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RESIDUAL-BASED A POSTERIORI ERROR ESTIMATES FOR A CONFORMING FINITE ELEMENT DISCRETIZATION OF THE STOKES-DARCY COUPLED PROBLEM

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Abstract

We consider in this paper, a new a posteriori residual type error estimators for the Stokes-Darcy coupled problem analyzed in [1] on isotropic meshes. Our analysis covers two-and three-dimensional domains, conforming discretizations as well as different elements. We derive a reliable and efficient residual-based a posteriori error estimator for this coupled problem. The proof of reliability makes use of suitable auxiliary problems, continuous inf-sup conditions satisfied by the bilinear forms involved, and local approximation properties. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution. It is proven that the a posteriori error estimate provided in this paper is both reliable and efficient.

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1. Introduction

The coupling of Stokes and Darcy flow problems has received significant attention over the past several years due to its importance in modelling problems such as surface fluid flow coupled with flow in porous media, see [6]-[9] and the references therein. Mathematical justification for the interface boundary condition was derived in Jäger and Mikelic [7] and Mardal et al. [8] for the robust finite element constructions. Well-posedness and convergence of the finite element method can be found in [1]. A posteriori error estimates are computable quantities in terms of the discrete solution of data that measure the actual discrete errors without the knowledge of exact solutions. They are essential for designing algorithms with adaptive mesh refinement which equidistribute the computational effort and optimize the approximation efficiency. It ensures a higher density of nodes in a certain area of the given domain, where the solution is more difficult to be approximated, using an a posteriori error indicator. Ever since the pioneering work of Bieterman and Babuska [10], the adaptive finite element method based on a posteriori error estimates has been extensively investigated. In [4], two a posteriori error estimators for the mini-element discretization of the Stokes equations were presented. Recently, a residual-based a posteriori error estimator for the Stokes-Darcy coupled problem was presented in [11], where Bernardi-Raugel and Raviart-Thomas elements for the velocity and piecewise constants for the pressures were considered. A posteriori error estimates for the finite element approximation of the distributed optimal control problems governed by the Stokes equations was derived in [12, 35, 36].

The purpose of this work is to derive a reliable and efficient residual-based a posteriori error estimator for the Stokes-Darcy coupled problem analyzed in [1]. Though one might think a priori that this should follow simply by combining the corresponding approaches already available for the Stokes and Darcy problems, the analysis below will show that this idea works only partially since further difficulties and several technical issues arise along the way. In this respect, it is important to remark that, on one hand, the transmission conditions stop us from splitting the analysis into the Stokes and Darcy parts, and, on the other hand, these conditions cannot be neglected since they also have to be incorporated into the resulting a posteriori error estimate.

The remainder of the paper is organized as follows: in Section 2, we recall from [1] the Stokes-Darcy coupled problem and its continuous and discrete mixed variational formulations. The kernel of the present work is given by Section 3, where we develop the a posteriori error analysis. We employ auxiliary problem, suitable continuous inf-sup conditions, and local approximation properties for to derive a reliable residual-based a posteriori error estimator (Theorem 3.1).

Next, we apply inverse inequalities, triangular inequality, Cauchy-Schwartz inequality, and the localization technique based on simplex-bubble and face-bubble functions to show the efficiency of the error estimator (Theorem 3.2).

In a forthcoming paper, we present the results of numerical tests with the finite element methods. Throughout the rest of the paper, we utilize the standard terminology for Sobolev spaces. In particular, if S is an open set, its closure, or a Lipschitz continuous curve, and $r \in \mathbb{R}$, then $|\cdot|_{r,S}$ and $\|\cdot\|_{r,S}$ stand for the seminorm and norm in the Sobolev spaces $H^r(S)$, $[H^r(S)]^d$, and $[H^r(S)]^{d\times d}$. Hereafter, given any normed space U, U^d and $U^{d\times d}$ denote, respectively, the space of vectors and square matrices of order d with entries in U. Also, we employ $\mathbf{0}$ as a generic null vector.

Finally, let $\mathbb{P}_k$ be the space of polynomial of total degree not larger than k. In order to avoid excessive use of constants, the abbreviations $x \lesssim y$ and $x \sim y$ stand for $x \le cy$ and $c_1 x \le y \le c_2 x$, respectively, with positive constants independent of x, y or $\mathcal{T}^h$.

2. The Stokes-Darcy Coupled Problem

2.1. The model problem

The model we consider consists of Stokes flow in the fluid region $\Omega_1 \subset \mathbb{R}^d$ and Darcy's law in the porous medium domain $\Omega_2 \subset \mathbb{R}^d$ (where $d = 2, 3$). These are separated by an interface Γ_I. Here $\Omega_j \subset \mathbb{R}^d$, $(j = 1, 2)$ are bounded domains with outward unit normal vectors $\mathbf{n}_j$, $j = 1, 2$. Let $\Gamma_j := \partial\Omega_j \setminus \Gamma_I$. Each interface and boundary is assumed to be polygonal.

The fluid velocities and pressures in Ω_1 and Ω_2 are denoted by:

$$\mathbf{u}_j : \Omega_j \to \mathbb{R}^d, \text{ fluid velocity in } \Omega_j,$$

$$p_j : \Omega_j \to \mathbb{R}, \text{ fluid pressure in } \Omega_j.$$

It is important to keep in mind that the velocities and pressures play different mathematical (and physical) roles in the fluid region and in the porous medium.

Recall that the deformation rate tensor $\mathbf{D}$ and stress tensor Φ associated with $(\mathbf{u}_1, p_1)$ are defined by:

$$\mathbf{D}(\mathbf{u}_1) := \frac{1}{2}\left(\frac{\partial \mathbf{u}_{1i}}{\partial x_j} + \frac{\partial \mathbf{u}_{1j}}{\partial x_i}\right)_{i, j \in \{1, \dots, d\}} \text{ in } \Omega_1, \tag{1}$$

$$\Phi(\mathbf{u}_1, p_1) := -p_1\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}_1) \text{ in } \Omega_1, \tag{2}$$

where μ is the viscosity of fluid. Assuming Stokes flow, $(\mathbf{u}_1, p_1)$ satisfies on Ω_1 :

$$\begin{cases} -\nabla \cdot \Phi(\mathbf{u}_1, p_1) = \mathbf{f}_1 & \text{in} \quad \Omega_1 \quad \text{(conservation of momentum)}, \\ \nabla \cdot \mathbf{u}_1 = \mathbf{0} & \text{in} \quad \Omega_1 \quad \text{(conservation of mass)}, \\ \mathbf{u}_1 = \mathbf{0} & \text{on} \quad \Gamma_1 \quad \text{(no slip)}, \end{cases} \tag{3}$$

where $\mathbf{f}_1$ is a data which belongs to the space $[L^2(\Omega_1)]^d$.

Assuming Darcy's law and no flow through Γ_2, $(\mathbf{u}_2, p_2)$ satisfies on Ω_2 :

$$\begin{cases} \mathbf{u}_2 & = & -\mathbf{K}\nabla p_2 & \text{in} & \Omega_2 & \text{(Darcy's law)}, \\ \nabla \cdot \mathbf{u}_2 & = & f_2 & \text{in} & \Omega_2 & \text{(conservation of mass)}, \\ \mathbf{u}_2 \cdot \mathbf{n}_2 & = & 0 & \text{on} & \Gamma_2 & \text{(no flow)}, \end{cases} \tag{4}$$

where $\mathbf{K}$ is a symmetric and uniformly positive definite tensor representing the rock permeability divided by the fluid viscosity. The source f_2 is assumed to satisfy the solvability condition:

$$\int_{\Omega_2} f_2 dx = 0, \tag{5}$$

which makes physical sense due to the no-flow boundary condition on $\partial\Omega_2$ and to (6) below. The mixed formulation (4) is the most natural one for computations in the porous medium region since it leads to direct approximation of the velocity.

Interface conditions

The problem (3)-(4) must be coupled across Γ_I by the correct interface conditions. Mass conservation across Γ_I is expressed by:

$$\mathbf{u}_1 \cdot \mathbf{n}_1 + \mathbf{u}_2 \cdot \mathbf{n}_2 = 0 \text{ on } \Gamma_I. \tag{6}$$

The second interface condition is balance of normal forces across Γ_I :

$$p_1 - 2\mu \mathbf{n}_1 \cdot \mathbf{D}(\mathbf{u}_1) \cdot \mathbf{n}_1 = p_2 \text{ on } \Gamma_I. \tag{7}$$

The back interface condition is now know as the Beavers-Joseph-Saffman law whose that:

$$2\mu \mathbf{n}_1 \cdot \mathbf{D}(\mathbf{u}_1) \cdot \tau_j = -\frac{\mu}{k_j} \mathbf{u}_1 \cdot \tau_j \text{ on } \Gamma_I, \text{ where } j = 1, \dots, d-1. \tag{8}$$

The τ_j are the tangentials vectors on Γ_I, $\kappa_j := \tau_h \cdot \mu\mathbf{K} \cdot \tau_j > 0$ is the friction constant, and the Beavers-Joseph-Saffman law that the slip velocity along Γ_I is proportional to the shear stress along Γ_I (assuming also, based on experimental evidences, that $\mathbf{u}_2 \cdot \tau_j$ is negligible).

Remark 2.1. We remark that on Γ_I :

$$\boldsymbol{n}_1 \cdot \Phi(\mathbf{u}_1, p_1) \cdot \boldsymbol{n}_1 = -p_1 + 2\mu \boldsymbol{n}_1 \cdot \mathbf{D}(\mathbf{u}_1) \cdot \boldsymbol{n}_1, \text{ and} \tag{9}$$

$$\tau_j \cdot \Phi(\mathbf{u}_1, p_1) \cdot \boldsymbol{n}_1 = -\frac{\mu}{\kappa_j} \mathbf{u}_1 \cdot \tau_j. \tag{10}$$

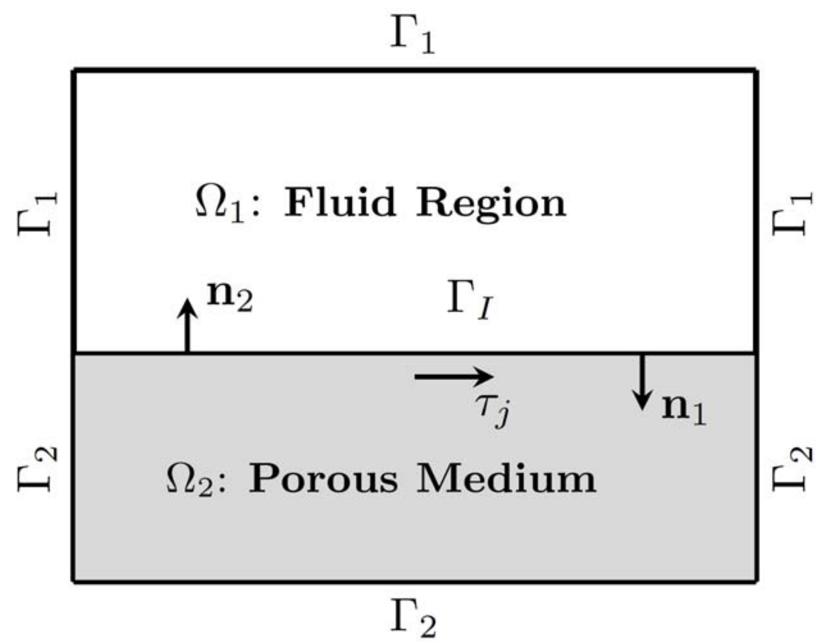

Figure 1. A sketch of the geometry of the problem (case: $\partial\Omega_d \neq \Gamma_I$).

