transseries describes families of solutions to the original equation in certain sectors of the complex plane.

2010 Mathematics Subject Classification: 34E05, 34E99.
Keywords and phrases: ordinary differential equation, solution, power series, exponential addition, exponential expansion, transseries.
Received September 3, 2019

1. Introduction

We consider a polynomial ODE of the order n in a neighbourhood of zero or of infinity of the independent variable. A method of calculation of its solutions in the form of a power series and an exponential addition, which contains one more power series, was described in [1]. The exponential addition has an arbitrary constant, exists in some set E_1 of sectors of the complex plane and can be found from a solution to an ODE of the order $n-1$. A hierarchical sequence of such exponential additions is possible, that each of these exponential additions is defined from an ODE of a lower order $n-i$ and exists in its own set E_i. Here we must check the non-emptiness of intersection of the sets $E_1 \cap \ldots \cap E_i$ [2]. Each exponential addition continues into its own exponential expansion, containing countable set of power series [3]. As a result we obtain an expansion of a solution into a transseries [4], containing countable set of power series, some of which are summable [5]. The transseries describes families of solutions to the initial ODE in some set of sectors of the complex plane. Examples from the Painlevé equations are given. We discuss also a connection of computation of these transseries with computer algebra.

The main result:

Theorem 4. *As* $x \to x_0$, *where* $x_0 = 0$ *or* $x_0 = \infty$, *the solution* $y(x)$ *of the polynomial ODE* $f(x, y) = 0$ *of order n can be expanded in a transseries*

$$y(x) = \varphi(x) + c_1 \exp\int y_1(x)dx + \sum_{k=2}^{\infty} B_{1,k}(x)\left[c_1 \exp\int y_1(x)dx\right]^k,$$

where

$$y_1(x) = \varphi_1(x) + c_2 \exp\int y_2(x)dx + \sum_{k=2}^{\infty} B_{2,k}(x)\left[c_2 \exp\int y_2(x)dx\right]^k,$$

where

…………………………………………………………………………………

$$y_i(x) = \varphi_i(x) + c_{i+1}\,\exp\int y_{i+1}(x)dx + \sum_{k=2}^{\infty} B_{i,k}(x)\left[c_{i+1}\,\exp\int y_{i+1}(x)dx\right]^k,$$

…………………………………………………………………………………

where

$$y_{\mu-1}(x) = \varphi_{\mu-1}(x) + c_{\mu}\,\exp\int y_{\mu}(x)dx + \sum_{k=2}^{\infty} B_{\mu,k}(x)\left[c_{\mu}\,\exp\int y_{\mu}(x)dx\right]^k,$$

where $y_\mu(x) = \varphi_\mu(x)$. *Here* $\varphi(x), \varphi_1(x), \dots, \varphi_\mu(x)$ *and all* $B_{ik}(x)$ *are power series in* x;

$$\varphi_i(x) = \beta_i x^{\alpha_i} + \dots,\ \alpha_i \in \mathbb{R},\ i = 1, \dots, \mu.$$

Then $E_i = \{x : \mathrm{Re}(\beta_i x^{\alpha_i + 1} / (\alpha_i + 1)) < 0\}$ *for* x *near* x_0, $D_i = E_1 \cap \dots \cap E_i$. *The arbitrary constants* c_i *are nonzero only in the sets* D_i. *The number* $\mu < n$ *is determined by the fact that the set* D_μ *is not empty, whereas the set* $D_{\mu+1}$ *is empty, and* $\mu = n$, *if the set* D_n *is not empty.*

2. Power Series and their Summability

Let $x \in \mathbb{C}$. We set $\omega = -1$ if $x \to 0$ and $\omega = 1$ if $x \to \infty$, i.e.,

$$x^{\omega} \to \infty.$$

Consider the power series

$$\varphi(x) = a_r x^r + \sum_{\omega s < \omega r} a_s x^s, \tag{1}$$

where r and s are rational numbers with a common denominator q, while a_r and a_s are complex constants, i.e., (1) are Puiseux-Laurent series. Series (1) with all coefficients a_s satisfying the inequalities

$$|a_s| \leqslant AB^{|s|}\Gamma^k(|s|),$$

where A, B, k are nonnegative constants and $\Gamma(|s|)$ is the gamma function (factorial), form a Gevrey class. Any power series (1) of the Gevrey class with $k \leqslant 1$ is summable in a neighbourhood of the point $x_0 = 0$ if $\omega = -1$ or the point $x_0 = \infty$ if $\omega = 1$ on a q-sheeted cover $\mathbb{C}_q$ of the complex plane $\mathbb{C}$ with several cuts starting at the point x_0 [5]. Here k is the Gevrey number.

Consider the ordinary differential equation (ODE)

$$f(x, y) \stackrel{\text{def}}{=} f(x, y, y', \dots, y^{(n)}) = 0, \tag{2}$$

where $y' = dy / dx$.

Theorem 1 ([6-8])**.** *If* $f(x, y)$ *in ODE* (2) *is a polynomial in the dependent variable y and all its derivatives* $y', \dots, y^{(n)}$ *and belongs to the Gevrey class with respect to the independent variable x, then its solution* $y = \varphi(x)$ *in the form of the power series* (1) *belongs to the Gevrey class.*

Remark 1. Power series representing solutions to algebraic and analytic equations always converge, i.e., they are analytic. Therefore, the class of analytic equations is closed on analytical solutions. For ODEs, solutions in the form of power series frequently diverge, but they belong to the Gevrey class, i.e., can be summable. According to Theorem 1, this class of ODEs is closed.

The following algorithm for computing all solutions (1) to Equation (2) of the Gevrey class with respect to x and of a polynomial equation in y and its derivatives was proposed in [1]. In ODE (2), the series $f(x, y)$ is considered as a sum of differential monomials $a_i(x, y)$, each being the product of a usual monomial const. $x^{\alpha} y^{\beta}$ with constants $\alpha, \beta \in \mathbb{R}$ and a finite number of derivatives $y^{(l)}$. Each differential monomial $a(x, y)$ is associated with its vector power exponent $Q(a) = (q_1, q_2) \in \mathbb{R}^2$ according to the following rules:

$$Q(\text{const}) = 0,\ Q(x^{\alpha} y^{\beta}) = (\alpha, \beta),\ Q(y^{(l)}) = (-l, 1),\ Q(a \cdot b) = Q(a) + Q(b).$$

