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STRONG INSERTION OF A CONTRA-BAIRE-1 (BAIRE-.5) FUNCTION

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Abstract

Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that F_σ-kernel of sets are F_σ-sets.

1. Introduction

A generalized class of closed sets was considered by Maki in 1986 [18]. He investigated the sets that can be represented as union of closed sets and called them V-sets. Complements of V-sets, i.e., sets that are intersection of open sets are called Λ-sets [18].

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Recall that a real-valued function f defined on a topological space X is called A-continuous [24] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of A-continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature, many topologists had focused their research in the direction of investigating different types of generalized continuity.

Dontchev in [6] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 4, 8, 9, 10, 12, 13, 22].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that F_σ-kernel of sets are F_σ-sets.

A real-valued function f defined on a topological space X is called *contra-Baire*-1 (*Baire*-.5) if the preimage of every open subset of $\mathbb{R}$ is a G_δ-set in X [25].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is a B–.5–*property* provided that any constant function has property P and provided that the sum of a function with property P and any Baire-.5 function also has property P. If P_1 and P_2 are B–.5–properties, the following terminology is used: (i) A space X has the *weak* $B-.5$*–insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g \leq h \leq f$. (ii) A space X has the *strong* B– .5*–insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space that F_σ-kernel of sets are F_σ-sets, is given a sufficient condition for the weak $B-.5$–insertion property. Also for a space with the weak $B-.5$–insertion property, we give necessary and sufficient conditions for the space to have the strong $B-.5$–insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^{Λ} and A^{V} as follows:

$$A^{\Lambda} = \bigcap\{O : O \supseteq A, O \in (X, \tau)\},$$

and

$$A^{V} = \bigcap\{F : F \subseteq A, F^c \in (X, \tau)\}.$$

In [7, 19, 21], A^{Λ} is called the *kernel* of A.

We define the subsets $G_\delta(A)$ and $F_\sigma(A)$ as follows:

$$G_\delta(A) = \bigcup\{O : O \subseteq A, O \text{ is } G_\delta - \text{set}\},$$

and

$$F_\sigma(A) = \bigcap\{F : F \supseteq A, F \text{ is } F_\sigma - \text{set}\},$$

where $F_\sigma(A)$ is called the F_σ-*kernel* of A.

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2. If ρ is a binary relation in a set S, then $\overline{\rho}$ is defined as follows: $x \overline{\rho} y$ if and only if $y \rho \nu$ implies $x \rho \nu$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

(1) If $A_i \rho B_j$ for any $i \in \{1, \dots, m\}$ and for any $j \in \{1, \dots, n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \dots, m\}$ and any $j \in \{1, \dots, n\}$.

(2) If $A \subseteq B$, then $A \overline{\rho} B$.

(3) If $A \rho B$, then $F_\sigma(A) \subseteq B$ and $A \subseteq G_\delta(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main results:

Theorem 2.1. *Let g and f be real-valued functions on the topological space X, that F_σ-kernel sets in X are F_σ-sets, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1)\ \rho\ A(g, t_2)$, then there exists a Baire-.5 function h defined on X such that $g \leq h \leq f$.*

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\, \bar{\rho}\, F(t_2)$, $G(t_1)\, \bar{\rho}\, G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [15], it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\, \rho\, H(t_2)$, $H(t_1)\, \rho\, H(t_2)$, and $H(t_1)\, \rho\, G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$, then x is in $G(t')$ for any $t' > t$; since x in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = G_\delta(H(t_2)) \setminus F_\sigma(H(t_1))$. Hence $h^{-1}(t_1, t_2)$ is a G_δ-set in X, i.e., h is a Baire-.5 function on X. □

The above proof used the technique of Theorem 1 of [14].

If a space has the strong B –.5–insertion property for (P_1, P_2), then it has the weak B –.5–insertion property for (P_1, P_2). The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak B –.5–insertion property to satisfy the strong B –.5–insertion property.

Theorem 2.2. *Let P_1 and P_2 be B –.5–property and X be a space that satisfies the weak B –.5–insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g \le f$, g has property P_1 and f has property P_2. The space X has the strong B –.5–insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f - g, 2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of X such that* (i) *for each n, F_n and $A(f - g, 2^{-n})$ are completely separated by Baire-.5 functions, and* (ii) $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.