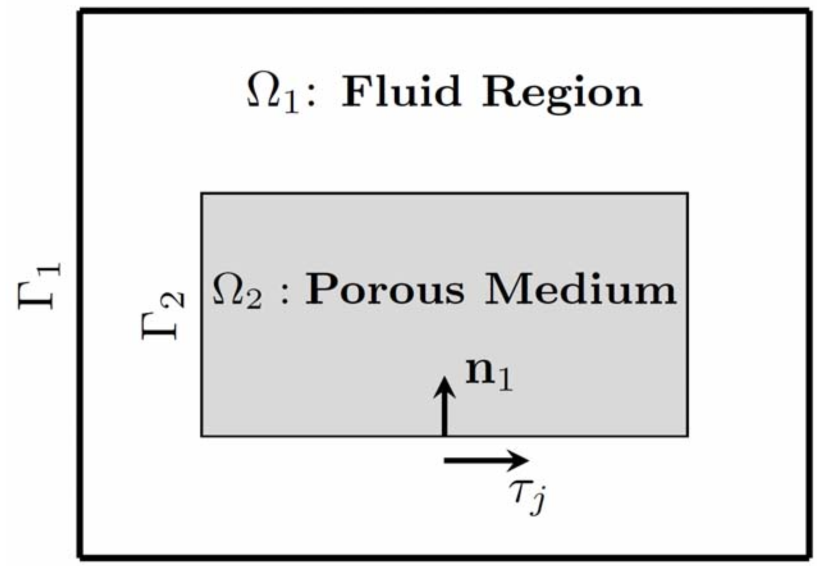

Figure 2. A sketch of the geometry of the problem (case: $\partial\Omega_d = \Gamma_I$).

2.2. Weak formulation of the coupled problem

In order to introduce the weak formulation of coupled problem, we define the spaces

$$\mathbf{H} := \mathbf{H}_1 \times \mathbf{H}_2, \tag{11}$$

$$M := L_0^2(\Omega_1) \times L_0^2(\Omega_2), \tag{12}$$

$$\Lambda := H^{1/2}(\Gamma_I), \tag{13}$$

where

$$\mathbf{H}_1 := \{\mathbf{v}_1 \in [H^1(\Omega_1)]^d : \mathbf{v}_1 = 0 \text{ on } \Gamma_1\},$$

$$\mathbf{H}_2 := \{\mathbf{v}_2 \in H(\text{div}, \Omega_2) : \mathbf{v}_2 \cdot \mathbf{n}_2 = 0 \text{ on } \Gamma_2\}.$$

The space $\mathbf{H}$ is equipped the product norm: $\|\mathbf{v}\|_{\mathbf{H}} := \|\mathbf{v}_1\|_{1,\Omega_1} + \|\mathbf{v}_2\|_{H(\text{div},\Omega_2)}$, for all $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}$.

For a connected open subset of the boundary $\Gamma \subset \partial\Omega_1 \cup \partial\Omega_2$, we write $\langle .,.\rangle_\Gamma$ for the $L^2(\Gamma)$ inner product (or duality pairing), that is, for scalar valued functions $\lambda, \eta \in L^2(\Gamma)$, one defines

$$\langle \lambda, \eta\rangle_\Gamma := \int_\Gamma \lambda(s)\eta(s)ds. \tag{14}$$

Also, we denote the global unknowns $\mathbf{u} := (\mathbf{u}_1, \mathbf{u}_2)$, $p := (p_1, p_2)$ and introduce the Lagrange multiplier $\lambda := p_2$ on Γ_I. Hence, we proceeding in the usual way (see [1], for example), we find that the mixed variational formulation of coupled problem reads as follows: Find $(\mathbf{u}, p, \lambda) \in \mathbf{H} \times M \times \Lambda$ such that

$$\begin{cases} \mathbf{a}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, p) + b_I(\mathbf{v}, \lambda) & = & l(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{H}, \\ \mathbf{b}(\mathbf{u}, q) & = & g(q), & \forall q \in M, \\ b_I(\mathbf{u}, \mu) & = & 0, & \forall \mu \in \Lambda, \end{cases} \tag{15}$$

where

$$b_I(\mathbf{v}, \lambda) :=< \mathbf{v}_1 \cdot \mathbf{n}_1 + \mathbf{v}_2 \cdot \mathbf{n}_2, \lambda >_{\Gamma_I} : \mathbf{H} \times \Lambda \to \mathbb{R};$$

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) := \sum_{i=1}^{2} a_i(\mathbf{u}_i, \mathbf{v}_i) : \mathbf{H} \times \mathbf{H} \to \mathbb{R};$$

$$\mathbf{b}(\mathbf{v}, q) := \sum_{i=1}^{2} b_i(\mathbf{v}_i, q_i) : \mathbf{H} \times M \to \mathbb{R};$$

$$l(\mathbf{v}) := (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1}, \quad g(q) := -(f_2, q_2)_{\Omega_2},$$

with

$$a_1(\mathbf{u}_1, \mathbf{v}_1) := 2\mu \int_{\Omega_1} \mathbf{D}(\mathbf{u}_1) : \mathbf{D}(\mathbf{v}_1) + \sum_{j=1}^{d-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} \int_{\Gamma_I} (\mathbf{u}_1, \tau_j)(\mathbf{v}_1, \tau_j),$$

$$a_2(\mathbf{u}_2, \mathbf{v}_2) := \int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}_2 \cdot \mathbf{v}_2,$$

$$b_1(\mathbf{v}_1, q_1) := -\int_{\Omega_1} q_1 \nabla \cdot \mathbf{v}_1, \qquad b_2(\mathbf{v}_2, q_2) := -\int_{\Omega_2} q_2 \nabla \cdot \mathbf{v}_2.$$

Here, we use the standard notation for the contraction of two matrices A and B, i.e.,

$$A : B := \sum_{i,j=1}^{d} A_{ij} B_{ij}.$$

The reason for keeping $H^{1/2}(\Gamma_I)$ as the right space for the Lagrange multiplier λ which differs from choice of $H_{00}^{1/2}(\Gamma_I) := [L^2(\Gamma_I), H_0^1(\Gamma_I)]_{1/2}$ adopted in [1], is that λ represents the trace of the porous pressure on Γ_I, and hence there is no physical reason to assume that λ vanishes in Γ_2 belong to $H^{1/2}(\partial\Omega_2)$. The present choice of $H^{1/2}(\Gamma_I)$ is also justified in Subsection 4.1 of [22]. We now recall that, given $\mathbf{v}_2 \in \mathbf{H}_2$, the boundary condition $\mathbf{v}_2 \cdot \mathbf{n}_2 = 0$ on Γ_I means:

$$\langle \mathbf{v}_2 \cdot \mathbf{n}_2, E_{00}(\xi) \rangle_{\partial\Omega_2} = 0, \quad \forall \xi \in H_{00}^{1/2}(\Gamma_I), \tag{16}$$

where $E_{00}(\xi)$ denotes the extension by zero in Γ_I of each $\xi \in H^{1/2}(\Gamma_2)$, and $\langle ., . \rangle_{\partial\Omega_2}$ stands for the duality pairing of $H^{-1/2}(\partial\Omega_2)$ and $H^{1/2}(\partial\Omega_2)$ with respect to the $L^2(\partial\Omega_2)$-inner product.

As a consequence, it is not difficult to show (see ([22], Section 2)) that the restriction of $\mathbf{v}_2 \cdot \mathbf{n}_2$ to Γ_I can be identified with an element of $H^{-1/2}(\Gamma_I)$:

$$\langle \mathbf{v}_2 \cdot \mathbf{n}_2, \xi \rangle_{\Gamma_I} := \langle \mathbf{v}_2 \cdot \mathbf{n}_2, E(\xi) \rangle_{\partial\Omega_2}, \quad \forall \xi \in H^{1/2}(\Gamma_I), \tag{17}$$

where $E : H^{1/2}(\Gamma_I) \to H^{1/2}(\partial\Omega_2)$ is the bounded linear operator defined by $E(\xi) := \gamma(z)$ for each $z \in H^{1/2}(\Gamma_I)$, $\gamma : H^1(\Omega_2) \to H^{1/2}(\partial\Omega_2)$ is the usual trace operator, and $z \in H^1(\Omega_2)$ is the unique solution of

$$\Delta z = 0 \text{ in } \Omega_2;\ z = \xi \text{ on } \Gamma_I \text{ and } \nabla z \cdot \mathbf{n}_2 = 0 \text{ on } \Gamma_2. \tag{18}$$

Moreover, thanks to (16) and (17), we may also write $\langle \mathbf{v}_2 \cdot \mathbf{n}_2, \xi \rangle_{\Gamma_I} := \langle \mathbf{v}_2 \cdot \mathbf{n}_2, \overline{\xi} \rangle_{\partial\Omega_2}$, with $\overline{\xi} \in H^{1/2}(\partial\Omega_2)$ such that $\overline{\xi} = \xi$ on Γ_I.

In fact, one can prove the following result ([1], Theorem 3.1 and Lemma 3.4).

Theorem 2.1. *There exists a unique solution* $(\mathbf{u}, p, \lambda)$ *to the problem* (15).

2.3. Finite element discretization

This under section considers the finite element discretization of the coupled problem. We let $(\mathcal{T}_j^h)_{h>0}(j = 1, 2)$ be members of shape-regular families of triangulations, that is, satisfying the minimum angle condition, of $\overline{\Omega}_1$ and $\overline{\Omega}_2$, respectively, by simplex T of diameter h_T (that is T = triangle if $d = 2$ and T = tetrahedral if $d = 3$). Next, we assume that the vertices of $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ coincide on the interface Γ_I. We define $(\mathcal{T}^h)_{h>0}$ a family regular of triangulation on $\overline{\Omega} := \overline{\Omega}_1 \cup \Gamma_I \cup \overline{\Omega}_2$, by $\mathcal{T}^h := \mathcal{T}_1^h \cup \mathcal{T}_2^h$, where $h := \max\{h_i;\, i = 1, 2\}$ which $h_i := \max\{h_T, T \in \mathcal{T}_i^h\}$ for each $i \in \{1, 2\}$. We use the notation

$$\varepsilon_h(T) := \text{ the set of all faces of the elements } K,$$

$$\varepsilon_h(\Gamma_I) := \text{ the set of all element faces } E \text{ with } E \subset \Gamma_I.$$

We now consider $\mathbf{H}_1^h, \mathbf{H}_2^h, M_1^h, M_2^h$ and Λ^h be finite dimensional subspaces of $\mathbf{H}_1, \mathbf{H}_2, L_0^2(\Omega_1), L_0^2(\Omega_2)$, and Λ, respectively. Then, we denote the products spaces as follow $\mathbf{H}^h := \mathbf{H}_1^h \times \mathbf{H}_2^h$ and $M^h = M_1^h \times M_2^h$.