The set $\mathbf{S}(f)$ of all vector power exponents $Q(a_i)$ of all differential monomials $a_i(x, y)$ involved in the sum $f(x, y)$ is called the support of the sum $f(x, y)$. The closure $\Gamma(f)$ of the convex hull of $\mathbf{S}(f)$ is called the polygon of the sum $f(x, y)$. The boundary $\partial\Gamma$ of the polygon Γ consists of vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$. Each vertex and edge $\Gamma_j^{(d)}$ are associated with the truncated sum

$$\hat{f}_j^{(d)}(x, y) = \sum a_i(x, y) \text{ over } Q(a_i) \in \Gamma_j^{(d)}, \tag{3}$$

and with the normal cone

$$\mathbf{U}_j^{(d)} = \{P = (p_1, p_2) : \langle Q, P\rangle = \langle Q', P\rangle,\ Q, Q' \in \Gamma_j^{(d)},$$
$$\langle Q, P\rangle > \langle Q'', P\rangle;\ Q'' \in \Gamma \setminus \Gamma_j^{(d)}\}, \tag{4}$$

lying in the dual plane $\mathbb{R}_*^2 = \{P = (p_1, p_2)\}$. Here, $\langle Q, P\rangle = q_1 p_1 + q_2 p_2$ is the scalar product. For each solution $y = \varphi(x)$ of Equation (2) in the form of power series (1), the first term $y = a_r x^r$ is the solution to the truncated equation (3) if

$$\omega(1, r) \subset \mathbf{U}_j^{(d)}.$$

If $\omega = -1$, i.e., $x \to 0$, then the vertices and edges $\Gamma_j^{(d)}$, including upper and lower ones, are taken on the left-hand side of the polygon $\Gamma(f)$. If $\omega = 1$, i.e., $x \to \infty$, then the vertices and edges $\Gamma_j^{(d)}$, including upper and lower ones, are taken on the right-hand side of the polygon $\Gamma(f)$.

3. Exponential Additions and Exponential Expansions

Let the series

$$y = \varphi(x) = a_r x^r + \sum_{\omega s < \omega r} a_s x^s, \tag{5}$$

be a formal solution to Equation (2). Substituting

$$y = \varphi(x) + z$$

into Equation (2) yields the equation

$$g(x, z) \overset{\text{def}}{=} f(x, \varphi(x) + z) = 0. \tag{6}$$

It has the solution $z = 0$. Therefore, the polygon $\tilde{\Gamma} = \Gamma(g)$ has a lower horizontal edge $\tilde{\Gamma}^{(1)}$ of height $m = q_2 \geqslant 1$. This edge is associated with the truncated equation

$$\hat{g}(x, z) = 0. \tag{7}$$

Lemma 1 ([1])**.** *If* $m = 1$, *then*

$$\hat{g}(x, z) = \left.\frac{\delta f}{\delta y}\right|_{y=\varphi(x)} z, \tag{8}$$

where $\left.\frac{\delta f}{\delta z}\right|_{y=\varphi(x)}$ *is an operator, namely, the first variation of the function f taken on* $y = \varphi(x)$. *This operator is applied to z.*

Lemma 2 ([1]). *The logarithmic transformation*

$$y_1 = \frac{d \ln z}{dx}, \tag{9}$$

for each integer $l \geqslant 0$ *yields*

$$z^{(l)} = z\left[y_1^l + P_{l-1}(y_1, y_1', \dots, y_1^{(l-1)})\right],$$

where P_{l-1} *is a polynomial of degree* $l-1$ *in the indicated variables with constant coefficients.*

Corollary 1. *Substitution* (9) *yields*

$$\hat{g}(x, z) = z^m f_1(x, y_1), \tag{10}$$

where $f_1(x, y_1)$ *is a differential polynomial in* y_1 *and its derivatives of order* $n-1$ *if* n *is the order of the differential polynomial* $\hat{g}(x, z)$. *Namely, if* $\hat{g}(x, z) \overset{\text{def}}{=} \hat{g}(x, z, z', \dots, z^{(n)})$, *then*

$$f_1(x, y_1, \dots, y_1^{(n-1)}) = \hat{g}\left(x, 1, y_1, \dots, y_1^n + P_{n-1}\left(y_1, \dots, y_1^{(n-1)}\right)\right).$$

Theorem 2. *If*

$$\left.\frac{\delta f}{\delta y}\right|_{y=\varphi(x)} = \sum_{k=1}^{n} A_k(x) \frac{d^k}{dx^k},$$

then, for $y_1 = d \ln z / dx$ *and* $m = 1$ *in* (10), *we have*

$$f_1(x, y_1) = \sum_{k=0}^{n} A_k(x) \left[y_1^k + P_{k-1}\left(y_1, \dots, y_1^{(k-1)}\right)\right].$$

Proof follows immediately from Lemmas 1, 2 and Corollary 1.

By using transformation (9), Equation (7) of order n is reduced to the equation

$$f_1(x, y_1) = 0 \tag{11}$$

of order $n-1$. By applying the above-mentioned method, we can find all solutions to Equation (11) in the form of power series

$$y_1 = \varphi_1(x) = b_\rho x^\rho + \sum_{\omega\sigma<\omega\rho} b_\sigma x^\sigma. \qquad (12)$$

Then the solutions to the truncated equation (7) are

$$z = z_0(x) = c_1 \exp\int \varphi_1(x)dx, \qquad (13)$$

where c_1 is an arbitrary constant.

Theorem 3 ([3]). *The solutions to the complete equation* (6) *have the form*

$$z = z_0(x) + \sum_{k=2}^{\infty} B_k(x) z_0^k, \qquad (14)$$

where $B_k(x)$ *are power series of form* (1).

Theorem 3 was formulated in [3] without proof. Its proof is given below in Section 6.

For each $B_k(x)$, using the original equation (2), we can set up its own ODE in a similar manner to what was done for functional coefficients of complicated and exotic expansions of solutions to ODEs [9] (see Section 6 below here).

Note that, as $x^\omega \to \infty$, the exponential addition (13) and expansion (14) must tend to zero. However, $\exp \alpha x^\beta \to 0$ only if $\mathrm{Re}(\alpha x^\beta) < 0$, i.e., according to (12), $\exp\int \varphi_1(x)dx \to 0$ for

$$\mathrm{Re}(b_\rho x^{\rho+1} / (\rho+1)) < 0.$$

This inequality single out several sectors on $\mathbb{C}_q$; their collection is denoted by E_1. This set is called the set of existence of exponential addition (13) and expansion (14).