Proof. Suppose that there is a sequence $(A(f - g, 2^{-n}))$ of lower cut sets for $f - g$ and suppose that there is a sequence (F_n) of subsets of X such that

$$\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n,$$

and such that for each n, there exists a Baire-.5 function k_n on X into $[0, 2^{-n}]$ with $k_n = 2^{-n}$ on F_n and $k_n = 0$ on $A(f - g, 2^{-n})$. The function k from X into $[0, 1/4]$ which is defined by

$$k(x) = 1/4\sum_{n=1}^{\infty} k_n(x)$$

is a Baire-.5 function by the Cauchy condition and the properties of Baire-.5 functions, (1) $k^{-1}(0) = \{x \in X : (f - g)(x) = 0\}$ and (2) if $(f - g)(x) > 0$, then $k(x) < (f - g)(x)$: In order to verify (1), observe that if $(f - g)(x) = 0$, then $x \in A(f - g, 2^{-n})$ for each n and hence $k_n(x) = 0$ for each n. Thus $k(x) = 0$. Conversely, if $(f - g)(x) > 0$, then there exists an n such that $x \in F_n$ and hence $k_n(x) = 2^{-n}$. Thus $k(x) \neq 0$ and this verifies (1). Next, in order to establish (2), note that

$$\{x \in X : (f - g)(x) = 0\} = \bigcap_{n=1}^{\infty} A(f - g, 2^{-n}),$$

and that $(A(f - g, 2^{-n}))$ is a decreasing sequence. Thus if $(f - g)(x) > 0$, then either $x \notin A(f - g, 1/2)$ or there exists a smallest n such that $x \notin A(f - g, 2^{-n})$ and $x \in A(f - g, 2^{-j})$ for $j = 1, \ldots, n - 1$. In the former case,

$$k(x) = 1/4\sum_{n=1}^{\infty} k_n(x) \leq 1/4\sum_{n=1}^{\infty} 2^{-n} < 1/2 \leq (f - g)(x),$$

and in the latter,

$$k(x) = 1/4\sum_{j=n}^{\infty} k_j(x) \leq 1/4\sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \leq (f - g)(x).$$

Thus $0 \leq k \leq f - g$ and if $(f - g)(x) > 0$ then $(f - g)(x) > k(x) > 0$. Let $g_1 = g + (1/4)k$ and $f_1 = f - (1/4)k$. Then $g \leq g_1 \leq f_1 \leq f$ and if $g(x) < f(x)$, then

$$g(x) < g_1(x) < f_1(x) < f(x).$$

Since P_1 and P_2 are B – .5–properties, then g_1 has property P_1 and f_1 has property P_2. Since by hypothesis X has the weak B – .5–insertion property for (P_1, P_2), then there exists a Baire-.5 function h such that $g_1 \leq h \leq f_1$. Thus $g \leq h \leq f$ and if $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. Therefore X has the strong B –.5–insertion property for (P_1, P_2). (The technique of this proof is by Lane [16].)

Conversely, assume that X satisfies the strong B – .5–insertion for (P_1, P_2). Let g and f be functions on X satisfying P_1 and P_2, respectively such that $g \leq f$. Thus there exists a Baire-.5 function h such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. We follow an idea contained in Powderly [23]. Now consider the functions 0 and $f - h$. 0 satisfies property P_1 and $f - h$ satisfies property P_2. Thus there exists a Baire-.5 function h_1 such that $0 \leq h_1 \leq f - h$ and if $0 < (f - h)(x)$ for any x in X, then $0 < h_1(x) < (f - h)(x)$. We next show that

$$\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}.$$

If x is such that $(f - g)(x) > 0$, then $g(x) < f(x)$. Therefore $g(x) < h(x) < f(x)$. Thus $f(x) - h(x) > 0$ or $(f - h)(x) > 0$. Hence $h_1(x) > 0$. On the other hand, if $h_1(x) > 0$, then since $(f - h) \geq h_1$ and $f - g \geq f - h$, therefore $(f - g)(x) > 0$. For each n, let

$$A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\},$$

$$F_n = \{x \in X : h_1(x) \geq 2^{-n+1}\},$$

and

$$k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}.$$

Since $\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}$, it follows that

$$\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n.$$

We next show that k_n is a Baire-.5 function which completely separates F_n and $A(f - g, 2^{-n})$, From its definition and by the properties of Baire-.5 functions, it is clear that k_n is a Baire-.5 function. Let $x \in F_n$. Then, from the definition of k_n, $k_n(x) = 2^{-n}$. If $x \in A(f - g, 2^{-n})$, then since $h_1 \leq f - h \leq f - g$, $h_1(x) \leq 2^{-n}$. Thus $k_n(x) = 0$, according to the definition of k_n. Hence k_n completely separates F_n and $A(f - g, 2^{-n})$.