In this way, the Galerkin schemes of (15) is given by: Find $(\mathbf{u}^h, p^h, \lambda^h) \in \mathbf{H}^h \times M^h \times \Lambda^h$ such that

$$\begin{cases} \mathbf{a}(\mathbf{u}^h, \mathbf{v}^h) + \mathbf{b}(\mathbf{v}^h, p^h) + b_I(\mathbf{v}^h, \lambda^h) & = & l(\mathbf{v}^h), & \forall \mathbf{v}^h \in \mathbf{H}^h, \\ \mathbf{b}(\mathbf{u}^h, q^h) & = & g(q^h), & \forall q^h \in M^h, \\ b_I(\mathbf{u}^h, \xi^h) & = & 0, & \forall \xi^h \in \Lambda^h. \end{cases} \tag{19}$$

Throughout the rest of the subsection, we assume the following hypotheses on the subspaces:

(**G**.1): For the discretization of the fluid's variables we choose finite element spaces $\mathbf{H}_1^h$, M_1^h which are assumed to be div-stable (also called LBB-stable) i.e., there exists $\beta_1 > 0$ independent of the h, such that for each $q_1^h \in M_1^h$ there holds

$$\sup_{\mathbf{v}_1^h \in \mathbf{H}_1^h} \frac{b_1(\mathbf{v}_1^h, q_1^h)}{\|\mathbf{v}_1^h\|_{1, \Omega_1}} \geqslant \beta_1 \|q_1^h\|_{0, \Omega_1}, \tag{20}$$

and to satisfy a discrete Korn inequality: there exists $\alpha_1 > 0$ independent of the h, such that,

$$\int_{\Omega_1} \mathbf{D}(\mathbf{v}_1^h) : \mathbf{D}(\mathbf{v}_1^h) \geqslant \alpha_1 |\mathbf{v}_1^h|_{1, \Omega_1}. \tag{21}$$

In addition, the space of constant functions on Ω_1 is contained in M_1^h.

(**G**.2): For the discretization of the porous medium problem in Ω_2, we choose finite element spaces $\mathbf{H}_2^h$, M_2^h which are assumed to be stable, that is, there exists $\beta_2 > 0$, independent of h, such that for each $q_2^h \in M_2^h$ there holds

$$\sup_{\mathbf{v}_2^h \in \mathbf{H}_2^h} \frac{b_2(\mathbf{v}_2^h, q_2^h)}{\|\mathbf{v}_2^h\|_{H(\mathrm{div}, \Omega_2)}} \geqslant \beta_2 \|q_2^h\|_{2, \Omega_2}. \tag{22}$$

(**G.**3): With (**G.**1) and (**G.**2), we assume that the space Λ^h satisfies the inf-sup condition, that is, there exists $\beta_4 > 0$ such that for each $\xi \in \Lambda^h$ there holds

$$\sup_{\mathbf{v} \in \mathbf{H}^h} \frac{b_I(\mathbf{v}, \xi)}{\|\mathbf{v}\|_{\mathbf{H}}} \geqslant \beta_4 \|\xi\|_{1/2, \Gamma_I}. \tag{23}$$

(**G.**4): Finally, we assume that, there exists an operator $\mathbf{I}^h = (\mathbf{I}_1^h, \mathbf{I}_2^h)$, with $\mathbf{I}_i^h : \mathbf{H}_i \to [H^1(\Omega_i)]^d$ such that

$$\mathbf{I}_i^h(\mathbf{H}_i) \subset \mathbf{H}_i^h, \quad i = 1, 2, \tag{24}$$

satisfying the local approximation properties:

$$\|\mathbf{v}_i - \mathbf{I}_i^h(\mathbf{v}_i)\|_{0, T} \leq C_1 h_T \|\mathbf{v}_i\|_{0, \Delta(T)}, \quad \forall T \in \mathcal{T}_i^h, \, i = 1, 2, \tag{25}$$

$$\|\mathbf{v}_1 - \mathbf{I}_1^h(\mathbf{v}_1)\|_{0, E} \leq C_2 h_E^{1/2} \|\mathbf{v}_1\|_{0, \Delta(E)}, \quad \forall E \in \varepsilon_h, \tag{26}$$

$$\|\mathbf{v}_2 \cdot \mathbf{n}_2 - \mathbf{I}_2^h(\mathbf{v}_2) \cdot \mathbf{n}_2\|_{0, E} \leq C_3 h_E^{1/2} \|\mathbf{v}_2\|_{1, \Delta(E)}, \forall E \in \mathcal{T}_2^h, \quad \forall \mathbf{v}_2 \in [H^1(\Omega_2)]^d, \tag{27}$$

where $\Delta(T) := \bigcup\{T' \in \mathcal{T}_i^h : T' \cap T \neq \emptyset\}$ and $\Delta(E) := \bigcup\{T' \in \mathcal{T}_i^h : T' \cap E \neq \emptyset\}$.

Theorem 2.2. *Assume that the hypotheses* (**G.**1), (**G.**2) *and* (**G.**3) *hold. Then the Galerkin scheme* (19) *has a unique solution* $(\mathbf{u}^h, p^h, \lambda^h) \in \mathbf{H}^h \times M^h \times \Lambda^h$.

Proof. Cf. [1, 22]. □

2.4. Examples of subspaces satisfying the hypotheses

There is a large variety of stable Stokes elements available in the literature: The Table 1 below provides a list of stable elements covered by our analysis. The first line gives alternative references where some equivalences between the error and the residual error estimator have been proved (over kinds of estimators are omitted). In Table 1, the space $\mathbf{S}_T$ is Stokes local space, $\mathbf{D}_T$ is Darcy local space, and $\mathbf{L}_T$ is Lagrange multiplier space. BDM is Brezzi-Douglas-Marini element, BDFM is Brezzi-Douglas-Fortin-Marini element, BDDF is Brezzi Douglas-Duran-Fortin element and CD is Chen-Douglas element.

Table 1. Stable isotropic elements covered

Spaces \ Examples	Example 1	Example 2	Example 3	Example 4
References	[3]	[25, 30, 31]	[33]	[21]
$\mathbf{S}_T$	Mini-Element (ABF): $[\mathbb{P}^1 - bulle]^d/\mathbb{P}^1$	Bernardi-Raugel (BR): $[\mathbb{P}_1]^d \oplus$ Enrichi	Taylor-Hood (TH): $[\mathbb{P}_2]^d/\mathbb{P}^1$	Bernardi-Raugel (BR): $[\mathbb{P}_1]^d \oplus$ Enrichi
$\mathbf{D}_T$	Raviart-Thomas (RT): $\mathbb{P}_k(T)^d \oplus \mathbb{P}_k(T)x$ $k \in \mathbb{N}$	BDM, BDFM or BDDF, CD	Raviart-Thomas (RT): $\mathbb{P}_k(T)^d \oplus \mathbb{P}_k(T)x$ $k \in \mathbb{N}$	Raviart-Thomas (RT): $\mathbb{P}_k(T)^d \oplus \mathbb{P}_k(T)x$ $k \in \mathbb{N}$
$\mathbf{L}_T$	$\mathbb{P}_1$ – Lagrange	$\mathbb{P}_1$ – Lagrange	$\mathbb{P}_1$ – Lagrange	$\mathbb{P}_1$ – Lagrange

3. Error Estimators

Here we present our main results, namely reliable and efficient error estimation on isotropic meshes. We will discuss the a posteriori error estimates for finite element approximations of the Darcy-Stokes systems. The upper error bound is derived in Subsection 3.3, whereas the lower error bound is proven in Subsection 3.4. We begin with some notations.

3.1. Notations

Given $i \in \{1, 2\}$ and $T \in \mathcal{T}_i^h$, we let $\varepsilon_h(T)$ be the set of faces of T and denote by ε_h let the set of all faces of $\mathcal{T}^h$. Then we write

$\varepsilon_h = \varepsilon_h(\Gamma_1) \cup \varepsilon_h(\Omega_1) \cup \varepsilon_h(\Gamma_I) \cup \varepsilon_h(\Omega_2) \cup \varepsilon_h(\Gamma_2)$, where $\varepsilon_h(\Gamma_i) := \{E \in \varepsilon_h : E \subset \Gamma_i\}$, $\varepsilon_h(\Omega_i) := \{E \in \varepsilon_h : E \subset \Omega_i\}$, $i = 1, 2$ and $\varepsilon_h(\Gamma_I) := \{E \in \varepsilon_h : E \subset \Gamma_I\}$.

Now, let $q \in L^2(\Omega_i)$ such that $q_{|T} \in C(T)$ for each $T \in \mathcal{T}_i^h$, and let $E \in \varepsilon_h(T) \cap \varepsilon_h(\Omega_i)$, we denote by $[q]_E$ the jump of q across E, that is, $[q]_E := (q_{|T} - q_{|T'})_{|E}$, where T' is the other element of $\mathcal{T}_i^h$ having E as face. Also, the jump of some (scalar or vector valued) function $\mathbf{v} \in [L^2(\Omega_i)]^d$, $i \in \{1, 2\}$ such that $\mathbf{v}_{|T} \in [C(T)]^d$ defined as $[\mathbf{v}]_E := (\mathbf{v}_{|T} - \mathbf{v}_{|T'})_{|E}$.

In this next following, we denote by $(\mathbf{u}, p, \lambda)$, with $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ and $p = (p_1, p_2)$ the unique solvability of continuous problem (15). Also, we denote by $(\mathbf{u}^h, p^h, \lambda^h)$, with $\mathbf{u}^h = (\mathbf{u}_1^h, \mathbf{u}_2^h)$ and $p^h = (p_1^h, p_2^h)$ the unique solvability of approach problem (19).

3.2. Residual error estimators

The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier, and that involve the data at hand.

Definition 3.1 (Residual error estimator). For each $T \in \mathcal{T}^h$, we define the based-residual errors indicator $\Theta_{i,T}$, $i \in \{1, 2\}$ by:

$$\begin{aligned}\Theta_{1,T} := {} & h_T \|\boldsymbol{f}_1^h + \nabla \cdot \Phi_{1,h}\|_{0,T} + \|\nabla \cdot \boldsymbol{u}_1^h\|_{0,T} \\ & + \sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_I)} h_E^{1/2} \|(\boldsymbol{n}_1 \cdot \Phi_{1,h} \cdot \boldsymbol{n}_1 + \lambda^h)\boldsymbol{n}_1\|_{0,E} \\ & + \sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_I)} h_E^{1/2} \left\| \sum_{j=1}^{d-1} \left(\boldsymbol{n}_1 \cdot \Phi_{1,h} \cdot \tau_j + \frac{\mu}{\kappa_j} \boldsymbol{u}_1^h \cdot \tau_j \right) \tau_j \right\|_{0,E} \\ & + \sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_I)} \|\boldsymbol{u}_1^h \cdot \boldsymbol{n}_1 + \boldsymbol{u}_2^h \cdot \boldsymbol{n}_2\|_{0,E} \\ & + \frac{1}{2} \sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Omega_1)} h_E^{1/2} \left\| [\Phi_{1,h} \cdot \boldsymbol{n}_1 + 2\mu \sum_{j=1}^{d-1} (\tau_j \cdot \Phi_{1,h} \cdot \boldsymbol{n}_1)\tau_j]_E \right\|_{0,E}, \ \forall T \in \mathcal{T}_1^h,\end{aligned} \tag{28}$$

where $\Phi_{1,h}$ is defined by

$$\Phi_{1,h} := -p_1^h I + 2\mu \mathbf{D}(\boldsymbol{u}_1^h) \text{ on } T \in \mathcal{T}_1^h, \tag{29}$$

and

$$\begin{aligned}\Theta_{2,T} := {} & h_T \|\nabla p_2^h + \mathbf{K}^{-1} \boldsymbol{u}_2^h\|_{0,T} + \|f_2^h - \nabla \cdot \boldsymbol{u}_2^h\|_{0,T} \\ & + \sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_I)} h_E^{1/2} \|(p_2^h - \lambda^h) n_2\|_{0,E} \\ & + \frac{1}{2} \sum_{E \in \varepsilon_h(\Omega_2) \cap \varepsilon_h(T)} h_E^{1/2} \|[p_2^h \boldsymbol{n}_2]_E\|_{0,E} \\ & + \sum_{E \in \varepsilon_h(\Gamma_2) \cap \varepsilon_h(T)} h_E^{1/2} \|p_2^h \boldsymbol{n}_2\|_{0,E}, \ \forall T \in \mathcal{T}_2^h.\end{aligned} \tag{30}$$

Then, we introduce the global a posteriori error estimator

$$\Theta := \left\{ \sum_{T \in \mathcal{T}_1^h} \Theta_{1,T}^2 + \sum_{T \in \mathcal{T}_2^h} \Theta_{2,T}^2 \right\}^{1/2}. \tag{31}$$

Remark 3.1. The residual character of each term on the right-hand sides of (28) and (30) is quite clear (see, the consistance property (38) and the residual equation (39) below).