4. Hierarchy of Exponential Additions and Transseries

Let power expansion (12) be a solution to Equation (11). Making the substitution $y_1 = \varphi_1(x) + z_1$ in Equation (11) yields the equation

$$g_1(x, z_1) \stackrel{\text{def}}{=} f_1(x, \varphi_1 + z_1) = 0. \tag{15}$$

The polygon $\Gamma(g_1)$ of the differential sum $g_1(x, z_1)$ has a lower horizontal edge of height $m_1 \geqslant 1,$ which is associated with the truncated equation

$$\hat{g}_1(x, z_1) = 0. \tag{16}$$

Applying the logarithmic transformation

$$\frac{d \ln z_1}{dx} = y_2,$$

we obtain

$$\hat{g}_1(x, z_1) = z_1^{m_1} f_2(x, y_2), \tag{17}$$

where $f_2(x, y_2)$ is a differential sum. Moreover, the truncated equation (16) of order $n - 1$ passes into the equation $f_2(x, y_2) = 0$ of order $n - 2.$ Let $y_2 = \varphi_2(x)$ be its solution in the form of a power series. Then the truncated equation (16) has the solutions

$$z_1 = c_2 \exp \int \varphi_2(x) dx.$$

As $x^{\omega} \to \infty,$ they tend to zero in a certain sectoral set E_2 of the q-sheeted cover $\mathbb{C}_q$ of the complex plane $\mathbb{C}.$ Moreover, the solutions to complete equation (6) have the form

$$z = c_1 \exp \int [\varphi_1(x) + c_2 \exp \int \varphi_2(x) dx + \dots] dx + \dots . \tag{18}$$

This expansion makes sense only in the intersection $E_1 \cap E_2 \overset{\text{def}}{=} D_2$. In it, there is a two-parameter family of expansions (18). Outside D_2, but within the set E_1, exists only a one-parameter family of asymptotic expansions of solutions (13). Outside E_1, there is only one solution (5).

Thus, we obtain the following sequence of equations and their solutions.

Step 0. The original equation $f(x, y) = 0$ of order n and its solution $y = \varphi(x)$, where $\varphi(x)$ is a power series.

Step 1. From $f(x, y)$ and $\varphi(x)$, by applying Corollary 1 and Theorem 2, we obtain: an equation $f_1(x, y_1) = 0$ of order $n - 1$, its solution $y_1 = \varphi_1(x)$ in the form of a power series, the exponential addition $z = c_1 \exp\int \varphi_1(x)dx$, and its set of existence $E_1 = \{x : \mathrm{Re}\int \varphi_1(x)dx < 0\}$. Moreover, $z = y - \varphi(x)$, $y_1 = d \ln z / dx$.

Step $i - 1$. We obtain an equation $f_{i-1}(x, y_{i-1}) = 0$ of order $n - i + 1$ and its solution $y_{i-1} = \varphi_{i-1}(x)$ in the form of a power series.

Step i. From $f_{i-1}(x, y_{i-1})$ and $\varphi_{i-1}(x)$, Corollary 1 and Theorem 2 yield: an equation $f_i(x, y_i) = 0$ of order $n - i$, its solution $y_i = \varphi_i(x)$ in the form of a power series, the exponential addition $z_{i-1} = c_i \exp\int \varphi_i(x)dx$, and its set of existence

$$E_i = \{x : \mathrm{Re}\int \varphi_i(x)dx < 0\}.$$

Moreover, $z_{i-1} = y_{i-1} - \varphi_{i-1}$, $y_i = d \ln z_{i-1} / dx$, and

$$D_i = E_1 \cap \ldots \cap E_i.$$

Step *n*. We obtain: an equation $f_n(x, y_n) = 0$ of order 0, i.e., without derivatives, its solution $y_n = \varphi_n(x)$ in the form of a power series, the exponential addition $z_{n-1} = c_n \exp\int \varphi_n(x)dx,$ and its set of existence E_n.

Let μ be a number $i < n$ such that $D_\mu \neq \emptyset,$ but $D_{\mu+1} = \emptyset$ and $\mu = n$ if $D_n \neq \emptyset.$ Then, for the equation

$$f_\mu(x, y_\mu) = 0,$$

we find its solution $y_\mu = \varphi_\mu(x)$ in the form of a power series. Next, by applying Theorem 3, for the equation

$$f_{\mu-1}(x, y_{\mu-1}) = 0,$$

we find its solution in the form of an exponential expansions

$$y_{\mu-1}(x) = \varphi_{\mu-1}(x) + c_\mu \exp\left(\int \varphi_\mu dx\right) + \sum_{k=2}^{\infty} B_{\mu k}(x)\left[c_\mu \exp\left(\int \varphi_\mu dx\right)\right]^k,$$

where $B_{\mu k}(x)$ are power series and c_μ is an arbitrary constant. After that, solutions to the equation

$$f_{\mu-2}(x, y_{\mu-2}) = 0$$

are found in the form

$$y_{\mu-2}(x) = \varphi_{\mu-2}(x) + c_{\mu-1} \exp\left(\int y_{\mu-1}(x)dx\right)$$

$$+ \sum_{k=2}^{\infty} B_{\mu-1,\, k}(x)\left[c_{\mu-1} \exp\left(\int y_{\mu-1}dx\right)\right]^k, \qquad (19)$$

where $B_{\mu-1, k}(x)$ are power series and $c_{\mu-1}$ is an arbitrary constant, etc. Finally, for the original equation $f(x, y) = 0,$ we obtain its solutions

$$y = \varphi(x) + c_1 \exp\int y_1(x)dx + \sum_{k=2}^{\infty} B_{1k}(x)\left[c_1 \exp\int y_1 dx\right]^k,$$

where $B_{1k}(x)$ are power series and c_1 is an arbitrary constant. Thus, we have obtained a transseries [4]. It describes i-parametric families of solutions to the original equation (2) in the domains D_i. In Theorem 4 below, this transseries is described in a reverse order.