□

Theorem 2.3. *Let P_1 and P_2 be $B - .5$–properties and assume that the space X satisfied the weak $B - .5$–insertion property for (P_1, P_2). The space X satisfies the strong $B - .5$–insertion property for (P_1, P_2) if and only if X satisfies the strong $B - .5$–insertion property for $(P_1, B - .5)$ and for $(B - .5, P_2)$.*

Proof. Assume that X satisfies the strong $B - .5$–insertion property for $(P_1, B - .5)$ and for $(B - .5, P_2)$. If g and f are functions on X such that $g \leq f$, g satisfies property P_1, and f satisfies property P_2, then since X satisfies the weak $B - .5$–insertion property for (P_1, P_2) there is a Baire-.5 function k such that $g \leq k \leq f$. Also, by hypothesis there exist Baire-.5 functions h_1 and h_2 such that $g \leq h_1 \leq k$ and if $g(x) < k(x)$ then $g(x) < h_1(x) < k(x)$ and such that $k \leq h_2 \leq f$ and if $k(x) < f(x)$ then $k(x) < h_2(x) < f(x)$. If a function h is defined by $h(x) = (h_2(x) + h_1(x))/2$, then h is a Baire-.5 function, $g \leq h \leq f$, and if $g(x) < f(x)$, then $g(x) < h(x) < f(x)$. Hence X satisfies the strong

$B-.5$–insertion property for (P_1, P_2). The converse is obvious since any Baire-.5 function must satisfy both properties P_1 and P_2. (The technique of this proof is by Lane [17].) □

3. Applications

Definition 3.1. A real-valued function f defined on a space X is *called contra-upper semi-Baire-*.5 (resp., *contra-lower semi-Baire-*.5) if $f^{-1}(-\infty, t)$ (resp., $f^{-1}(t, +\infty)$) is a G_δ-set for any real number t.

The abbreviations usc, lsc, cus$B$.5 and cls$B$.5 are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

Remark 1 ([14, 15])**.** A space X has the weak c-insertion property for (usc, lsc) if and only if X is normal.

Before stating the consequences of Theorems 2.1, 2.2 and 2.3 we suppose that X is a topological space that F_σ-kernel of sets are F_σ-sets.

Corollary 3.1. *For each pair of disjoint F_σ-sets F_1, F_2, there are two G_δ-sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$, and $G_1 \cap G_2 = \emptyset$ if and only if X has the weak $B-.5$–insertion property for* ($cusB-.5$, $clsB-.5$).

Proof. Let g and f be real-valued functions defined on the X, such that f is lsB_1, g is usB_1, and $g \leq f$. If a binary relation ρ is defined by $A \,\rho\, B$ in case $F_\sigma(A) \subseteq G_\delta(B)$, then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a F_σ-set and since $\{x \in X : g(x) < t_2\}$ is a G_δ-set, it follows that $F_\sigma(A(f, t_1)) \subseteq G_\delta(A(g, t_2))$. Hence $t_1 < t_2$ implies that $A(f, t_1)\,\rho\, A(g, t_2)$. The proof follows from Theorem 2.1.

On the other hand, let F_1 and F_2 are disjoint F_σ-sets. Set $f = \chi_{F_1^c}$ and $g = \chi_{F_2}$, then f is cls$B - .5$, g is cus$B - .5$, and $g \leq f$. Thus there exists Baire-.5 function h such that $g \leq h \leq f$. Set $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$, then G_1 and G_2 are disjoint G_δ-sets such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. □

Remark 2 ([26])**.** A space X has the weak c-insertion property for (lsc, usc) if and only if X is extremally disconnected.

Corollary 3.2. *For every G of G_δ-set, $F_\sigma(G)$ is a G_δ-set if and only if X has the weak $B - .5$–insertion property for* ($clsB - .5$, $cusB - .5$).

Before giving the proof of this corollary, the necessary lemma is stated.

Lemma 3.1. *The following conditions on the space X are equivalent*:

(i) *For every G of G_δ-set we have $F_\sigma(G)$ is a G_δ-set.*

(ii) *For each pair of disjoint G_δ-sets as G_1 and G_2, we have* $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$.

