

A GENERALIZED REVIEW ON HOMOGENIZATION AND TWO-SCALE CONVERGENCE

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Abstract

We study homogenization and two-scale convergence. Homogenization is a mathematical concept that makes it possible to develop a global model of the behaviour of a physical structure evolving in a heterogeneous structure. The behaviour of this physical structure will therefore be studied in a homogeneous environment, which greatly facilitates calculations. The two-scale convergence method was introduced by Nguetseng and later developed by Allaire. It is a particular form of weak convergence, a convergence between weak convergence

2010 Mathematics Subject Classification: 35B27, 76M50, 34E10.

Keywords and phrases: two-scale convergence, multiscale convergence, weak convergence, strong convergence, homogenization.

Received July 16, 2019; Revised August 27, 2019

and strong convergence. The two-scale convergence simplifies the proof of homogenization theory. The method evolved very quickly and has been extended to several cases depending on the functional space.

1. Introduction

The objective of this article is to present a general review on the theme of homogenization and two-scale convergence.

The idea of homogenization was introduced by Ivo M. Babuska in 1974, see [14-16]. After, it has been studied by many other authors. We can cite Bensoussan et al. in 1978 [19], Srinivasan Kesavan in 1979 [47], Sanchez-Palencia in 1980 [63], Bakhvalov and Panasenko in 1989 [18].

Homogenization is a mathematical theory that consists in replacing a very heterogeneous medium by a homogeneous medium which is easy to study. The two-scale convergence gives a rigorous mathematical justification of homogenization. It has been defined following an idea of Nguetseng [55] in the L^2 space. In 1992, Allaire renamed it in «Two-scale convergence». Allaire set out the essential properties and applied it to several homogenization problems. This convergence has been extended to the multiscale case by Ene and Saint Jean Paulin in 1995 [37], and by Allaire and Briane in 1996 [6]. The periodic unfolding method was introduced in 2002 by Cioranescu et al. [29, 30, 36]. It generalises the notion of multiscale convergence. Nguetseng extended the theory to the Sobolev space $W^{1,2}$ in 2002 [49]. In 2003, he extended it to the non periodic case (Σ -convergence) [56-58].

Before introducing these two notions, we will dwell on the functional spaces and the notations which intervene here.

The classic notation for a periodic function space consists in using # hashtag.

$(0, T)$ is an interval with $T > 0$.

Ω : periodic domain of \mathbb{R}^N .

Let's be \mathbb{R}_y^N (N integer ≥ 1) the numerical space on \mathbb{R}^N of variables (called microscopic) $y = (y_1, \dots, y_N)$, \mathbb{R}_x^N the numerical space on \mathbb{R}^N of variables (called macroscopic) $x = (x_1, \dots, x_N)$ and $Y = \left(-\frac{1}{2}; \frac{1}{2}\right)$ the open unit cube in \mathbb{R}_y^N .

\mathbb{R}_y^N : with Lebesgue's measure.

Y : unit cell.

$\mathcal{C}(\overline{\Omega})$ is the space of continuous functions on $\overline{\Omega}$.

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable} : \|f\|_p = \left(\int_{\Omega} \|f\|^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

$C_{\#}^0(Y)$ is a set of continuous functions and Y -periodic on \mathbb{R}^N .

$C_{\#}^k(Y)$ is a set of functions C^k -continuous and Y -periodic on \mathbb{R}^N .

$L_{\#}^p(Y)$ for $1 \leq p < \infty$ fixed, is the space of functions $w \in L_{\text{loc}}^p(\mathbb{R}_y^N)$ which is Y -periodic, i.e., which verifies $w(y+k) = w(y)$ a.a. $y \in \mathbb{R}^N$, for all $k \in \mathbb{Z}^N$.

$L_{\#}^2(Y)/\mathbb{R}$ is the set of functions $u(y)$, in $L^2(Y)$, Y -periodic and defined to the nearest constant.

$W_{\#}^{1,p}(Y)$ denotes the subset of $W^{1,p}(Y)$ of all functions u with mean value zero which have the same trace on opposite faces of Y .

$H_{\#}^1(Y)$ is the space of function $H_{\text{loc}}^1(\mathbb{R}^N)$ and Y -periodic.