Theorem 4. *As* $x \to x_0$, *where* $x_0 = 0$ *or* $x_0 = \infty$, *the solution* $y(x)$ *of the polynomial ODE* $f(x, y) = 0$ *of order* n *can be expanded in a transseries*

$$y(x) = \varphi(x) + c_1 \exp\int y_1(x)dx + \sum_{k=2}^{\infty} B_{1,k}(x)\left[c_1 \exp\int y_1(x)dx\right]^k,$$

where

$$y_1(x) = \varphi_1(x) + c_2 \exp\int y_2(x)dx + \sum_{k=2}^{\infty} B_{2,k}(x)\left[c_2 \exp\int y_2(x)dx\right]^k,$$

where

....

$$y_i(x) = \varphi_i(x) + c_{i+1} \exp\int y_{i+1}(x)dx + \sum_{k=2}^{\infty} B_{i,k}(x)\left[c_{i+1} \exp\int y_{i+1}(x)dx\right]^k,$$

....

where

$$y_{\mu-1}(x) = \varphi_{\mu-1}(x) + c_\mu \exp\int y_\mu(x)dx + \sum_{k=2}^{\infty} B_{\mu,k}(x)\left[c_\mu \exp\int y_\mu(x)dx\right]^k,$$

where $y_\mu(x) = \varphi_\mu(x)$. *Here* $\varphi(x), \varphi_1(x), \ldots, \varphi_\mu(x)$ *and all* $B_{ik}(x)$ *are power series in* x;

$$\varphi_i(x) = \beta_i x^{\alpha_i} + \ldots, \ \alpha_i \in \mathbb{R}, \ i = 1, \ldots, \mu.$$

Then $E_i = \{x : \mathrm{Re}(\beta_i x^{\alpha_i+1}/(\alpha_i+1)) < 0\}$ *for x near* x_0, $D_i = E_1 \cap \ldots \cap E_i$. *The arbitrary constants* c_i *are nonzero only in the sets* D_i. *The number* $\mu < n$ *is determined by the fact that the set* D_μ *is not empty, whereas the set* $D_{\mu+1}$ *is empty, and* $\mu = n$, *if the set* D_n *is not empty.*

Remark 2. In fact, the infinite power series $\varphi_i(x)$ cannot be calculated. It suffices to calculate their segments ensuring that the equations $f_i(x, y_i) = 0$ contain the nearest with respect to q_1 differential monomials of the highest order $n - i$.

In [2], for Equation (2) of order $n = 4$, a sequence of segments of the series $\varphi(x)$, $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ was calculated and it was shown that $D_2 \neq \emptyset$ and $D_3 = \emptyset$, i.e., $\mu = 2$.

5. Examples

Let us consider the fourth Painlevé equation with zero values of two its parameters

$$y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2x^2y.$$

Multiplying it by $2y$ and transferring its left part into right, we obtain the polynomial ODE

$$f(x, y) \stackrel{\mathrm{def}}{=} -2yy'' + y'^2 + 3y^4 + 8xy^3 + 4x^2y^2 = 0. \tag{20}$$

Its support $\mathbf{S}(f)$ and polygon $\Gamma(f)$ are shown in Figure 1.

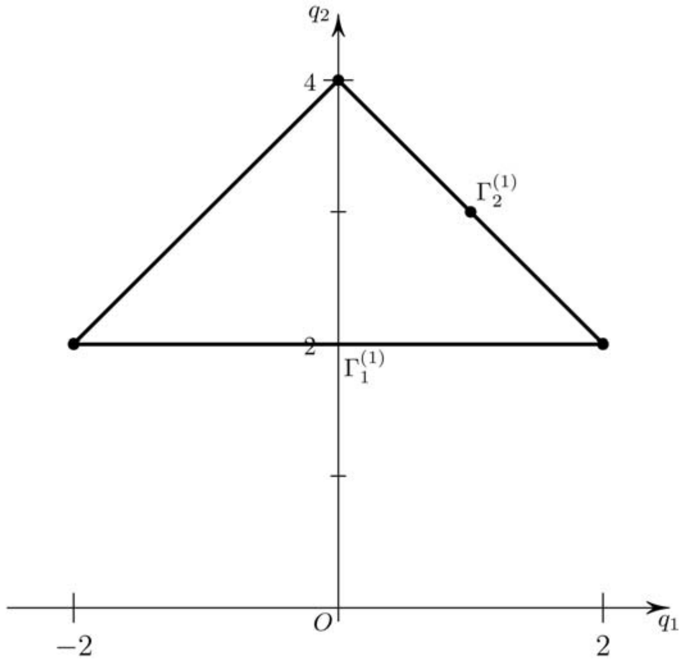

Figure 1. Support and polygon of Equation (20).

Example 1. The lower horizontal edge $\Gamma_1^{(1)}$ of Figure 1 corresponds to the truncated equation

$$\hat{f}_1^{(1)}(x,\ y) \stackrel{\text{def}}{=} -2yy'' + y'^2 + 4x^2y^2 = 0. \tag{21}$$

Here $\varphi \equiv 0$, $z = y$ and $g(x,\ z) \equiv f(x,\ y)$. We put

$$y_1 = \frac{d \ln z}{dx} = \frac{d \ln y}{dx},$$

i.e., $y = \exp\int y_1 dx$. Then

$$y' = y_1 y, \quad y'' = (y_1' + y_1^2)y, \tag{22}$$

that corresponds to Lemma 2, and the truncated equation (21) takes the form

$$y^2[-2(y_1' + y_1^2) + y_1^2 + 4x^2] = 0,$$

i.e.,

$$f_1(x, y_1) \stackrel{\text{def}}{=} -2y_1' - y_1^2 + 4x^2 = 0. \qquad (23)$$

It is the non-integrable Riccati equation. Its support and polygon $\Gamma(f_1)$ are shown in Figure 2.

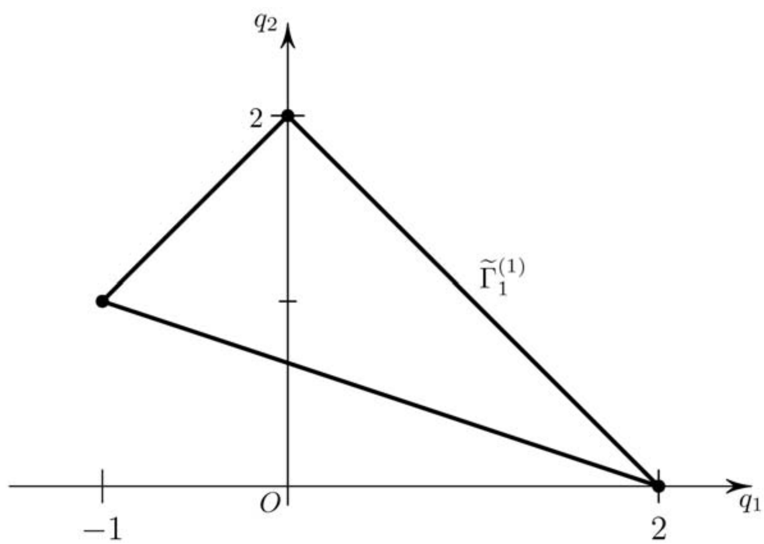

Figure 2. Support and polygon of Equation (23).