The proof of Lemma 3.1 is a direct consequence of the definition F_σ-kernel of sets.

We now give the proof of Corollary 3.2.

Proof. Let g and f be real-valued functions defined on the X, such that f is cls$B - .5$, g is cus$B - .5$, and $f \leq g$. If a binary relation ρ is defined by $A \rho B$ in case $F_\sigma(A) \subseteq G \subseteq F_\sigma(G) \subseteq G_\delta(B)$ for some G_δ-set g in X, then by hypothesis and Lemma 3.1 ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(g, t_1) = \{x \in X : g(x) < t_1\} \subseteq \{x \in X : f(x) \leq t_2\};$$

$$= A(f, t_2);$$

since $\{x \in X : g(x) < t_1\}$ is a G_δ-set and since $\{x \in X : f(x) \leq t_2\}$ is a F_σ-set, by hypothesis it follows that $A(g, t_1)\, \rho\, A(f, t_2)$. The proof follows from Theorem 2.1.

On the other hand, let G_1 and G_2 are disjoint G_δ-sets. Set $f = \chi_{G_2}$ and $g = \chi_{G_1^c}$, then f is cls$B - .5$, g is cus$B - .5$, and $f \leq g$.

Thus there exists Baire-.5 function h such that $f \leq h \leq g$. Set $F_1 = \{x \in X : h(x) \leq \frac{1}{3}\}$ and $F_2 = \{x \in X : h(x) \geq 2/3\}$, then F_1 and F_2 are disjoint F_σ-sets such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$. Hence $F_\sigma(F_1) \cap F_\sigma\ (F_2) = \emptyset$. □

Before stating the consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.2. *The following conditions on the space X are equivalent*:

(i) *Every two disjoint* F_σ*-sets of X can be separated by* G_δ*-sets of X.*

(ii) *If F is a* F_σ*-set of X which is contained in a* G_δ*-set* G, *then there exists a* G_δ*-set H such that* $F \subseteq H \subseteq F_\sigma(H) \subseteq G$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $F \subseteq G$, where F and G are F_σ-set and G_δ-set of X, respectively. Hence, G^c is a F_σ-set and $F \cap G^c = \emptyset$.

By (i) there exists two disjoint G_δ-sets G_1, G_2 such that $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c,$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G,$$

and since G_2^c is a F_σ-set containing G_1 we conclude that $F_\sigma(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq F_\sigma(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that F_1, F_2 are two disjoint F_σ-sets of X.

This implies that $F_1 \subseteq F_2^c$ and F_2^c is a G_δ-set. Hence by (ii), there exists a G_δ-set H such that, $F_1 \subseteq H \subseteq F_\sigma(H) \subseteq F_2^c$.

But

$$H \subseteq F_\sigma(H) \Rightarrow H \cap (F_\sigma(H))^c = \emptyset,$$

and

$$F_\sigma(H) \subseteq F_2^c \Rightarrow F_2 \subseteq (F_\sigma(H))^c.$$

Furthermore, $(F_\sigma(H))^c$ is a G_δ-set of X. Hence $F_1 \subseteq H$, $F_2 \subseteq (F_\sigma(H))^c$ and $H \cap (F_\sigma(H))^c = \emptyset$. This means that condition (i) holds. □

Lemma 3.3. *Suppose that X is the topological space such that we can separate every two disjoint F_σ-sets by G_δ-sets. If F_1 and F_2 are two disjoint F_σ-sets of X, then there exists a Baire-.5 function $h : X \to [0, 1]$ such that $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

Proof. Suppose F_1 and F_2 are two disjoint F_σ-sets of X. Since $F_1 \cap F_2 = \emptyset$, hence $F_1 \subseteq F_2^c$. In particular, since F_2^c is a G_δ-set of X containing F_1, by Lemma 3.2, there exists a G_δ-set $H_{1/2}$ such that,

$$F_1 \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq F_2^c.$$

Note that $H_{1/2}$ is a G_δ-set and contains F_1, and F_2^c is a G_δ-set and contains $F_\sigma(H_{1/2})$. Hence, by Lemma 3.2, there exists G_δ-set $H_{1/4}$ and $H_{3/4}$ such that,

$$F_1 \subseteq H_{1/4} \subseteq F_\sigma(H_{1/4}) \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq H_{3/4} \subseteq F_\sigma(H_{3/4}) \subseteq F_2^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain G_δ-sets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_2$ and $h(x) = 1$ for $x \in F_2$.