$H^{-k}(\mathbb{R}^n)$ is the space of linear forms u on $H^k(\mathbb{R}^n)$.

$L^2((0, T) \times \Omega)$ is the set of functions $u(t, x)$ of square summable in $(0, T) \times \Omega$.

$L^1(\Omega; C_{\sharp}(Y))$ is the space of functions $f : \Omega \rightarrow C_{\sharp}(Y)$ which are measurable and satisfy that $\int_{\Omega} \|f(x)\|_{C_{\sharp}(Y)} dx < \infty$.

$L^2(\Omega; H_{\sharp}^1(Y) / \mathbb{R})$ is the set of summable square functions in Ω with values in $H_{\sharp}^1(Y) / \mathbb{R}$.

$C^{\infty}(\Omega)$: the space of functions indefinitely differentiable on Ω where each derivative is continuous on Ω .

$\mathcal{D}(\Omega; C_{\sharp}^{\infty}(Y))$ is the space of functions indefinitely regular with compact support in Ω to values in the space $C_{\sharp}^{\infty}(Y)$.

We say that a function $f : \Omega \rightarrow \mathbb{C}$ is with compact support if there is a compact $K \subseteq \Omega$ such that $f = 0$ on $\Omega \setminus K$.

$\mathcal{C}_c(\Omega)$ is the space of continuous functions on Ω with a compact support.

$\mathcal{C}_c^{\infty}(\Omega) = C^{\infty}(\Omega) \cap \mathcal{C}_c(\Omega)$ (some authors write $\mathcal{D}(\Omega)$ instead of $\mathcal{C}_c^{\infty}(\Omega)$): the space of functions indefinitely differentiable on Ω with compact support in Ω .

After this introduction, in the Section 2, we will define the theory of homogenization and give simple example of homogenized problem. In Section 3, we will define two-scale convergence, give some examples of problems solved with this theory and an example of solving problem with this theory.

We will end this survey by a conclusion in Section 4.

2. Homogenization Theory

The classical method of asymptotic analysis by homogenization is the asymptotic development method described, for example, in [17, 19, 62]. It is a formal method which does not provide a convergence result to the homogenized model but which, nevertheless, makes it possible to have an explicit form of this model and its effective parameters.

2.1. Problem position

We consider a diffusion or conductivity model in a periodic medium (see Figure 1). Let Ω periodic domain of \mathbb{R}^N with $N \geq 1$. Let's note by ε the period of the domain (a positive number which is supposed to be very small compared to the size of the domain), the periodic cell of the rescaled unit $Y = (0, 1)^N$. Conductivity isn't constant in Ω , but varies with a period ε in each direction. We note by $A(y)$ the second order tensor matrix, where $y = \frac{x}{\varepsilon} \in Y$ is the *rapid variable* also called *microscopic variable*, while $x \in \Omega$ is the *slow variable* also called *macroscopic variable*.

Let A be a tensor under the classic tensor assumptions, namely the existence of $0 < c < C$ such that

$$\forall \xi \in \mathbb{R}^N, c|\xi|^2 \leq (A\xi, \xi) \leq C|\xi|^2. \quad (1)$$

We note by $f(x) \in L^2(\Omega)$ the source term. The conductivity equation is as follows:

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) & = f \text{ in } \Omega, \\ u_\varepsilon & = 0 \text{ on } \partial\Omega. \end{cases} \quad (2)$$

u_ε is an unknown function.

The domain is represented below

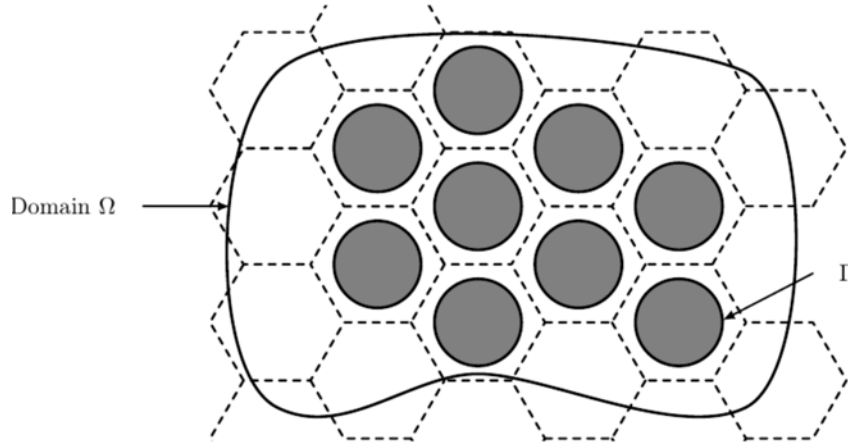


Figure 1. A periodic domain.