The polygon has the right inclined edge $\tilde{\Gamma}_1^{(1)}$ with the truncated equation

$$\hat{f}_1(x, y_1) \stackrel{\text{def}}{=} -y_1^2 + 4x^2 = 0. \qquad (24)$$

Here $\omega = 1$, i.e., $x \to \infty$. Equation (24) has 2 solutions $y_1 = \pm 2x$, which are continued into series

$$y_1 = \varphi_1(x) = \pm 2x - \frac{1}{x} \mp \frac{3}{4x^3} + O(x^{-5}), \qquad (25)$$

for solutions to Equation (24). For them

$$D_{1\pm} = E_{1\pm} = \{x : \pm \operatorname{Re} x^2 < 0\}.$$

The set D_{1+} is shaded in Figure 3.

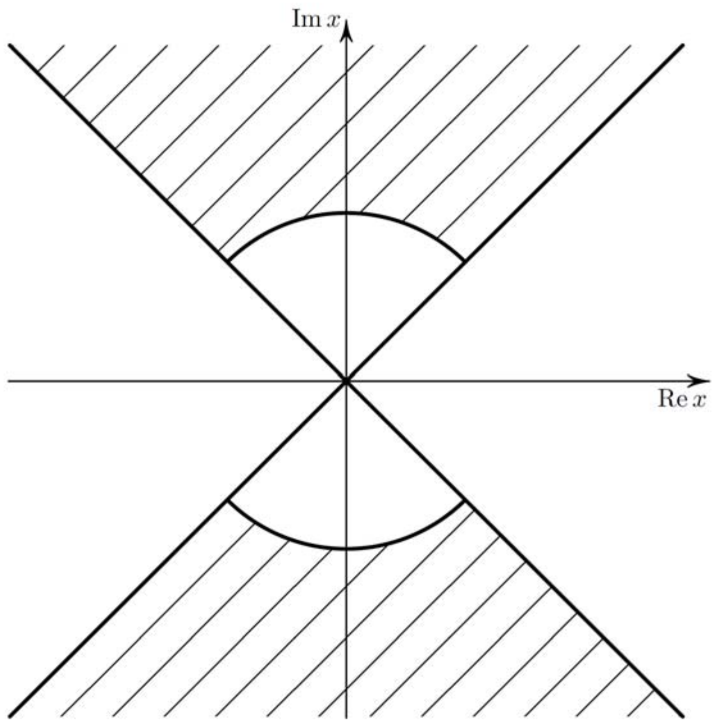

Figure 3. Sectors of the set E_{1+} near $x = \infty$ are dashed.

Let us apply Theorem 2 to calculate $f_2(x, y_2)$. According to (23)

$$\frac{\delta f_1}{\delta y_1} = -2\frac{d}{dx} - 2y_1,$$

i.e., on $y_1 = \varphi_1(x)$

$$\left.\frac{\delta f_1}{\delta y_1}\right|_{y_1=\varphi_1(x)} = -2\frac{d}{dx} - 2\varphi_1(x) = -2\left[\frac{d}{dx} + \varphi_1(x)\right].$$

Then

$$f_2(x,\ y_2) = -\ 2[y_2 + \varphi_1(x)],$$

and

$$y_2 = -\ \varphi_1(x) = \mp 2x + \dots,$$

i.e.,

$$\int y_2 dx = \mp x^2 + \dots.$$

Thus

$$E_{2\pm} = \{x : \mp \operatorname{Re} x^2 < 0\} \text{ and } E_{1\pm} \cap E_{2\pm} = \emptyset.$$

Hence, $\mu = 1$ and for solutions of Equation (20), we obtain 2 one-parameter families of expansions

$$y = c_1 \exp \int \varphi_1(x)dx + \sum_{k=2}^{\infty} B_k(x)\left[c_1 \exp \int \varphi_1(x)dx\right]^k \tag{26}$$

in two sectors $E_{1\pm} = \{x : \pm \operatorname{Re} x^2 < 0\}$ of the complex plane (Figure 3).

Example 2. The edge $\Gamma_2^{(1)}$ of Figure 1 corresponds to the truncated equation

$$\hat{f}_2^{(1)} \overset{\text{def}}{=} 3y^4 + 8xy^3 + 4x^2y^2 = 0.$$

It has 2 truncated solutions $y = -\ 2x$ and $y = -\frac{2}{3}x$. $\Gamma_2^{(1)}$ is the right edge, so $x \to \infty$ and $\omega = 1$. The first truncated solution is continued into a solution of the full equation (20) as

$$y = \varphi(x) = -\ 2x + \frac{1}{4x^3} + O(x^{-7}). \tag{27}$$

Let us calculate $f_1(x,\ y_1)$ using Theorem 2. Here

$$\frac{\delta f}{\delta y} = -\ 2y\frac{d^2}{dx^2} + 2y'\frac{d}{dx} - 2y'' + 12y^3 + 24xy^2 + 8x^2y.$$

According (22) and Theorem 2, we obtain

$$\begin{aligned} f_1(x,\ y_1) &= -\ 2\varphi(y_1' + y_1^2) + 2\varphi' y_1 - 2\varphi'' + 12\varphi^3 + 24x\varphi^2 + 8x^2\varphi \\ &= -\ 2\left(-\ 2x + \frac{1}{4x^3} + \ldots\right)(y_1' + y_1^2) + 2\left(-\ 2 - \frac{3}{4x^4} + \ldots\right)y_1 \\ &\quad - \frac{2}{3x^5} - 12\cdot 8x^3 + 24\cdot 4x^3 - 16x^3 + \ldots \\ &= 4x(y_1' + y_1^2) - 4y_1 - 16x^3 + \ldots. \qquad (28) \end{aligned}$$

The right parts of support $\mathbf{S}(f_1)$ and polygon $\Gamma(f_1)$ are shown in Figure 4.

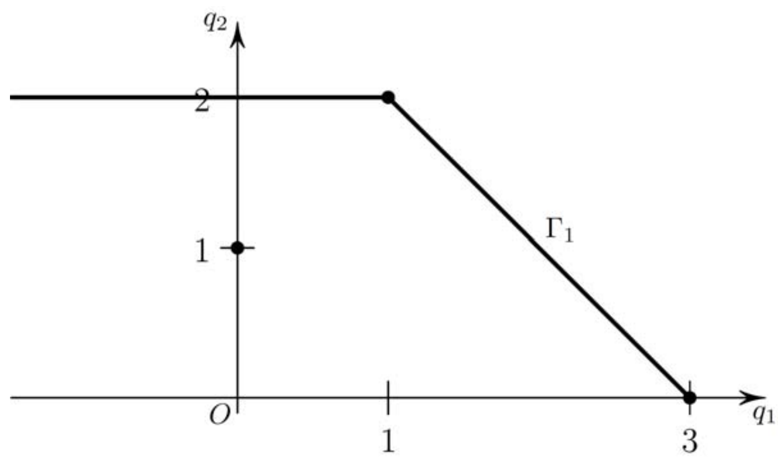

Figure 4. The right sides of support and of polygon of the sum (28).