Note that for every $x \in X$, $0 \leq h(x) \leq 1$, i.e., h maps X into $[0, 1]$. Also, we note that for any $t \in D$, $F_1 \subseteq H_t$; hence $h(F_1) = \{0\}$. Furthermore, by definition, $h(F_2) = \{1\}$. It remains only to prove that h is a Baire-.5 function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \bigcup\{H_t : t < \alpha\}$, hence, they are G_δ-sets of X. Similarly, if $\alpha < 0$, then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$, then $\{x \in X : h(x) > \alpha\} = \bigcup\{(F_\sigma(H_t))^c : t > \alpha\}$ hence, every of them is a G_δ-set. Consequently h is a Baire-.5 function. □

Lemma 3.4. *Suppose that X is the topological space such that we can separate every two disjoint F_σ-sets by G_δ-sets. If F_1 and F_2 are two disjoint F_σ-sets of X and F_1 is a countable intersection of G_δ-sets, then there exists a Baire-.5 function h on X into $[0, 1]$ such that $h^{-1}(0) = F_1$ and $h(F_2) = \{1\}$.*

Proof. Suppose that $F_1 = \bigcap_{n=1}^{\infty} G_n$, where G_n is a G_δ-set of X. We can suppose that $G_n \cap F_2 = \emptyset$, otherwise we can substitute G_n by $G_n \setminus F_2$. By Lemma 3.3, for every $n \in \mathbb{N}$, there exists a Baire-.5 function h_n on X into $[0, 1]$ such that $h_n(F_1) = \{0\}$ and $h_n(X \setminus G_n) = \{1\}$. We set $h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$.

Since the above series is uniformly convergent, it follows that h is a Baire-.5 function from X to $[0, 1]$. Since for every $n \in \mathbb{N}$, $F_2 \subseteq X \setminus G_n$, therefore $h_n(F_2) = \{1\}$ and consequently $h(F_2) = \{1\}$. Since $h_n(F_1) = \{0\}$, hence $h(F_1) = \{0\}$. It suffices to show that if $x \notin F_1$, then $h(x) \neq 0$.

Now if $x \notin F_1$, since $F_1 = \bigcap_{n=1}^{\infty} G_n$, therefore there exists $n_0 \in \mathbb{N}$ such that $x \notin G_{n_0}$, hence $h_{n_0}(x) = 1$, i.e., $h(x) > 0$. Therefore $h^{-1}(0) = F_1$.

□

Lemma 3.5. *Suppose that X is the topological space such that we can separate every two disjoint F_σ-sets by G_δ-sets. The following conditions are equivalent*:

(i) *For every two disjoint F_σ-sets F_1 and F_2, there exists a Baire-.5 function h on X into* $[0, 1]$ *such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.*

(ii) *Every F_σ-set is a countable intersection of G_δ-set.*

(iii) *Every G_δ-set is a countable union of F_σ-set.*

Proof. (i) $\Rightarrow$ (ii). Suppose that F is a F_σ-sets. Since $\emptyset$ is a F_σ-set, by (i) there exists a Baire-.5 function h on X into $[0, 1]$ such that $h^{-1}(0) = F$. Set $G_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, G_n is a G_δ-set and $\bigcap_{n=1}^{\infty} G_n = \{x \in X : h(x) = 0\} = F$.

(ii) $\Rightarrow$ (i). Suppose that F_1 and F_2 are two disjoint F_σ-sets. By Lemma 3.4, there exists a Baire-.5 function f on X into $[0, 1]$ such that $f^{-1}(0) = F_1$ and $f(F_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two F_σ-sets and $(G \cup F) \cap F_2 = \emptyset$. By Lemma 3.4, there exists a Baire-.5 function g on X into $[\frac{1}{2}, 1]$ such that $g^{-1}(1) = F_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define h by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. h is well-defined and a Baire-.5 function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence h defined on X and maps to $[0, 1]$. Also, we have $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

(ii) $\Leftrightarrow$ (iii) By De Morgan law and noting that the complement of every F_σ-set is a G_δ-set and complement of every G_δ-set is a F_σ-set, the equivalence is hold. $\square$

Remark 3 ([20]). A space X has the strong c-insertion property for (usc, lsc) if and only if X is perfectly normal.