Remark 1. The problem (2) is well posed because $f \in L^2$, therefore according to Lax-Milgram's lemma, there is a unique solution $u_\varepsilon \in H_0^1(\Omega)$. In addition, the estimation energy gives

$$\|u_\varepsilon\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C,$$

where C is a constant independent of ε .

There are different methodologies between the traditional physical approach to homogenization and the mathematical theory of homogenization. In literature, the so-called representative volume element (RVE) method is often used (see [28] or Chapter 1 in [45]).

In the sub-section below, we will give an example of mathematical homogenization in a periodic medium.

2.2. Ansatz

The asymptotic expansion method is a heuristic method¹; for further clarification, see [18, 19, 63]. As previously stated, the starting point is asymptotic two-scale expansion, for a solution u_ε of the Equation (2)

¹a heuristic is a calculation method that provides a fast (in polynomial time) feasible, not necessarily optimal, solution for an optimization problem.

$$u_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^i u_i\left(x, \frac{x}{\varepsilon}\right)$$

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots, \quad (3)$$

where $u_i(x, y)$ is a function of two variables x and y , periodic in y with a period $Y = (0, 1)^N$ (u_i is called function Y -periodic with respect to y). The derivative of u_i and u_ε gives:

$$\nabla\left(u_i\left(x, \frac{x}{\varepsilon}\right)\right) = (\varepsilon^{-1}\nabla_y u_i + \nabla_x u_i)\left(x, \frac{x}{\varepsilon}\right), \quad (4)$$

and

$$\nabla u_\varepsilon(x) = \varepsilon^{-1}\nabla_y u_0\left(x, \frac{x}{\varepsilon}\right) + \sum_{i=0}^{+\infty} \varepsilon^i (\nabla_y u_{i+1} + \nabla_x u_i)\left(x, \frac{x}{\varepsilon}\right). \quad (5)$$

Applying that to the Equation (2) we obtain

$$\begin{aligned} & -\operatorname{div}\left(A(y)\left(\varepsilon^{-1}\nabla_y u_0(x, y) + \sum_{i=0}^{+\infty} \varepsilon^i (\nabla_y u_{i+1} + \nabla_x u_i)(x, y)\right)\right) = f(x) \\ & -\frac{1}{\varepsilon}\operatorname{div}_y\left[A(y)\left[\varepsilon^{-1}\nabla_y u_0(x, y) + \sum_{i=0}^{+\infty} \varepsilon^i (\nabla_y u_{i+1} + \nabla_x u_i)(x, y)\right]\right] \\ & -\operatorname{div}_x\left[A(y)\left[\varepsilon^{-1}\nabla_y u_0(x, y) + \sum_{i=0}^{+\infty} \varepsilon^i (\nabla_y u_{i+1} + \nabla_x u_i)(x, y)\right]\right] \\ & = f(x). \end{aligned}$$

That gives the following series in ε which will give a cascade equation:

$$\begin{aligned} & -\varepsilon^{-2}\operatorname{div}_y[A(y)[\nabla_y u_0(x, y)]] \\ & -\varepsilon^{-1}\operatorname{div}_y[A(y)[(\nabla_y u_1 + \nabla_x u_0)(x, y)]] - \varepsilon^{-1}\operatorname{div}_x[A(y)[\nabla_y u_0(x, y)]] \end{aligned}$$

$$\begin{aligned}
& - \varepsilon^0 [\operatorname{div}_y [A(y) [(\nabla_y u_2 + \nabla_x u_1)(x, y)]] + \operatorname{div}_x [A(y) [(\nabla_y u_1 + \nabla_x u_0)(x, y)]]] \\
& - \sum_{i=0}^{+\infty} \varepsilon^i [\operatorname{div}_y [A(y) [(\nabla_y u_{i+2} + \nabla_x u_{i+1})(x, y)]] \\
& + \operatorname{div}_x [A(y) [(\nabla_y u_{i+1} + \nabla_x u_i)(x, y)]]] = f(x).
\end{aligned}$$