The right edge Γ_1 of Figure 4 corresponds to the truncated equation

$$\hat{f}_{1_1} \stackrel{\text{def}}{=} 4xy_1^2 - 16x^3 = 0.$$

It has two truncated solutions

$$y_1 = \pm 2x,$$

which are continued into solutions of the full equation $f_1 = 0$ from (28) as

$$y_1 = \varphi_1(x) = \pm 2x + \frac{7}{4x} + O\left(x^{-3}\right). \tag{29}$$

Here $E_{1\pm} = \{x : \pm \operatorname{Re} x^2 < 0\}$. According to (28), the first variation

$$\frac{\delta}{\delta y_1} f_1(x, y_1) = -2\varphi \frac{d}{dx} - 4\varphi y_1 + 2\varphi'.$$

According to (27), (29) and to Theorem 2, we obtain

$$f_2(x, y_2) \stackrel{\text{def}}{=} -2\varphi y_2 - 4\varphi\varphi_1 + 2\varphi' = 0,$$

hence

$$y_2 = -2\varphi_1 + \frac{\varphi'}{\varphi} = \mp 4x \mp \frac{1}{x} + \dots.$$

Thus

$$E_{2\pm} = \{x : \mp \operatorname{Re} x^2 < 0\},\ E_{1\pm} \cap E_{2\pm} = \emptyset,$$

i.e., $\mu = 1$, and we obtain one-parameter families of expansions of solutions to Equation (20)

$$y = \varphi(x) + c_1 \exp\int \varphi_1 dx + \sum_{k=2}^{\infty} B_k(x)\left[c_1 \exp\int \varphi_1(x) dx\right]^k \tag{30}$$

in sectors of Figure 3 of the complex plane $\mathbb{C}$.

6. Constructive Proof of Theorem 3

Let $g(x, z)$ be a differential polynomial of order n and a power series in x, i.e., it is a polynomial in $z, z', \ldots, z^{(n)}$, where $z' = dz/dx$. Let us write

$$g(x, z) = \sum_j a_j(x, z),$$

where $a_j(x, z)$ are differential monomials. Let $\mathbf{S}(g)$ be the support of $g(x, z)$. Denote

$$g_i(x, z) = \sum a_j(x, z) \text{ over } Q(a_j) = (q_1, q_2) \text{ with } q_2 = i.$$

Then

$$g(x, z) = \sum_{i=m}^{M} g_i(x, z), \quad 0 \leqslant m \leqslant M < \infty. \tag{31}$$

Let us consider the expansion

$$z = \psi(x) + \sum_{k=2}^{\infty} B_k(x)\psi^k \overset{\text{def}}{=} \psi(x) + \Delta(x), \tag{32}$$

where $\psi(x) = c \exp\int \varphi_1(x)dx$, $\varphi_1(x)$ is the power series (12) and all $B_k(x)$ are power series in x. Then

$$\Delta(x) = \sum_{k=2}^{\infty} B_k(x)\psi^k, \quad \Delta^j = \sum_{k=2j}^{\infty} C_{jk}(x)\psi^k,$$

where $j > 0$, coefficients $C_{jk}(x)$ are definite sums of productions of j coefficients $B_l(x)$ and corresponding multinomial coefficients [10]. Here $C_{1k}(x) = B_k(x)$. Each term $g_i(\psi + \Delta)$ is expanded in the analogue of the Taylor series

$$g_i(\psi + \Delta) = \sum_{j=0}^{\infty} \frac{1}{j!} \left.\frac{\delta^j g_i}{\delta z^j}\right|_{z=\psi} \Delta^j,$$

where $\frac{\delta^j g_i}{\delta z^j}$ is the j-variation or the j-derivative by Frechet/Gateaux. Here $\delta^j g_i / \delta z^j \equiv 0$, if $j > M$. So after substitution of expansion (32) in the differential polynomial (31), we obtain the expansion

$$g(x, z) = \sum_{i=m}^{M} \sum_{j=0}^{M} \frac{1}{j!} \left.\frac{\delta^j g_i}{\delta z^j}\right|_{z=\psi} \sum_{k \geqslant 2j}^{\infty} C_{jk}(x)\psi^k.$$

Lemma 3. *If n is the order of* $g(x, z)$, *then*

$$\frac{\delta^j g_i(x, z)}{\delta z^j} = \sum_{l=0}^{jn} A_l(x, z) \frac{d^l}{dx^l},$$

where all $A_l(x, z)$ *are homogeneous polynomals in* $z, z', \ldots, z^{(n)}$ *of degree* $i - j \geqslant 0$.

Lemma 4.

$$\frac{d^l}{dx^l}[C_{jk}(x)\psi^k] = \psi^k \sum_{s=0}^{l} \binom{l}{s} C_{jk}^{(l-s)}(x) \left[(k\varphi_1)^s + P_{s-1}(k\varphi_1, \ldots, k\varphi_1^{(s-1)})\right],$$

where $\binom{l}{s}$ *are the binomial coefficients and* P_{s-1} *are polynomials from Lemma* 2.

According to Lemmas 3 and 4,

$$\left.\frac{\delta^j g_i}{\delta z^j}\right|_{z=\psi} \cdot C_{jk}(x)\psi^k = \psi^{i-j+k} G_{i,j,k}(x, \varphi_1, B_2, \ldots, B_{k-2(j-1)}),$$

where $G_{i,j,k}$ is a polynomial in series $\varphi_1, B_2, \ldots, B_{k-2(j-1)}$ and their derivatives.

Hence,

$$\left.\frac{\delta^j g_i}{\delta z^j}\right|_{z=\psi} \cdot C_{jk}\psi^k,$$

is a product of ψ^N and a polynomial from power series, where

$$N = i - j + k. \tag{33}$$

If the expansion (32) is a solution to equation $g(x, z) = 0$, then for the each fixed N, we obtain its own equation

$$\sum_{i=m}^{M}\sum_{j=0}^{M}\frac{1}{j!}\left.\frac{\delta^j g_i}{\delta z^j}\right|_{z=\psi}\sum_{k\geqslant 2j} C_{jk}(x)\psi^k = 0, \tag{34}$$

where the equality (33) is satisfied. Here

$$1 \leqslant m \leqslant i \leqslant M, \quad 0 \leqslant j \leqslant M, \quad k \geqslant 2j \text{ and } k = 0, \text{ if } j = 0.$$

The set of such points $(i, j, k) \in \mathbb{Z}^3$ we denote $\mathbf{M}$. Its subset, satisfied equality (33), we denote as $\mathbf{M}(N)$.