Corollary 3.3. *For every two disjoint F_σ-sets F_1 and F_2, there exists a Baire-.5 function h on X into $[0, 1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$ if and only if X has the strong B –.5–insertion property for* ($cusB$ –.5, $clsB$ – .5).

Proof. Since for every two disjoint F_σ-sets F_1 and F_2, there exists a Baire-.5 function h on X into $[0, 1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$, define $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then

G_1 and G_2 are two disjoint G_δ-sets that contain F_1 and F_2, respectively. This means that, we can separate every two disjoint F_σ-sets by G_δ-sets. Hence by Corollary 3.1, X has the weak $B - .5$–insertion property for (cus$B - .5$, cls$B - .5$). Now, assume that g and f are functions on X such that $g \leq f$, g is cus$B - .5$ and f is cls$B - .5$. Since $f - g$ is cls$B - .5$, therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$ is a F_σ-set. By Lemma 3.5, we can choose a sequence $\{F_n\}$ of F_σ-sets such that $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$ and for every $n \in \mathbb{N}$, F_n and $A(f - g, 2^{-n})$ are disjoint. By Lemma 3.3, F_n and $A(f - g, 2^{-n})$ can be completely separated by Baire-.5 functions. Hence by Theorem 2.2, X has the strong $B - .5$–insertion property for (cus$B - .5$, cls$B - 5$).

On the other hand, suppose that F_1 and F_2 are two disjoint F_σ-sets. Since $F_1 \cap F_2 = \emptyset$, hence $F_2 \subseteq F_1^c$. Set $g = \chi_{F_2}$ and $f = \chi_{F_1^c}$. Then f is cls$B - .5$ and g is cus$B - .5$ and furthermore $g \leq f$. By hypothesis, there exists a Baire-.5 function h on X such that $g \leq h \leq f$ and whenever $g(x) < f(x)$ we have $g(x) < h(x) < f(x)$. By definitions of f and g, we have $h^{-1}(1) = F_2 \cap F_1^c = F_2$ and $h^{-1}(0) = F_1 \cap F_2^c = F_1$. □

Remark 4 ([2])**.** A space X has the strong c-insertion property for (lsc, usc) if and only if each open subset of X is closed.

Corollary 3.4. *Every G_δ-set is a F_σ-set if and only if X has the strong $B - .5$–insertion property for* (*cls*$B - .5$, *cus*$B - .5$).

Proof. By hypothesis, for every G of G_δ-set, we have $F_\sigma(G) = G$ is a G_δ-set. Hence by Corollary 3.2, X has the weak $B - .5$–insertion property for (cls$B - .5$, cus$B - .5$). Now, assume that g and f are functions on X such that $g \leq f$, g is cls$B - .5$ and f is $B - .5$. Set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) < 2^{-n}\}$. Then, since $f - g$ is cus$B - .5$, we can say that $A(f - g, 2^{-n})$ is a G_δ-set. By hypothesis, $A(f - g, 2^{-n})$ is a F_σ-set. Set $G_n = X \setminus A(f - g, 2^{-n})$. Then G_n is a G_δ-set. This means that G_n and $A(f - g, 2^{-n})$ are disjoint G_δ-sets and also are two disjoint F_σ-set. Therefore G_n and $A(f - g, 2^{-n})$ can be completely separated by Baire-.5 functions. Now, we have $\bigcup_{n=1}^{\infty} G_n = \{x \in X : (f - g)(x) > 0\}$. By Theorem 2.2, X has the strong B – .5–insertion property for (cls$B$ – .5, B –.5). By an analogous argument, we can prove that X has the strong B – .5–insertion property for (B – .5, cus$B$ – .5). Hence, by Theorem 2.3, X has the strong B – .5–insertion property for (cls$B$ – .5, cus$B$ – .5).

On the other hand, suppose that X has the strong B – .5–insertion property for (cls$B$ – .5, cus$B$ – .5). Also, suppose that G is a G_δ-set. Set $f = 1$ and $g = \chi_G$. Then f is cus$B$ – .5, g is cls$B$ – .5 and $g \leq f$. By hypothesis, there exists a Baire-.5 function h on X such that $g \leq h \leq f$ and whenever $g(x) < f(x)$, we have $g(x) < h(x) < f(x)$. It is clear that $h(G) = \{1\}$ and for $x \in X \setminus G$ we have $0 < h(x) < 1$. Since h is a Baire-.5 function, therefore $\{x \in X : h(x) \geq 1\} = G$ is a F_σ-set, i.e., G is a F_σ-set. □

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