To solve the cascade equations, we need a technical lemma:

Lemma 1 (Alternative of Fredholm ([5], p. 7)). *Let f be a function in $L^2_{\#}(Y)$ Y -periodic and A a matrix Y -periodic, coercive and bounded (in norm L^∞) of $\mathbb{R}^{N \times N}$, then the problem*

$$\begin{cases} - \operatorname{div}_y A(y) \nabla_y v(y) = f(y) & \text{in } Y \\ y \rightarrow v(y) & \text{is } Y\text{-periodic} \end{cases} \quad (6)$$

admits a single solution up to an additive constant in $H^1_{\#}(Y) / \mathbb{R}$ if and only if

$$\int_Y f(y) dy = 0.$$

Cascade equations:

- The first equation to solve is:

$$\begin{cases} \operatorname{div}_y A(y) \nabla_y u_0(x, y) = 0 & \text{in } Y, \\ y \mapsto u_0(x, y) & \text{is } Y\text{-periodic.} \end{cases}$$

In this equation y is a variable and x acts as a parameter. According to the Lemma 1, there is a single solution to this equation up to a constant. u_0 is therefore a function that does not depend on y , i.e., there is a function $u(x)$ such that

$$u_0(x, y) \equiv u(x).$$

- The second equation to solve is:

$$\begin{cases} \operatorname{div}_y A(y) (\nabla_x u_0(x, y) + \nabla_y u_1(x, y)) = 0 & \text{in } Y, \\ y \mapsto u_1(x, y) & \text{is } Y\text{-periodic.} \end{cases}$$

This equation can be written as

$$- \operatorname{div}_y A(y) \nabla_y u_1(x, y) = \operatorname{div}_y A(y) \nabla_x u_0(x, y),$$

which gives

$$- \operatorname{div}_y A(y) \nabla_y u_1(x, y) = \operatorname{div}_y A(y) \nabla_x u(x). \quad (7)$$

It's an equation of unknown u_1 in the periodic cell Y and $u_1(x, y)$ depends linearly on the first derivative $\nabla_x u(x)$. By Lemma 1, this equation also admits a unique solution up to a constant.

- The third equation to solve is:

$$\begin{cases} - \operatorname{div}_y A(y) \nabla_y u_2(x, y) = \operatorname{div}_y A(y) \nabla_x u_1(x, y) \\ \quad + \operatorname{div}_x A(y) (\nabla_x u_0(x, y) + \nabla_y u_1(x, y)) + f(x) & \text{in } Y, \\ y \mapsto u_2(x, y) & \text{is } Y\text{-periodic.} \end{cases}$$

This gives

$$\begin{aligned} - \operatorname{div}_y A(y) \nabla_y u_2(x, y) &= \operatorname{div}_y A(y) \nabla_x u_1(x, y) \\ &\quad + \operatorname{div}_x A(y) (\nabla_x u(x) + \nabla_y u_1(x, y)) + f(x). \end{aligned} \quad (8)$$

According to the Lemma 1, this problem in u_2 admits a unique solution up to a constant.

By integrating the right part of the Equation (8) on Y , and using the boundary condition for u_2 , we get

$$\int_Y \operatorname{div}_y A(y) \nabla_y u_2(x, y) dy = \int_{\partial Y} [A(y) \nabla_y u_2(x, y)] \cdot ndy = 0,$$

and

$$-\operatorname{div}_x \left(\int_Y A(y) (\nabla_x u(x) + \nabla_y u_1(x, y)) dy \right) = f(x) \text{ in } \Omega. \quad (9)$$