Example 3. Let us consider several cases.

(1) $N = m$. Then $i = m$, $j = k = 0$. Equation (34) is $g_m(x, \psi) = 0$.

(2) $N = m + 1$. Then Equation (33) has two solutions:

(a) $i = m$, $j = 1$, $k = 2$, and

(b) $i = m + 1$, $j = k = 0$.

Equation (34) is

$$\left.\frac{\delta g_m}{\delta z}\right|_{z=\psi} C_{12}\psi^2 + g_{m+1}(\psi) = 0. \tag{35}$$

Here $C_{12} = B_2$.

(3) $N = m + 2$. Equation (33) has 4 solutions:

$$i = m,\ j = 1,\ k = 3;$$

$$i = m,\ j = 2,\ k = 4;$$

$$i = m + 1,\ j = 1,\ k = 3;$$

$$i = m + 2,\ j = k = 0.$$

Hence the Equation (34) is

$$\frac{\delta g_m}{\delta z} C_{13}\psi^3 + \frac{1}{2}\frac{\delta^2 g_m}{\delta z^2}(C_{12}\psi^2)^2 + \frac{\delta g_{m+1}}{\delta z} C_{12}\psi^2 + g_{m+2} = 0.$$

Here all g_i and their variations are taken for $z = \psi(x)$, $C_{12} = B_2$, $C_{13} = B_3$.

□

For each N, the Equation (34) begins by term $\frac{\delta g_m}{\delta z} B_{N-m}\psi^{N-m}$ and finishes by term $g_N(\psi)$. After cancellation by ψ^N, it becomes a linear ODE for $B_{N-m}(x)$, depending from φ_1 and from B_j with $j < N - m$. Theorem 3 is proved.

7. Examples of Computation of the Coefficients $B_{ik}(x)$

Applying Theorem 3, here we will calculate initial coefficients $B_k(x)$ in expansions (26) and (30).

Example 4 (Continuation of Example 1)**.** Here $\varphi \equiv 0$, so $z = y$ and $g(x, z) = f(x, z) = -2zz'' + z'^2 + 4x^2z^2 + 8xz^3 + 3z^4$, according to (20).

In notations of Section 5 here

$$g_0(x, z) \equiv g_1(x, z) \equiv 0, \quad g_2(x, z) = -2zz'' + z'^2 + 4x^2z^2,$$

$$g_3(x, z) = 8xz^3, \quad g_4(x, z) = 3z^4. \tag{36}$$

So here $m = 2$ and the Equation (35) for the expansion (26) is

$$\left.\frac{\delta g_2}{\delta z}\right|_{z=\psi}(B_2\psi^2) + g_3(x, \psi) = 0, \tag{37}$$

where $\psi = c\exp\int \varphi_1(x)dx$. Hence

$$\psi' = \varphi_1\psi, \quad \psi'' = (\varphi_1' + \varphi_1^2)\psi. \tag{38}$$

We have

$$\frac{\delta g_2}{\delta z} = -2z\frac{d^2}{dx^2} + 2z'\frac{d}{dx} - 2z'' + 8x^2 z.$$

According to that and (38),

$$\left.\frac{\delta g_2}{\delta z}\right|_{z=\psi}(B_2\psi^2) = \left[-2\psi\frac{d^2}{dx^2} + 2\varphi_1\psi\frac{d}{dx} - 2(\varphi_1' + \varphi_1^2)\psi + 8x^2\psi\right](B_2\psi^2).$$

According to (38), we have

$$\frac{d}{dx}(B_2\psi^2) = B_2'\psi^2 + 2B_2\psi^2\varphi_1, \tag{39}$$

$$\frac{d^2}{dx^2}(B_2\psi^2) = B_2''\psi^2 + 4B_2'\psi^2\varphi_1 + 4B_2\psi^2\varphi_1^2 + 2B_2\psi^2\varphi_1'. \tag{40}$$

Hence

$$\begin{aligned}\left.\frac{\delta g_2}{\delta z}\right|_{z=\psi}(B_2\psi^2) &= \psi^3[-2(B_2'' + 4B_2'\varphi_1 + 4B_2\varphi_1^2 + 2B_2\varphi_1') + 2\varphi_1(B_2' + 2B_2\varphi_1) \\ &\qquad - 2B_2(\varphi_1' + \varphi_1^2) + 8x^2B_2] \\ &= \psi^3[-2B_2'' - 6B_2'\varphi_1 - 6B_2\varphi_1^2 - 6B_2\varphi_1' + 8x^2B_2].\end{aligned}$$

According to (36) $g_3(x, \psi) = 8x\psi^3$. After cancellation by ψ^3, the Equation (37) takes the form

$$-2B_2'' - 6B_2'\varphi_1 - 6B_2\varphi_1^2 - 6B_2\varphi_1' + 8x^2B_2 + 8x = 0. \tag{41}$$

According to (25)

$$\varphi_1(x) = \pm 2x - \frac{1}{x} \mp \frac{3}{4x^3} + O\left(\frac{1}{x^5}\right).$$

Taking in account that equality, we obtain, that the right parts of support and polygon of Equation (41) have the forms shown in Figure 5. The right inclined edge of the polygon corresponds to the truncated equation

$$-24x^2 B_2 + 8x^2 B_2 + 8x = 0.$$

Its root $B_2 = \frac{1}{2x}$ is the truncated solution of the Equation (41). Corresponding full solutions of that equation are

$$B_2(x) = \frac{1}{2x} \pm \frac{3}{4x^3} + \frac{17}{8x^5} + O\left(\frac{1}{x^7}\right).$$

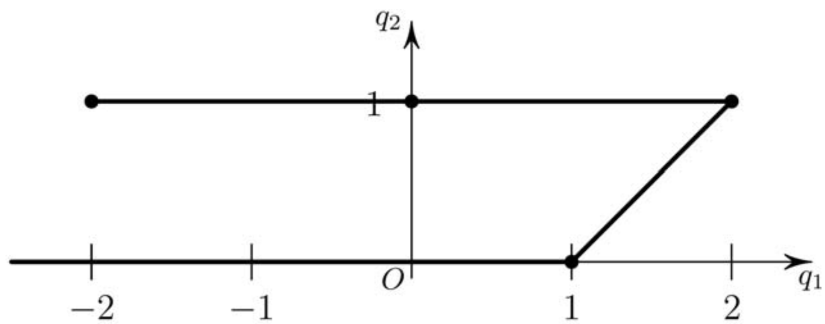

Figure 5. Right parts of support and of polygon of Equation (41).