3.3. Proof of the upper error bound

Global upper error bound is given by the theorem:

Theorem 3.1 (Global upper error bound). *The following global upper error bound holds*:

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_{\boldsymbol{H}} + \|p - p^h\|_M + \|\lambda - \lambda^h\|_{\boldsymbol{H}^{1/2}(\Gamma_I)} \lesssim \Theta + \zeta, \tag{32}$$

with

$$\zeta := \left(\sum_{T \in \mathcal{T}_1^h} h_T^2 \|\boldsymbol{f}_1 - \boldsymbol{f}_1^h\|_{0,T}^2 + \sum_{T \in \mathcal{T}_2^h} h_T^2 \|f_2 - f_2^h\|_{0,T}^2 \right)^{1/2}. \tag{33}$$

The constant intervenying in this inequality (*i.e.*, (32)) *depends of parameter of regularity of the triangulation.*

Proof. For all $U = (\mathbf{u}, p, \lambda) \in \mathbf{Y}$ and $V = (\mathbf{v}, q, \xi) \in \mathbf{Y}$, we define the continuous bilinear form $A : \mathbf{Y} \times \mathbf{Y} \to \mathbb{R}$ by:

$$A(U, V) := \mathbf{a}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, p) + b_I(\mathbf{v}, \lambda) + \mathbf{b}(\mathbf{u}, q) + b_I(\mathbf{u}, \xi). \tag{34}$$

Hence, the problem $(\mathcal{P})$ is equivalent to problem $(\mathcal{Q})$: Find $U \in \mathbf{Y}$ such that

$$A(U, V) = G(V),\ AV \in \mathbf{Y}, \text{ where } G(V) = l(\mathbf{v}) + g(q). \tag{35}$$

Thus, approach problem $(\mathcal{Q}^h)$: Find $U_h \in \mathbf{Y}^h$ such that

$$A(U^h, V^h) = G(V),^h \quad \forall V^h \in \mathbf{Y}^h. \tag{36}$$

Then, (cf. [5], Theorem 2.4, pp. 32): the bilinear form A satisfy the inf-sup condition on $\mathbf{Y} \times \mathbf{Y}$, i.e., there exists a constant $\beta > 0$ such that:

$$\sup_{V \in \mathbf{Y}} \frac{A(U, V)}{\|V\|_{\mathbf{Y}}} \geq \beta \|U\|_{\mathbf{Y}}, \quad \forall U \in \mathbf{Y}. \tag{37}$$

And, we have the consistance property or Galerkin orthogonality relation

$$A(U - U^h, V^h) = 0, \quad \forall V^h \in \mathbf{Y}^h. \tag{38}$$

Applying the definition of residual, the consistance property (38) and proceeding by integration by parts on each element of meshes, we obtain the residual equations which is given by:

$$\begin{aligned}
A(U - U_h, V) = \sum_{T \in \mathcal{T}_1^h} \Bigg\{ & \int_T (\mathbf{f}_1 - \mathbf{f}_1^h) \cdot (\mathbf{v}_1 - \mathbf{v}_1^h) + \int_T (\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}) \cdot (\mathbf{v}_1 - \mathbf{v}_1^h) \\
& + \int_T q_1 \nabla \cdot \mathbf{u}_1^h - \sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_I)} \int_E (\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h) \mathbf{n}_1 \cdot (\mathbf{v}_1 - \mathbf{v}_1^h) \\
& - \sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_I)} \int_E \sum_{j=1}^{d-1} (\tau_j \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \frac{\mu}{\kappa_j} \mathbf{u}_1^h \cdot \tau_j) \tau_j \cdot (\mathbf{v}_1 - \mathbf{v}_1^h) \\
& - \frac{1}{2} \sum_{E \in \varepsilon_h(\Omega_1) \cap \varepsilon_h(T)} \int_E [\Phi_{1,h} \mathbf{n}_1 + 2\mu \sum_{j=1}^{d-1} (\tau_j \cdot \Phi_{1,h} \cdot \mathbf{n}_1) \tau_j]_E \cdot (\mathbf{v}_1 - \mathbf{v}_1^h) \\
& - \sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_I)} < \mathbf{u}_1^h \cdot \mathbf{n}_1 + \mathbf{u}_2^h \cdot \mathbf{n}_2, \xi >_E \Bigg\} \\
& + \sum_{T \in \mathcal{T}_2^h} \Bigg\{ \int_T (f_2 - f_2^h) q_2 - \int_T (\nabla p_2^h + \mathbf{K}^{-1} \mathbf{u}_2^h) \cdot (\mathbf{v}_2 - \mathbf{v}_2^h)
\end{aligned}$$

$$+\int_T(-f_2^h+\nabla\cdot\mathbf{u}_2^h)q_2+\sum_{E\in\varepsilon_h(T)\cap\varepsilon_h(\Gamma_I)}\int_E(p_2^h-\lambda^h)\mathbf{n}_2\cdot(\mathbf{v}_2-\mathbf{v}_2^h)$$

$$+\frac{1}{2}\sum_{E\in\varepsilon_h(\Omega_2)\cap\varepsilon_h(T)}\int_E[p_2^h\mathbf{n}_2]_E\cdot(\mathbf{v}_2-\mathbf{v}_2^h)+\sum_{E\in\varepsilon_h(\Gamma_2)\cap\varepsilon_h(T)}\int_E p_2^h\mathbf{n}_2\cdot(\mathbf{v}_2-\mathbf{v}_2^h)\Bigg\}, \tag{39}$$

additionally, we have

$$\mathbf{b}(\mathbf{u}-\mathbf{u}_h,q)=\int_{\Omega_1}q_1\nabla\cdot\mathbf{u}_1^h+\int_{\Omega_2}(\nabla\cdot\mathbf{u}_2^h-f_2)q_2, \tag{40}$$

$$b_I(\mathbf{u}-\mathbf{u}_h,\xi)=-<\mathbf{u}_1^h\cdot\mathbf{n}_1+\mathbf{u}_2^h\cdot\mathbf{n}_2,\xi>_{\Gamma_I}, \tag{41}$$

$\mathbf{f}_1^h$ is the approximation of the data $\mathbf{f}_1$ in $[L^2(\Omega_1)]^d$ space of functions polynomial on each element $T\in\mathcal{T}_1^h$ and f_2^h is the approximation of the data f_2 in $L^2(\Omega_2)$ spaces polynomial on each element $T\in\mathcal{T}_2^h$.

We apply respectively the triangular inequality and Cauchy-Schwartz inequality to the residual equation (39). Next, we use respectively the interpolation operators of the assumption (**G.**4) and the inf-sup condition (37), replacing U by $U-U^h$. We omit the details. □

3.4. Proof of the lower error bound

In order to derive the upper bounds for the remaining terms defining the a posteriori error indicator $\Theta_{i,T}$ $i\in\{1,2\}$, we proceed similarly as in [19] and [20] (see also [21]), and apply inverse inequalities, and the localization technique based on simplex-bubble and face-bubble functions. To this end, we now recall some notation and introduce further preliminary results. Given $T\in\mathcal{T}^h$, and $E\in\varepsilon_h(T)$, we let b_T and b_E be the usual simplex-bubble and face-bubble functions, respectively (see (1.5) and (1.6) in [3]). In particular, b_T satisfies $b_T\in\mathbb{P}_3(T)$, $\mathrm{supp}(b_T)\subseteq T$, $b_T=0$ over ∂T, and $0\le b_T\le 1$ on T.

Similarly, $b_E \in \mathbb{P}_2(T)$, $\text{supp}(b_E) \subseteq W_E := \bigcup\{T' : E \in \varepsilon_h(T')\}$, $b_E = 0$ on $\partial T \setminus E$ and $0 \leq b_E \leq 1$ in W_E. We also recall from [4] that, given $k \in \mathbb{N}$, there exists an extension operator $L : C(E) \to C(T)$ that satisfies $L(p) \in \mathbb{P}_k(T)$ and $L(p)_{|E} = p$, $\forall p \in \mathbb{P}_k(E)$. A corresponding vectorial version of L, that is, the componentwise application of L, is denoted by $\mathbf{L}$. Additional properties of b_T, b_E and L are collected in the following lemma [4].

Lemma 3.1. *Given $k \in \mathbb{N}^*$, there exist positive constants depending only on k and shape-regularity of the triangulations (minimum angle condition), such that for each simplex T and $E \in \varepsilon_h(T)$ there hold*

$$\|q\|_{0,T} \lesssim \|q b_T^{1/2}\|_{0,T} \lesssim \|q\|_{0,T}, \ \forall q \in \mathbb{P}_k(T), \tag{42}$$

$$|q|_{1,T} \lesssim h_T^{-1}\|q\|_{0,T}, \ \forall q \in \mathbb{P}_k(T), \tag{43}$$

$$\|p\|_{0,E} \lesssim \|b_E^{1/2} p\|_{0,E} \lesssim \|p\|_{0,E}, \ \forall p \in \mathbb{P}_k(E), \tag{44}$$

$$\|L(p)\|_{0,T} + h_E|L(p)|_{1,T} \lesssim h_E^{1/2}\|p\|_{0,E}, \ \forall p \in \mathbb{P}_k(E). \tag{45}$$

3.4.1. Lower error bound in Ω_1

The lower error bound in Ω_1 is given by the following proposition:

Proposition 3.1 (Local lower error bound in Ω_1). *For each $T \in \mathcal{T}_1^h$, the following local lower error bound holds*:

$$\begin{aligned}\Theta_{1,T} \lesssim \Big\{ &\|\boldsymbol{u}_1 - \boldsymbol{u}_1^h\|_{1,W_1^T} + \|\boldsymbol{u}_2 - \boldsymbol{u}_2^h\|_{0,W_1^T} + \|p_1 - p_1^h\|_{0,W_1^T} \\ &+ \sum_{k \in W_1^T} h_k(\|\lambda - \lambda^h\|_{0,\Gamma_1 \cup \partial k} + \|\boldsymbol{u}_1 - \boldsymbol{u}_1^h\|_{1,k} + \|p_1 - p_1^h\|_{0,k} \\ &+ \|\nabla \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_2^h)\|_{0,k} + \|\boldsymbol{f}_1 - \boldsymbol{f}_1^h\|_{0,k})\Big\},\end{aligned} \tag{46}$$

where W_1^T *is defined as follow*:

$$W_1^T := \{T' \in \mathcal{T}_1^h : \partial T \cap \partial T' \in \varepsilon_h(\overline{\Omega}_1)\}. \tag{47}$$

Proof. We begin by bounding each term of the residuals separately.