Since u_1 depends linearly on $\nabla_x u$, it corresponds to the following homogenized problem:

$$\begin{cases} -\operatorname{div}_x (A^* \nabla_x u_0(x)) & = f(x) \text{ in } \Omega, \\ u_0(x) & = 0 \text{ on } \partial\Omega, \end{cases} \quad (10)$$

where A^* is the homogenized conductivity matrix. Its coefficients are given by:

$$A_{i,j}^* = \int_Y A(y) (e_i + \nabla_y \omega_i) \cdot (e_j + \nabla_y \omega_j) dy = \int_Y A(y) (e_i + \nabla_y \omega_i) \cdot e_j dy, \quad (11)$$

where $(e_i)_i$ is the canonical basis of \mathbb{R}^n and w_i is the unique solution of the cell problem below

$$\begin{cases} -\operatorname{div}_y A(y) (e_i + \nabla_y \omega_i(y)) = 0 & \text{in } Y, \\ y \mapsto \omega_i(y) & \text{is } Y\text{-periodic.} \end{cases} \quad (12)$$

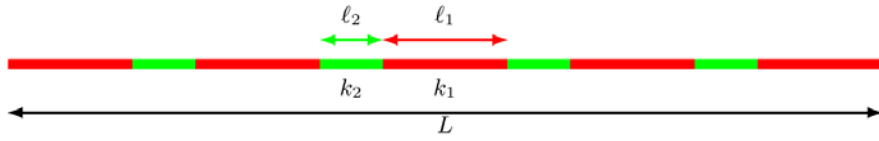
2.3. Example of homogenized problem in one dimensional case ([24], pp. 4-18)

The study of the monodimensional case is interesting because it makes it possible to calculate the exact solution and to see its evolution when the number of periods increases.

2.3.1. Presentation of the problem

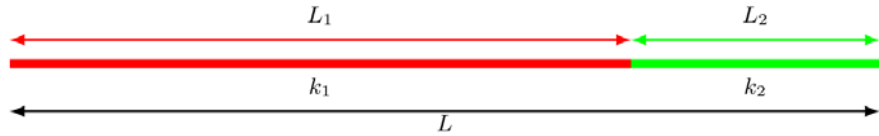
Let a periodic elastic bar, that is to say constituted by the regular juxtaposition of N_c identical patterns made up of two homogeneous

materials. The stiffnesses are k_1 and k_2 of respective lengths ℓ_1 and ℓ_2 . The pattern length is $\ell = \ell_1 + \ell_2$ and the bar length is $L = N_c \ell$.



The bar being made up of two different media is heterogeneous, its equivalent rigidity is a periodic function x , where x varies by taking the points of the bar. For asymptotic developments we will consider a function $k(y)$, set in \mathbb{R} . The period is divided into two parts, the first having the length $L_1 = N_c \ell_1$ and the second $L_2 = N_c \ell_2$ where the function $k(y)$ takes the respective values k_1 or k_2 . The bar is reconstituted by grouping together the equally stiff mediums. One obtains a bar in two parts : that of rigidity k_1 and that of rigidity k_2 . We define $k(y)$ as follows :

$$k(y) = \begin{cases} k_1 & 0 < y < L_1, \\ k_2 & L_1 < y < L. \end{cases} \quad (13)$$



The bar has at point x a stiffness which is $k(N_c)$. Indeed this function is periodic of period $\frac{L}{N_c} = \ell$ and it takes the value k_1 on the part of the length of the bar $\frac{L_1}{N_c} = \ell_1$ and k_2 on the part of the length of the bar $\frac{L_2}{N_c} = \ell_2$. Let us now consider several bars of the same length L ,

composed of a number N_c of variable but large basic patterns. With a very large N_c , let's pose the small variable

$$\varepsilon = \frac{1}{N_c}. \quad (14)$$

So we have $\ell = \varepsilon L$, $\ell_1 = \varepsilon L_1$ and $\ell_2 = \varepsilon L_2$ and the stiffness of the bar at point x is $k\left(\frac{x}{\varepsilon}\right)$.

Remark 2. The bar stiffness also applies if the periodic stiffness of the bar is not constant in pieces. To do this we can simply change the definition (13) of the periodic function $k(y)$ on \mathbb{R} . The period of the function $k(y)$ is $[0; L]$ on \mathbb{R} .

Now let us take a bar fixed in $x = 0$, subjected to a longitudinal force density f and a force F at point $x = L$.