Example 5 (Continuation of Example 2)**.** According to (27) here

$$\varphi(x) = -2x + \frac{1}{4x^3} + O\left(\frac{1}{x^7}\right).$$

In notations of Section 5

$$g(x, z) = f(x, \varphi + z) = \sum_{i=0}^{4} g_i(x, z),$$

where

$$g_0(x, z) = -2\varphi\varphi'' + \varphi'^2 + 3\varphi^4 + 8x\varphi^3 + 4x^2\varphi^2,$$

$$g_1(x, z) = -2(\varphi z'' + \varphi'' z) + 2\varphi' z' + 12\varphi^3 z + 24x\varphi^2 z + 8x^2\varphi z,$$

$$g_2(x, z) = -2zz'' + z'^2 + 18\varphi^2 z^2 + 24x\varphi z^2 + 4x^2 z^2,$$

$$g_3(x, z) = 12\varphi z^3 + 8xz^3, \quad g_4(x, z) = 3z^4.$$

As $\varphi(x)$ is a solution to Equation (20), then $g_0(x, z) \equiv 0$. As $g_1(x, z) \not\equiv 0$, then $m = 1$ and Equation (35) takes the form

$$\left.\frac{\delta g_1}{\delta z}\right|_{z=\psi}(B_2\psi^2) + g_2(z, \psi) = 0, \tag{42}$$

where $\psi = c_1 \exp\int \varphi_1 dx$, $y_1 = \varphi_1(x)$ is a solution to equation $f_1(x, y_1) = 0$ from (28) and takes the form (29). Here

$$\frac{\delta g_1}{\delta z} = -2\varphi\frac{d^2}{dx^2} + 2\varphi'\frac{d}{dx} - 2\varphi'' + 12\varphi^3 + 24x\varphi^2 + 8x\varphi.$$

According to that, (38), (39) and (40), after cancellation by ψ^2, Equation (42) takes the form

$$-2\varphi B_2'' + B_2'(-8\varphi\varphi_1 + 2\varphi') + B_2(-8\varphi\varphi_1^2 - 4\varphi\varphi_1' + 4\varphi'\varphi_1 - 2\varphi'' + 12\varphi^3$$

$$+ 24x\varphi^2 + 8x\varphi) - 2\varphi_1' - \varphi_1^2 + 18\varphi^2 + 24x\varphi + 4x^2 = 0.$$

Taking in account Equalities (27) and (29), we obtain that the right parts of support and polygon of the Equation (43) have shapes shown in Figure 6. The right inclined edge of polygon of Figure 6 corresponds to the truncated equation

$$64B_2x^3 + 24x^2 = 0.$$

Hence,

$$B_2(x) = -\frac{3}{8x} + O\left(\frac{1}{x^2}\right).$$

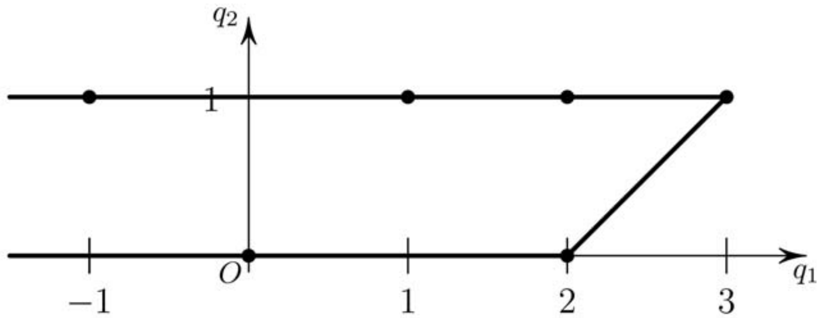

Figure 6. Right parts of support and of polygon of Equation (43).

8. Conclusion

Computation of a transseries of Theorem 4 uses 3 algorithms:

(1) Computation of the solution (1) of the Equation (2) according to [1].

(2) Computation of series $\varphi_1(x)$ in exponential addition (13) according to Section 3.

(3) Computation of series $B_{ik}(x)$ in exponential expansions according to Section 6.

Now a program for item 1 exists and programs for items 2 and 3 can be written using computer algebra.

References

[1] A. D. Bruno, Asymptotic behaviour and expansions of solutions to an ordinary differential equation, Russian Mathem. Surveys 59(3) (2004), 429-480.

DOI: https://doi.org/10.1070/RM2004v059n03ABEH000736

[2] A. D. Bruno and N. A. Kudryashov, Expansions of solutions to the equation P_1^2 by algorithms of power geometry, Ukrainian Mathematical Bulletin 6(3) (2009), 311-337.

[3] A. D. Bruno, Exponential expansions of solutions to an ordinary differential equation, Doklady Mathematics 85(2) (2012), 259-264.

DOI: https://doi.org/10.1134/S1064562412020287

[4] G. A. Edgar, Transseries for Beginners, arXiv 0801.4877v5 (2009).

http://arxiv.org/abs/0801.4877v5

[5] J.-P. Ramis, Séries Divergentes et Théories Asymptotiques, Panoramas and Synthéses, Société Mathématique de France 121 (1993).

[6] J. Cano, On the series defined by differential equations, with an extension of the Puiseux polygon construction to these equations, Analysis 13(1-2) (1993), 103-119.

DOI: https://doi.org/10.1524/anly.1993.13.12.103

[7] O. Costin, Topological construction of transseries and introduction to generalized Borel summability, Contemporary Mathematics 373 (2005), 137-177.

DOI: http://dx.doi.org/10.1090/conm/373/06918

[8] J. Thomann, Resommation des series formelles, Numerische Mathematik 58(1) (1990), 503-535.

DOI: https://doi.org/10.1007/BF01385638

[9] A. D. Bruno, Complicated and exotic expansions of solutions to the Painlevé equations, in: Formal and Analytic Solutions of Differential Equations, G. Filipuk et al. (Editors), Springer Proceedings in Mathematics & Statistics, Springer, Heidelberg 256 (2018), 103-145.

DOI: https://doi.org/10.1007/978-3-319-99148-1_7

[10] M. Hazewinkel, Editor, Multinomial Coefficient, Encyclopedia of Mathematics, 2001.

http://www.encyclopediaofmath.org/index.php?title=p/m065320

■