• To estimate $h_T \|\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}\|_{0,T}$. For each $T \in \mathcal{T}_1^h$, we choose in residual equation (39), $V = (\mathbf{v}^T, 0, 0)$ and $\mathbf{v}^h = (\mathbf{0}, \mathbf{0})$, with $\mathbf{v}^T = (\mathbf{v}_1^T, \mathbf{v}_2^T)$ and $\mathbf{v}_2^T = \mathbf{0}$ on Ω_2,

$$\mathbf{v}_1^T := \begin{cases} (\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}) b_T & \text{on} \quad T \in \mathcal{T}_1^h, \\ \mathbf{0} & \text{on} \quad \Omega \setminus T. \end{cases} \tag{48}$$

We have well $V \in \mathbf{Y}$ and the residual equation (39) becomes:

$$A(U - U_h, V) = \int_T (\mathbf{f}_1 - \mathbf{f}_1^h) \cdot \mathbf{v}_1^T + \int_T (\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}) \cdot \mathbf{v}_1^T,$$

because the bubble-function b_T is vanish on ∂T.

$$\int_T (\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}) \cdot \mathbf{v}_1^T = A(U - U_h, V) - \int_T (\mathbf{f}_1 - \mathbf{f}_1^h) \cdot \mathbf{v}_1^T. \tag{49}$$

On the other hand, using the definition of operator A, we have

$$\int_T (\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}) \cdot \mathbf{v}_1^T = 2\mu \int_T \mathbf{D}(\mathbf{u}_1 - \mathbf{u}_1^h) : \mathbf{D}(\mathbf{v}_1^T) - \int_T (p_1 - p_1^h) \nabla \cdot \mathbf{v}_1^T$$
$$- \int_T (\mathbf{f}_1 - \mathbf{f}_1^h) \cdot \mathbf{v}_1^T. \tag{50}$$

Applying respectively the triangular inequality and the Cauchy-Schwartz inequality to (50), we have

$$\int_T (\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}) \cdot \mathbf{v}_1^T \lesssim (\|\mathbf{u}_1 - \mathbf{u}_1^h\|_{1,T} + \|p_1 - p_1^h\|_{0,T})$$
$$\times \|\mathbf{v}_1^T\|_{1,T} + \|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,T} \times \|\mathbf{v}_1^T\|_{0,T}.$$

Next, we use the inverse inequality (43), we have

$$\|(\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h})b_T^{1/2}\|_{0,T}^2 \lesssim \left\{h_T^{-1}(|\mathbf{u}_1 - \mathbf{u}_1^h|_{1,T} + \|p_1 - p_1^h\|_{0,T}) + \|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,T}\right\} \times \|\mathbf{v}_1^T\|_{0,T}.$$

Applying the inverse inequality (42), we have the estimation

$$\|(\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h})\|_{0,T}^2 \lesssim \{h_T^{-1}(|\mathbf{u}_1 - \mathbf{u}_1^h|_{1,T} + \|p_1 - p_1^h\|_{0,T}) + \|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,T}\right\} \times \|\mathbf{v}_1^T\|_{0,T}. \tag{51}$$

Finally, the property $0 \le b_T \le 1$ and the inequality (51), lead to

$$h_T\|\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}\|_{0,T} \lesssim \left(|\mathbf{u}_1 - \mathbf{u}_1^h|_{1,T} + \|p_1 - p_1^h\|_{0,T} + h_T\|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,T}\right). \tag{52}$$

- To estimate $\|\nabla \cdot \mathbf{u}_1^h\|_{0,T}$.

$$\|\nabla \cdot \mathbf{u}_1^h\|_{0,T} \lesssim |\mathbf{u}_1 - \mathbf{u}_1^h|_{1,T}. \tag{53}$$

- To estimate $\sum_{E\in\varepsilon_h(T)\cap\varepsilon_h(\Gamma_I)} h_E^{1/2}\|(\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h)\mathbf{n}_1\|_{0,E}$. For each $T \in \mathcal{T}^h$ and for each $E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_I)$, we choose in residual equation (39), $V = (\mathbf{v}^E, 0, 0)$, $\mathbf{v}^h = (\mathbf{0}, \mathbf{0})$, with $\mathbf{v}^E = (\mathbf{v}_1^E, \mathbf{0})$, where

$$\mathbf{v}_1^E := \begin{cases} \mathbf{L}(\mathbf{n}_1 \cdot \Phi_{1,h}.\mathbf{n}_1 + \lambda^h)\mathbf{n}_1 b_E) & \text{on} \quad \mathrm{T}, \\ \mathbf{0} & \text{on} \quad \Omega \setminus \mathrm{T}. \end{cases} \tag{54}$$

We noted that the tangential component of $\mathbf{v}_1^E$ on E are vanish. In this case, the residual equation become:

$$A(U - U_h, V) = \int_T (\mathbf{f}_1 - \mathbf{f}_1^h) \cdot \mathbf{v}_1^E + \int_T (\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}).\mathbf{v}_1^E - \int_E (\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h)\mathbf{n}_1 \cdot \mathbf{v}_1^E.$$

Hence,

$$\int_E (\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h)\mathbf{n}_1 \cdot \mathbf{v}_1^E = \int_T (\mathbf{f}_1 - \mathbf{f}_1^h) \cdot \mathbf{v}_1^E + \int_T \mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}).\mathbf{v}_1^E$$

$$- A(U - U_h, V).$$

On the other hand, by definition of the operator A, we have

$$\int_E (\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h)\mathbf{n}_1 \cdot \mathbf{v}_1^E = \int_T (\mathbf{f}_1 - \mathbf{f}_1^h) \cdot \mathbf{v}_1^E + \int_T (\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}).\mathbf{v}_1^E$$

$$+ 2\mu \int_T \mathbf{D}(\mathbf{u}_1 - \mathbf{u}_1^h) : \mathbf{D}(\mathbf{v}_1^E) - \int_T (p_1 - p_1^h)\nabla \cdot \mathbf{v}_1^E$$

$$- < \mathbf{v}_1^E, \lambda - \lambda^h >_E . \qquad (55)$$

We apply respectively the triangular inequality and the Cauchy-Schwartz inequality to (55):

$$\int_E (\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h)\mathbf{n}_1 \cdot \mathbf{v}_1^E \leq \left(\|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,T} + \|\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}\|_{0,T} \right) \times \|\mathbf{v}_1^E\|_{0,T}$$

$$(2\mu|\mathbf{u}_1 - \mathbf{u}_1^h|_{1,T} + \|p_1 - p_1^h\|_{0,T}) \times |\mathbf{v}_1^E|_{1,T} + \|\mathbf{v}_1^E \cdot \mathbf{n}_1\|_{0,E} \times \|\lambda - \lambda^h\|_{1/2,E}.$$

$$(56)$$

We apply the inverse inequality (45) to $\mathbf{v}_1^E$, it comes

$$h_E^{1/2} \int_E (\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h)\mathbf{n}_1 \cdot \mathbf{v}_1^E \leq C h_E \|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,T} + h_E \|\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}\|_{0,T}$$

$$+ \|p_1 - p_1^h\|_{0,T} + |\mathbf{u}_1 - \mathbf{u}_1^h|_{1,T}$$

$$+ h_T \|\lambda - \lambda^h\|_{1/2,T} \times \|\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h)\mathbf{n}_1\|_{0,E}. \qquad (57)$$

Also, by definition of the operator $\mathbf{L}$, we have

$$(\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h)\mathbf{n}_1 \cdot \mathbf{v}_1^E = (\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h)\mathbf{n}_1 \times \mathbf{L}[(\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 \mathbf{n}_1$$

$$+ \lambda^h)\mathbf{n}_1 b_E] = [(\mathbf{n}_1 \cdot \Phi_{1,h} \cdot \mathbf{n}_1 + \lambda^h)\mathbf{n}_1 b_E]^2.$$

We use the inverse inequality (44), and we have

$$h_E^{1/2}\|(\mathbf{n}_1\cdot\Phi_{1,h}\cdot\mathbf{n}_1+\lambda^h)\mathbf{n}_1\|_{0,E}\lesssim h_E\|\mathbf{f}_1-\mathbf{f}_1^h\|_{0,q}+h_E\|\mathbf{f}_1^h+\nabla\cdot\Phi_{1,h}\|_{0,T}$$

$$+\|p_1-p_1^h\|_{0,T}+|\mathbf{u}_1-\mathbf{u}_1^h|_{0,T}$$

$$+h_T\|\lambda-\lambda^h\|_{1/2,T}\cdot\|(\mathbf{n}_1\cdot\Phi_{1,h}\mathbf{n}_1+\lambda^h)\mathbf{n}_1\|_{0,E}. \qquad (58)$$

Using estimation (52) and the fact that $h_E\le h_T$ in (58), we lead to

$$\sum_{E\in\varepsilon_h(T)\varepsilon_h(\Gamma_I)} h_E^{1/2}\|(\mathbf{n}_1.\Phi_{1,h}.\mathbf{n}_1+\lambda^h)\mathbf{n}_1\|_{0,E}\lesssim\Bigg\{|\mathbf{u}_1-\mathbf{u}_1^h|_{1,W_1^T}+\|\lambda-\lambda^h\|_{1/2,\partial T\cap\Gamma_I}$$

$$+\|p_1-p_1^h\|_{0,W_1^T}+\sum_{k\in W_1^T}h_k\Big(\|\mathbf{f}_1-\mathbf{f}_1^h\|_{0,k}\Big)\Bigg\}. \qquad (59)$$

- To estimate $\sum_{E\in\varepsilon_h(T)\cap\varepsilon_h(\Gamma_I)}h_E^{1/2}\|\sum_{j=1}^{d-1}\left(\tau_j\cdot\Phi_{1,h}\cdot\mathbf{n}_1+\frac{\mu}{\kappa_j}\mathbf{u}_1\cdot\tau_j\right)$

$\tau_j\|_{0,E}$. For each $T\in\mathcal{T}^h$ and for each $E\in\varepsilon_h(T)\cap\varepsilon_h(\Gamma_I)$, we choose in residual equation (39), $V=(\mathbf{v}^E,0,0)$, $\mathbf{v}^h=\mathbf{0}$ with $\mathbf{v}^E=(\mathbf{v}_1^E,\mathbf{0})$ and

$$\mathbf{v}_1^E:=\begin{cases}\mathbf{L}\left(\sum_{j=1}^{d-1}(\tau_j\cdot\Phi_{1,h}\cdot\mathbf{n}_1+\frac{\mu}{\kappa_j}\mathbf{u}_1\cdot\tau_j)\tau_j b_E\right) & \text{on} \quad T,\\ \mathbf{0} & \text{on} \quad \Omega\setminus T.\end{cases} \qquad (60)$$

We noted that the normal component of $\mathbf{v}_1^E$ on E are vanish. Hence, the residual equation (39) become:

$$A(U-U_h,V)=\int_T(\mathbf{f}_1-\mathbf{f}_1^h)\cdot\mathbf{v}_1^E+\int_T(\mathbf{f}_1^h+\nabla\cdot\Phi_{1,h})\cdot\mathbf{v}_1^E$$

$$-\sum_{j=1}^{d-1}\int_E\left(\tau_j\cdot\Phi_{1,h}\cdot\mathbf{n}_1+\frac{\mu}{\kappa_j}\mathbf{u}_1^h\cdot\tau_j\right)\tau_j\mathbf{v}_1^E.$$

Then,

$$\sum_{j=1}^{d-1}\int_E\left(\tau_j\cdot\Phi_{1,h}\cdot\mathbf{n}_1+\frac{\mu}{\kappa_j}\mathbf{u}_1^h\cdot\tau_j\right)\tau_j.\mathbf{v}_1^E=\int_T(\mathbf{f}_1-\mathbf{f}_1^h)\cdot\mathbf{v}_1^E$$

$$+\int_T(\mathbf{f}_1^h+\nabla\cdot\Phi_{1,h})\cdot\mathbf{v}_1^E-A(U-U_h,V),$$

$$A(U-U_h,V)=2\mu\int_T\mathbf{D}(\mathbf{u}_1-\mathbf{u}_1^h):\mathbf{D}(\mathbf{v}_1^E)-\int_T(p_1-p_1^h)\nabla\cdot\mathbf{v}_1^E$$

$$+<\mathbf{v}_1^E\cdot\mathbf{n}_1,\lambda-\lambda^h>_E+\sum_{j=1}^{d-1}\frac{\mu}{\kappa_j}\int_E(\mathbf{u}_1-\mathbf{u}_1^h).\tau_j(\mathbf{v}_1^E\cdot\tau_j).$$

Thus,

$$\sum_{j=1}^{d-1}\int_E\left(\tau_j\cdot\Phi_{1,h}\cdot\mathbf{n}_1+\frac{\mu}{\kappa_j}\mathbf{u}_1^h\cdot\tau_j\right)\tau_j.\mathbf{v}_1^E=\int_T(\mathbf{f}_1-\mathbf{f}_1^h)\cdot\mathbf{v}_1^E+\int_T(\mathbf{f}_1^h$$

$$+\nabla\cdot\Phi_{1,h})\cdot\mathbf{v}_1^E-2\mu\int_T\mathbf{D}(\mathbf{u}_1-\mathbf{u}_1^h):\mathbf{D}(\mathbf{v}_1^E)+\int_T(p_1-p_1^h)\nabla\cdot\mathbf{v}_1^E$$

$$+\sum_{j=1}^{d-1}\frac{\mu}{\kappa_j}\int_E(\mathbf{u}_1-\mathbf{u}_1^h)\cdot\tau_j(\mathbf{v}_1^E\cdot\tau_j)+<\mathbf{v}_1^E\cdot\mathbf{n}_1,\lambda-\lambda^h>_E.$$

We apply respectively the triangular inequality and the Cauchy-Schwartz inequality:

$$\sum_{j=1}^{d-1}\int_E\left(\tau_j\cdot\Phi_{1,h}\cdot\mathbf{n}_1+\frac{\mu}{\kappa_j}\mathbf{u}_1^h\cdot\tau_j\right)\tau_j\cdot\mathbf{v}_1^E\le\left(\|\mathbf{f}_1-\mathbf{f}_1^h\|_{0,T}+\|\mathbf{f}_1+\nabla.\Phi_{1,h}\|_{0,T}\right)$$

$$\times\|\mathbf{v}_1^E\|_{0,E}+(2\mu|\mathbf{u}_1-\mathbf{u}_1^h|_{1,T}+\|p_1-p_1^h\|_{0,T}\times|\mathbf{v}_1^E|_{1,T}$$

$$+\sum_{j=1}^{d-1}\frac{\mu}{\kappa_j}\|(\mathbf{u}_1-\mathbf{u}_1^h).\tau_j\|_{0,E}\|\mathbf{v}_1^E\cdot\tau_j\|_{0,E}+\|\mathbf{v}_1^E\cdot\mathbf{n}_1\|_{0,E}\times\|\lambda-\lambda^h\|_{1/2,E}.$$

Let

$$\sum_{j=1}^{d-1}\int_E\left(\tau_j\cdot\Phi_{1,h}\cdot\mathbf{n}_1+\frac{\mu}{\kappa_j}\mathbf{u}_1^h\cdot\tau_j\right)\tau_j\cdot\mathbf{v}_1^E\le(\|\mathbf{f}_1-f_1^h\|_{0,T}+\|\mathbf{f}_1+\nabla\cdot\Phi_{1,h}\|_{0,T})$$

$$\times\|\mathbf{v}_1^E\|_{0,E}+(2\mu|\mathbf{u}_1-\mathbf{u}_1^h|_{1,T}+\|p_1-p_1^h\|_{0,T})\times|\mathbf{v}_1^E|_{1,T}$$

$$+\left(\sum_{j=1}^{d-1}\frac{\mu}{\kappa_j}\|(\mathbf{u}_1-\mathbf{u}_1^h)\cdot\tau_j\|_{0,E}+\|\lambda-\lambda^h\|_{1/2,E}\right)\times\|\mathbf{v}_1^E\|_{0,E}.$$

Next, we apply the inverse inequalities (44) and (42) to $\mathbf{v}_1^E$, it comes:

$$\sum_{j=1}^{d-1}\int_E\left(\tau_j\cdot\Phi_{1,h}\cdot\mathbf{n}_1+\frac{\mu}{\kappa_j}\mathbf{u}_1^h\cdot\tau_j\right)\tau_j\cdot\mathbf{v}_1^E\lesssim h_T\|\mathbf{f}_1-\mathbf{f}_1^h\|_{0,T}+h_T\|\mathbf{f}_1+\nabla$$

$$\cdot\Phi_{1,h}\|_{0,T}+\|\lambda-\lambda^h\|_{1/2,T}+2\mu|\mathbf{u}_1-\mathbf{u}_1^h|_{1,T}+\|p_1-p_1^h\|_{0,T}$$

$$+\sum_{j=1}^{d-1}\frac{\mu}{\kappa_j}h_T\|\mathbf{u}_1-\mathbf{u}_1^h\|_{0,T}.$$

Using the estimation (52), we deduce finally:

$$\sum_{E\in\varepsilon_h(T)\cap\varepsilon_h(\Gamma_I)}h_E^{1/2}\left\|\sum_{j=1}^{d-1}\left(\tau_j\cdot\Phi_{1,h}\mathbf{n}_1+\frac{\mu}{\kappa_j}\mathbf{u}_1\cdot\tau_j\right)\tau_j\right\|_{0,E}$$

$$\lesssim\Bigg\{\|p_1-p_1^h\|_{0,W_1^T}+|\mathbf{u}_1-\mathbf{u}_1^h|_{1,W_1^T}+\|\lambda-\lambda^h\|_{1/2,\partial T\cap\Gamma_I}$$

$$+\sum_{k\in W_1^T}h_k\left(\|\mathbf{f}_1-\mathbf{f}_1^h\|_{0,k}+\|\mathbf{u}_1-\mathbf{u}_1^h\|_{0,k}\right)\Bigg\}. \qquad (61)$$

• To estimate $\frac{1}{2}\sum_{E\in\varepsilon_h(T)\cap\varepsilon_h(\Omega_1)} h_E^{1/2}\| [\Phi_{1,h}\mathbf{n}_1 + 2\mu\sum_{j=1}^{d-1}(\tau_j \cdot \Phi_{1,h} \cdot \mathbf{n}_1)\tau_j]_E\|_{0,E}$. For each $T \in \mathcal{T}_1^h$ and for each $E \in \varepsilon_h(T)\cap\varepsilon_h(\Omega_1)$, we choose in residual equation (39), $V = (\mathbf{v}^E, 0, 0)$ and $\mathbf{v}^h = (\mathbf{0}, \mathbf{0})$, with $\mathbf{v}_2^E = \mathbf{0}$ on Ω_2 and

$$\mathbf{v}_1^E := \begin{cases} \mathbf{L}\left([\Phi_{1,h}.\mathbf{n}_1 + 2\mu\sum_{j=1}^{d-1}(\tau_j \cdot \Phi_{1,h}\mathbf{n}_1)\tau_j]_E b_E\right) & \text{on } \ k \in \{T, T'\}, \ (\partial T' \cap \partial T = E), \\ \mathbf{0} & \text{on } \ \Omega \setminus T \cup T', \end{cases}$$

$$A(U - U_h, V) = \int_{T\cup T'}(\mathbf{f}_1 - \mathbf{f}_1^h)\cdot \mathbf{v}_1^E + \int_{T\cup T'}(\mathbf{f}_1^h + \nabla\cdot T_{1,h})\cdot\mathbf{v}_1^E$$

$$- \frac{1}{2}\int_E [\Phi_{1,h}\mathbf{n}_1 + 2\mu\sum_{j=1}^{d-1}(\tau_j \cdot \Phi_{1,h} \cdot \mathbf{n}_1)\tau_j]_E \cdot \mathbf{v}_1^E.$$

Hence,

$$\frac{1}{2}\int_E [\Phi_{1,h}\mathbf{n}_1 + 2\mu\sum_{j=1}^{d-1}(\tau_j \cdot \Phi_{1,h} \cdot \mathbf{n}_1)\tau_j]_E \cdot \mathbf{v}_1^E = \int_{T\cup T'}(\mathbf{f}_1 - \mathbf{f}_1^h)\cdot\mathbf{v}_1^E$$

$$+ \int_{T\cup T'}(\mathbf{f}_1^h + \nabla\cdot\Phi_{1,h})\cdot\mathbf{v}_1^E - A(U - U_h, V).$$

By definition of operator A, we have:

$$A(U - U_h, V) = 2\mu\int_{T\cup T'}\mathbf{D}(\mathbf{u}_1 - \mathbf{u}_1^h) : \mathbf{D}(\mathbf{v}_1^E) - \int_{T\cup T'}(p_1 - p_1^h)\nabla.\mathbf{v}_1^E,$$

$$\frac{1}{2}\int_E [\Phi_{1,h}\mathbf{n}_1 + 2\mu\sum_{j=1}^{d-1}(\tau_j \cdot \Phi_{1,h} \cdot \mathbf{n}_1)\tau_j]_E \cdot \mathbf{v}_1^E = \int_{T\cup T'}(\mathbf{f}_1 - \mathbf{f}_1^h)\cdot\mathbf{v}_1^E$$

$$+ \int_{T\cup T'}(\mathbf{f}_1^h + \nabla.\Phi_{1,h})\cdot\mathbf{v}_1^E - 2\mu\int_{T\cup T'}\mathbf{D}(\mathbf{u}_1 - \mathbf{u}_1^h) : \mathbf{D}(\mathbf{v}_1^E)$$

$$+ \int_{T\cup T'}(p_1 - p_1^h)\nabla\cdot\mathbf{v}_1^E. \tag{62}$$

We apply respectively triangular inequality and Cauchy-Schwartz inequality:

$$\frac{1}{2}\int_E [\Phi_{1,h}\mathbf{n}_1 + 2\mu\sum_{j=1}^{d-1}(\tau_j \cdot \Phi_{1,h}\mathbf{n}_1)\tau_j]_E \cdot \mathbf{v}_1^E \lesssim \|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,\, T\cup T'} \times \|\mathbf{v}_1^E\|_{0,\, T\cup T'}$$

$$+ \|\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}\|_{0,\, T\cup T'} \times \|\mathbf{v}_1^E\|_{0,\, T\cup T'}$$

$$+ 2\mu|\mathbf{u}_1 - \mathbf{u}_1^h|_{1, T\cup T'} \times |\mathbf{v}_1^E|_{1,\, T\cup T'}$$

$$+ \|p_1 - p_1^h\|_{0, T\cup T'} \times |\mathbf{v}_1^E|_{1,\, T\cup T'},$$

$$\frac{1}{2}\int_E [\Phi_{1,h}\mathbf{n}_1 + 2\mu\sum_{j=1}^{d-1}(\tau_j \cdot \Phi_{1,h} \cdot \mathbf{n}_1)\tau_j]_E \cdot \mathbf{v}_1^E \lesssim \sum_{k\in\{T,T'\}} \{\|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,k}\|\mathbf{v}_1^E\|_{0,k}$$

$$+ \|\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}\|_{0,\, k} \times \|\mathbf{v}_1^E\|_{0,\, k}$$

$$+ 2\mu|\mathbf{u}_1 - \mathbf{u}_1^h|_{1,\, k} \times |\mathbf{v}_1^E|_{1,k} + \|p_1 - p_1^h\|_{0,\, k} \times |\mathbf{v}_1^E|_{1,k}\}.$$

The inverse inequality (45) gives:

$$\frac{1}{2}h_E^{1/2}\int_E [\Phi_{1,h}\mathbf{n}_1 + 2\mu\sum_{j=1}^{d-1}(\tau_j \cdot \Phi_{1,h}\mathbf{n}_1)\tau_j]_E.\mathbf{v}_1^E \lesssim \left(\sum_{k\in\{T,T'\}}\{h_E\|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,\, k}\right.$$

$$\left. + h_E\,\|\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}\|_{0,\, k} + |\mathbf{u}_1 - \mathbf{u}_1^h|_{0,k} + \|p_1 - p_1^h\|_{0,\, k}\}\right)$$

$$\times \|[\Phi_{1,h}\mathbf{n}_1 + 2\mu\sum_{j=1}^{d-1}(\tau_j \cdot \Phi_{1,h} \cdot \mathbf{n}_1)\tau_j]_E\|_{0,\, E}.$$

Next, we apply the inverse inequality (44) and the definition of the operator $\mathbf{L}$:

$$\frac{1}{2}h_E^{1/2}\|[\Phi_{1,h}\mathbf{n}_1 + 2\mu\sum_{j=1}^{d-1}(\tau_j \cdot \Phi_{1,h} \cdot \mathbf{n}_1)\tau_j]_E\|_{0,\, E} \lesssim \left(\sum_{k\in\{T,T'\}}\{h_k\|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,\, k}\right.$$

$$\left. + h_k\|\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}\|_{0,\, k} + |\mathbf{u}_1 - \mathbf{u}_1^h|_{1,\, k} + \|p_1 - p_1^h\|_{0,\, k}\}\right).$$

We use the estimation of $h_k \|\mathbf{f}_1^h + \nabla \cdot \Phi_{1,h}\|_{0,k}$ (cf. (52)), and we have the estimation:

$$\frac{1}{2} \sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Omega_1)} h_E^{1/2} \| [\Phi_{1,h}\mathbf{n}_1 + 2\mu \sum_{j=1}^{d-1} (\tau_j \cdot \Phi_{1,h} \cdot \mathbf{n}_1)\tau_j]_E \|_{o,E}$$

$$\lesssim \left(|\mathbf{u}_1 - \mathbf{u}_1^h|_{1,W_1^T} + \|p_1 - p_1^h\|_{0,W_1^T} + \sum_{k \in W_1^T} h_k \|\mathbf{f}_1 - \mathbf{f}_1^h\|_{0,k} \right),$$

where W_1^T is given by (47).

• The following estimation holds for each $T \in \mathcal{T}^h$ (cf. [1], Lemma 4.7, pp. 519):

$$\sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_1)} h_E^{1/2} \|\mathbf{u}_1^h \cdot \mathbf{n}_1 + \mathbf{u}_2^h \cdot \mathbf{n}_2\|_{0,E} \lesssim \left\{ \|\mathbf{u}_1 - \mathbf{u}_1^h\|_{0,W_1^T} + \|\mathbf{u}_2 - \mathbf{u}_2^h\|_{0,W_2^T} \right.$$

$$\left. + \sum_{k \in W_1^T} h_k |\mathbf{u}_1 - \mathbf{u}_1^h|_{1,k} + \sum_{k \in W_2^T} \|\nabla \cdot (\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,W_2^T} \right\}.$$

Thus, the Proposition 3.1 is proved.

□

3.4.2. Local error bound in Ω_2

The local error bound in Ω_2 is given by the following proposition:

Proposition 3.2 (Local lower error bound in Ω_2). *For each $T \in \mathcal{T}_2^h$, the following local lower error bound holds*:

$$\Theta_{2,T} \lesssim \Bigg(\|\mathbf{u}_2 - \mathbf{u}_2^h\|_{H(\mathrm{div}, W_2^T)} + \|p_2 - p_2^h\|_{0, W_2^T}$$

$$+ \sum_{k \in W_2^T} h_k (\|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,k} + \|p_2 - p_2^h\|_{0,k} + \|f_2 - f_2^h\|_{0,k} + \|\lambda - \lambda^h\|_{0, \Gamma_I \cap \partial k}) \Bigg). \tag{63}$$

Proof. We begin also by bounding each term of the residuals separately.

- To estimate $h_T \|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,T}$. For each $T \in \mathcal{T}_2^h$, we choose in the residual equation (39), $V = (\mathbf{v}^T, 0, 0)$, with $\mathbf{v}^T = (\mathbf{0}, \mathbf{v}_2^T)$ and $\mathbf{v}^h = (\mathbf{0}, \mathbf{0})$

$$\mathbf{v}_2^T := \begin{cases} (\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h) b_T & \text{on} \quad T \in \mathcal{T}_2^h, \\ \mathbf{0} & \text{on} \quad \Omega \setminus T. \end{cases} \tag{64}$$

Hence,

$$A(U - U_h, V) = -\int_T (\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h) \cdot \mathbf{v}_2^T.$$

Thus

$$\|(\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h) b_T^{1/2}\|_{0,T}^2 = -A(U - U_h, V),$$

$$\|(\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h) b_T^{1/2}\|_{0,T}^2 = -\int_T \mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h).\mathbf{v}_2^T + \int_T (p_2 - p_2^h) \nabla \cdot \mathbf{v}_2^T$$

$$\lesssim \|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,T} \times \|\mathbf{v}_2^T\|_{0,T} + \|p_2 - p_2^h\|_{0,T} \times \|\nabla \cdot \mathbf{v}_2^T\|_{0,T}.$$

We apply the inverse inequality (42) and we get the estimation (65), i.e., the estimate

$$h_T\|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,T} \lesssim h_T\|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,T} + \|p_2 - p_2^h\|_{0,T}. \tag{65}$$

- To estimate $\|f_2^h - \nabla\cdot\mathbf{u}_2^h\|_{0,T}$. For each $T \in \mathcal{T}_2^h$, we choose in the residual equation (39), $V = (\mathbf{0}, q^T, 0)$ and $\mathbf{v}^h = (\mathbf{0}, \mathbf{0})$, with $q^T = (0, q_2^T)$ and

$$q_2^T := \begin{cases} (-f_2^h + \nabla\cdot\mathbf{u}_2^h)b_T & \text{on} \quad T \in \mathcal{T}_2^h, \\ \mathbf{0} & \text{on} \quad \Omega\setminus T. \end{cases} \tag{66}$$

Then, we have

$$A(U - U_h, V) = \int_T (f_2^h - f_2)q_2^T + \int_T(-f_2^h + \nabla\cdot\mathbf{u}_2^h)q_2^T.$$

Let

$$\begin{aligned}\int_T(-f_2^h + \nabla\cdot\mathbf{u}_2^h)q_2^T &= A(U - U_h, V) - \int_T (f_2^h - f_2)q_2^T \\ &= -\int_T q_2^T\nabla\cdot(\mathbf{u}_2 - \mathbf{u}_2^h) - \int_T (f_2 - f_2^h)q_2^T \\ &\le (\|\nabla\cdot(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,T} + \|f_2 - f_2^h\|_{0,T})\|q_2^T\|_{0,T}.\end{aligned}$$

Hence,

$$\|(-f_2^h + \nabla.\mathbf{u}_2^h)b_T^{1/2}\|_{0,T}^2 \le (\|\mathbf{u}_2 - \mathbf{u}_2^h\|_{H(\mathrm{div},T)} + \|f_2 - f_2^h\|_{0,T} \times \|(-f_2^h + \nabla.\mathbf{u}_2^h)b_T)\|_{0,T}$$

The inverse inequality (42) and the property $0 \le b_T \le 1$ give estimation (67), i.e.,

$$\|f_2^h - \nabla\cdot\mathbf{u}_2^h\|_{0,T} \lesssim (\|\mathbf{u}_2 - \mathbf{u}_2^h\|_{H(\mathrm{div},T)} + \|f_2 - f_2^h\|_{0,T}). \tag{67}$$

- To estimate $\sum_{E\in\varepsilon_h(T)\cap\varepsilon_h(\Gamma_I)} h_E^{1/2}\|(p_2^h-\lambda^h)\mathbf{n}_2\|_{0,E}$. For each $T\in\mathcal{T}^h$ and for each $E\in\varepsilon_h(T)\cap\varepsilon_h(\Gamma_I)$, we choose in the residual equation (39), $V=(\mathbf{v}^E, 0, 0)$, and $\mathbf{v}^h=(\mathbf{0},\mathbf{0})$, with $\mathbf{v}^E=(\mathbf{0},\mathbf{v}_2^E)$,

$$\mathbf{v}_2^E := \begin{cases} \mathbf{L}((p_2^h-\lambda^h)\mathbf{n}_2 b_E) & \text{on} \quad \mathrm{T}, \\ \mathbf{0} & \text{on} \quad \Omega\setminus\mathrm{T}. \end{cases} \tag{68}$$

Then

$$A(U-U_h, V) = -\int_T(\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h)\cdot\mathbf{v}_2^E + \int_E (p_2^h-\lambda^h)\mathbf{n}_2\cdot\mathbf{v}_2^E.$$

Let

$$\int_E (p_2-\lambda^h)\mathbf{n}_2.\mathbf{v}_2^E = A(U-U_h, V) + \int_T(\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h)\cdot\mathbf{v}_2^E. \tag{69}$$

On the other hand,

$$A(U-U_h, V) = -\int_T (p_2-p_2^h)\nabla\cdot\mathbf{v}_2^E + <\mathbf{v}_2^E\cdot\mathbf{n}_2, \lambda-\lambda^h>_E. \tag{70}$$

Combining (69) and (70), we have

$$\int_E (p_2-\lambda^h)\mathbf{n}_2\cdot\mathbf{v}_2^E = -\int_T (p_2-p_2^h)\nabla\cdot\mathbf{v}_2^E + <\mathbf{v}_2^E\cdot\mathbf{n}_2, \lambda-\lambda^h>_E$$

$$+\int_T(\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h)\cdot\mathbf{v}_2^E.$$

We use the triangular inequality and the Cauchy-Schwartz inequality:

$$\int_E (p_2-\lambda^h)\mathbf{n}_2\cdot\mathbf{v}_2^E \lesssim \|p_2-p_2^h\|_{0,T}\cdot|\mathbf{v}_2^E|_{1,T} + \|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,T}$$

$$\times\|\mathbf{v}_2^E\|_{0,T} + \|\lambda-\lambda^h\|_{1/2,E}\times\|\mathbf{v}_2^E\|_{0,E}.$$

Next, we apply the inverse inequality (45) and $(p_2^h - \lambda^h)\mathbf{n}_2$, it comes

$$h_E^{1/2}\int_E (p_2 - \lambda^h)\mathbf{n}_2 \cdot \mathbf{v}_2^E \lesssim \{\|p_2 - p_2^h\|_{0,T} + \|(\lambda - \lambda^h)\|_{0,E}$$

$$+ h_E\|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,T}\} \times \|(p_2^h - \lambda^h)\mathbf{n}_2\|_{0,E}. \tag{71}$$

We combine the inverse inequality (44) with (71), and we get the estimation (72), i.e.,

$$\sum_{E\in\varepsilon_h(T)\cap\varepsilon_h(\Gamma_I)} h_E^{1/2}\|(p_2^h - \lambda^h)\mathbf{n}_2\|_{0,E} \lesssim \|p_2 - p_2^h\|_{0,W_2^T} + \|\lambda - \lambda^h\|_{1/2,\partial T\cap\Gamma_I}$$

$$+ \sum_{k\in W_2^T} h_k(\|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,k} + \|p_2 - p_2^h\|_{0,k}).$$

- To estimate $\frac{1}{2}\sum_{E\in\varepsilon_h(\Omega_2)\cap\varepsilon_h(T)}\|[p_2^h\mathbf{n}_2]_E\|_{0,E}$. For each $T \in \mathcal{T}_2^h$ and for each $E \in \varepsilon_h(\Omega_2)\cap\varepsilon_h(T)$, we choose in the residual equation (39), $V = (\mathbf{v}^E, 0, 0)$ and $\mathbf{v}^h = (\mathbf{0}, \mathbf{0})$ with $\mathbf{v}^E = (\mathbf{0}, \mathbf{v}_2^E)$ and

$$\mathbf{v}_2^E := \begin{cases} \mathbf{L}([p_2^h\mathbf{n}_2]_E b_E) & \text{on} \quad k \in \{T, T'\}, \quad (\text{where } \partial T \cap \partial T' = E), \\ \mathbf{0} & \text{on} \quad \Omega \setminus T \cup T'. \end{cases} \tag{72}$$

Then, we have

$$A(U - U_h, V) = -\int_{T\cup T'}(\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h)\cdot\mathbf{v}_2^E + \frac{1}{2}\int_E [p_2^h\mathbf{n}_2]_E \cdot \mathbf{v}_2^E.$$

Hence

$$\frac{1}{2}\int_E [p_2^h\mathbf{n}_2]_E \cdot \mathbf{v}_2^E = A(U - U_h, V) + \int_{T\cup T'}(\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h)\cdot\mathbf{v}_2^E.$$

Using the definition of the operator A, we have

$$\frac{1}{2}\int_E [p_2^h \mathbf{n}_2]_E \cdot \mathbf{v}_2^E = \int_{T\cup T'} (\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h) \cdot \mathbf{v}_2^E$$

$$- \int_{T\cup T'} \mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h) \cdot \mathbf{v}_2^E + \int_{T\cup T'} (\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h) \cdot \mathbf{v}_2^E. \tag{73}$$

We apply respectively to (73), the triangular inequality and the Cauchy-Schwartz inequality:

$$\frac{1}{2}\int_E [p_2^h \mathbf{n}_2]_E . \mathbf{v}_2^E \leq \|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,T\cup T'} \times \|\mathbf{v}_2^E\|_{0,T\cup T'}$$

$$+ \|p_2 - p_2^h\|_{0,T\cup T'} \times |\mathbf{v}_2^E|_{0,T\cup T'} + \|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,T\cup T'} \times \|\mathbf{v}_2^E\|_{0,T\cup T'}. \tag{74}$$

Let

$$\frac{1}{2}\int_E [p_2^h \mathbf{n}_2]_E \cdot \mathbf{v}_2^E \leq \sum_{k\in\{T,T'\}} \{\|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,k} \times \|\mathbf{v}_2^E\|_{0,k}$$

$$+ \|p_2 - p_2^h\|_{0,k} |\mathbf{v}_2^E|_{0,k} + \|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,k} \times \|\mathbf{v}_2^E\|_{0,k}\}. \tag{75}$$

We apply inverse inequality (45):

$$\frac{1}{2} h_E^{1/2} \int_E [p_2^h \mathbf{n}_2]_E \cdot \mathbf{v}_2^E \lesssim \sum_{k\in\{T,T'\}} \{h_k \|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,k}$$

$$+ \|p_2 - p_2^h\|_{0,k} + h_k \|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,k}\} \| [p_2^h \mathbf{n}_2]_E \|_{0,E}. \tag{76}$$

On the other hand, $[p_2^h \mathbf{n}_2]_E \cdot \mathbf{v}_2^E = [p_2^h \mathbf{n}_2]_E^2 b_E$. Hence

$$\frac{1}{2} h_E^{1/2} \| [p_2^h \mathbf{n}_2]_E b_E^{1/2} \|_{0,E}^2 \leq C \sum_{k\in\{T,T'\}} \{h_k \|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,k} + \|p_2 - p_2^h\|_{0,k}$$

$$+ h_k \|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,k}\} \| [p_2^h \mathbf{n}_2]_E \|_{0,E}. \tag{77}$$

Next, we apply inverse inequality (44):

$$\frac{1}{2} h_E^{1/2} \| [p_2^h \mathbf{n}_2]_E \|_{0,E} \leq \sum_{k \in \{T, T'\}} \{h_k \|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,k} + \|p_2 - p_2^h\|_{0,k}$$

$$+ h_k \|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,k}\}. \tag{78}$$

Finally, we use (65) and adding on $E \in \varepsilon_h(\Omega_2) \cap \varepsilon_h(T)$, we get the estimation (79), i.e.,

$$\frac{1}{2} \sum_{E \in \varepsilon_h(\Omega_2) \cap \varepsilon_h(T)} \| [p_2^h \mathbf{n}_2]_E \|_{0,E} \lesssim \Bigg\{ \|p_2 - p_2^h\|_{0, W_2^T}$$

$$+ \sum_{k \in W_2^T} h_k (\|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,k} + \|p_2 - p_2^h\|_{0,k}) \Bigg\}, \tag{79}$$

where W_2^T is defined as follow:

$$W_2^T := \{T' \in \mathcal{T}_2^h : \partial T \cap \partial T' \in \varepsilon_h(\overline{\Omega}_2)\}. \tag{80}$$

• To estimate $\sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_2)} h_E^{1/2} \|p_2 \mathbf{n}_2\|_{0,E}$. For each $T \in \mathcal{T}_2^h$ and for each $E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_2)$, we choose in the residual equation (39), $V = (\mathbf{v}^E, 0, 0)$, $\mathbf{v}^h = (\mathbf{0}, \mathbf{0})$, with $\mathbf{v}^E = (\mathbf{0}, \mathbf{v}_2^E)$, where

$$\mathbf{v}_2^E := \begin{cases} \mathbf{L}(p_2^h \mathbf{n}_2 b_E) & \text{on} \quad T, \\ \mathbf{0} & \text{on} \quad \Omega \setminus T. \end{cases} \tag{81}$$

Then, the residual equation becomes:

$$A(U - U_h, V) = -\int_T (\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h) \cdot \mathbf{v}_2^E + \int_E p_2^h \mathbf{n}_2 \cdot \mathbf{v}_2^E. \tag{82}$$

Hence,

$$\int_E p_2^h \mathbf{n}_2 \cdot \mathbf{v}_2^E = A(U - U_h, V) + \int_T (\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h) \cdot \mathbf{v}_2^E.$$

We use the definition of the operator A:

$$\int_E p_2^h \mathbf{n}_2 \cdot \mathbf{v}_2^E = -\int_T \mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h) \cdot v_2^E + \int_T (p_2 - p_2^h) \nabla \cdot v_2^E + \int_T (\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h) \cdot v_2^E.$$

We apply respectively, the triangular inequality and the Cauchy-Schwartz inequality:

$$\begin{aligned} \|p_2^h \mathbf{n}_2 b_E^{1/2}\|_{0,E}^2 \leq & \|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,T} \times \|v_2^E\|_{0,T} \\ & + \|p_2 - p_2^h\|_{0,T} \times \|\nabla \cdot v_2^E\|_{0,T} \\ & + \|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,T} \times \|v_2^E\|_{0,T}. \end{aligned} \tag{83}$$

Next, we apply inverse inequality (45) to (83):

$$\begin{aligned} h_E^{1/2}\|p_2^h \mathbf{n}_2 b_E^{1/2}\|_{0,E}^2 \lesssim & \ h_E^{1/2}\|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,T} \\ & \times \|v_2^E\|_{0,T} + \|p_2 - p_2^h\|_{0,T} \times \|p_2^h \mathbf{n}_2 b_E\|_{0,E} \\ & + h_E^{1/2}\|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,T} \times \|v_2^E\|_{0,T}. \end{aligned}$$

The inverse inequality (45) gives:

$$\begin{aligned} h_E^{1/2}\|p_2^h \mathbf{n}_2 b_E^{1/2}\|_{0,E}^2 \lesssim & \ (h_E\|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,T} + h_E\|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,T} \\ & + \|p_2 - p_2^h\|_{0,T}) \times \|p_2^h \mathbf{n}_2 b_E\|_{0,E}. \end{aligned}$$

The inverse inequality (44) and inequalities $0 \leq b_E \leq 1$, $h_E \leq h_T$ lead to:

$$h_E^{1/2}\|p_2^h \mathbf{n}_2\|_{0,E} \lesssim h_T\|\mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h)\|_{0,T} + h_T\|\nabla p_2^h + \mathbf{K}^{-1}\mathbf{u}_2^h\|_{0,T} + \|p_2 - p_2^h\|_{0,T}. \tag{84}$$

We combine the inequalities (84) and (65), and we have (85), i.e.,

$$\sum_{E \in \varepsilon_h(T) \cap \varepsilon_h(\Gamma_2)} h_E^{1/2} \| p_2 \mathbf{n}_2 \|_{0,E} \lesssim \sum_{k \in W_2^T} h_k \| \mathbf{K}^{-1}(\mathbf{u}_2 - \mathbf{u}_2^h) \| \|_{0,k} + \| p_2 - p_2^h \|_{0,W_2^T}. \tag{85}$$

Thus, the Proposition 3.2 is proved. □

Theorem 3.2 (Global lower error bound). *The following estimation holds*:

$$\Theta \lesssim \| \boldsymbol{u} - \boldsymbol{u}^h \|_H + \| p - p^h \|_M + \| \lambda - \lambda^h \|_{H^{1/2}(\Gamma_I)} + \zeta,$$

where ζ *is defined by* (33).

Proof. Follows directly from the Proposition 3.1 and the Proposition 3.2. □

Corollary 3.1 (Main result).

$$\frac{|\boldsymbol{Error} - \Theta|}{\zeta} = \mathcal{O}(1), \tag{86}$$

where

$$\boldsymbol{Error} := \| \boldsymbol{u} - \boldsymbol{u}^h \|_H + \| p - p^h \| + \| \lambda - \lambda^h \|_{H^{1/2}(\Gamma_I)}. \tag{87}$$

4. Conclusion and Further Works

In this paper, we have proposed and rigorously analyzed a new a posteriori residual type error estimators for the Stokes-Darcy coupled problem on isotropic meshes. Our investigations cover conforming discretization in 2D and 3D domains. The residual type a posteriori error estimator is provided. It is proven that the a posteriori error estimate provided in this paper is both reliable and efficient. There are many issues to be addressed in this area such as the other types of a posteriori error estimates, extend the residual error estimator methods to anisotropic meshes [34] and related implementation of the adaptive finite element methods.

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