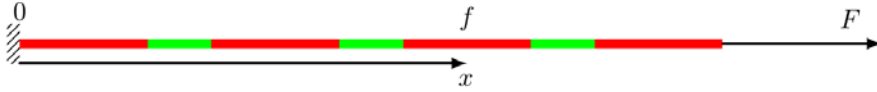


Figure 2. Loading of the bar.

2.3.2. Equations of the problem

Note N the normal force in the bar, u the points of the bar displacement. They check the following equations:

Equation of equilibrium:

$$\frac{dN}{dx} + f = 0. \quad (15)$$

Law of behaviour:

$$N = k\left(\frac{x}{\varepsilon}\right) \frac{du}{dx}. \quad (16)$$

Boundary conditions:

$$u(0) = 0 \quad \text{and} \quad N(L) = F. \quad (17)$$

2.3.3. Analytical solutions

The problems (15), (16), (17) being one-dimensional, it is easy to determine the exact solution, in order to study the dependence of this solution on the number of patterns in the bar.

After an integration of Equation (15) taking into account the boundary conditions (17), we obtain

$$N(x) = \int_x^L f(\xi) d\xi + F, \quad (18)$$

by integrating this expression in (16), we have

$$\frac{du}{dx} = \frac{1}{k\left(\frac{x}{\varepsilon}\right)} \left(\int_x^L f(\xi) d\xi + F \right).$$

In a general case, we integrate this equation by using a primitive; in the considered case where the stiffness is constant in pieces on a segment $[a, b]$ we take for stiffness k_i . So we have

$$u(x) = u(a) + \frac{1}{k_i} (P_N(x) - P_N(a)),$$

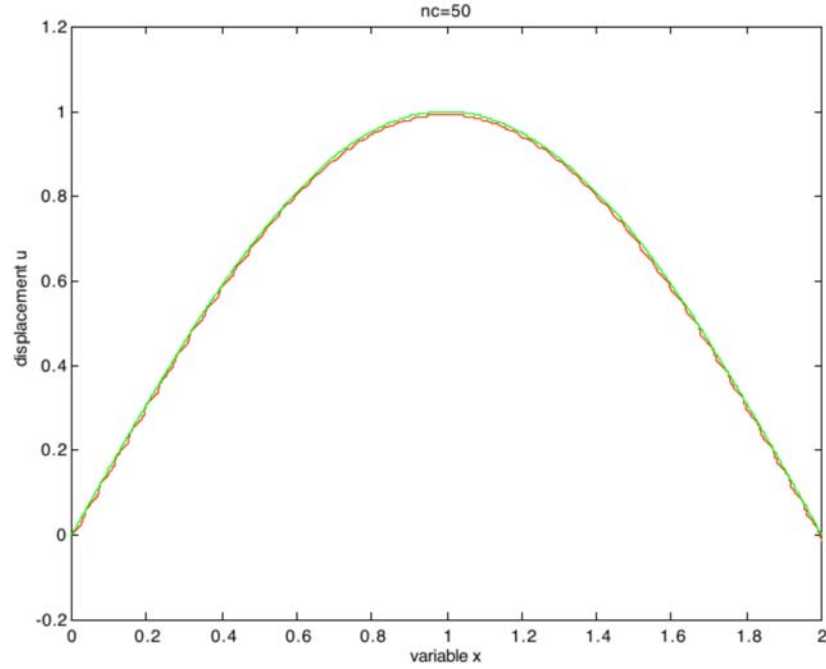
where $P_N(x)$ is a primitive of N on the segment $[a, b]$. In the segment $[0, \ell_1]$, we have $u(\ell_1) = \frac{1}{k_1} (P_N(\ell_1) - P_N(0))$ which allows to determine u on the segment $[\ell_1, \ell_2]$ and we continue step by step.

2.3.4. Numerical examples

Let us give values to the different geometrical and mechanical data:

$$L = 2; L_1 = 1.4; L_2 = 0.6; k_1 = 1.5; k_2 = 0.562.$$

Let us take $N_c = 50$, $f = -\frac{\pi^2}{L^2} \sin\left(\frac{\pi}{L} x\right)$, $F = -\frac{\pi}{L}$, by plotting on the same graph the movements of the heterogeneous bar and the homogeneous global stiffness bar k^G , we obtain the following figure:

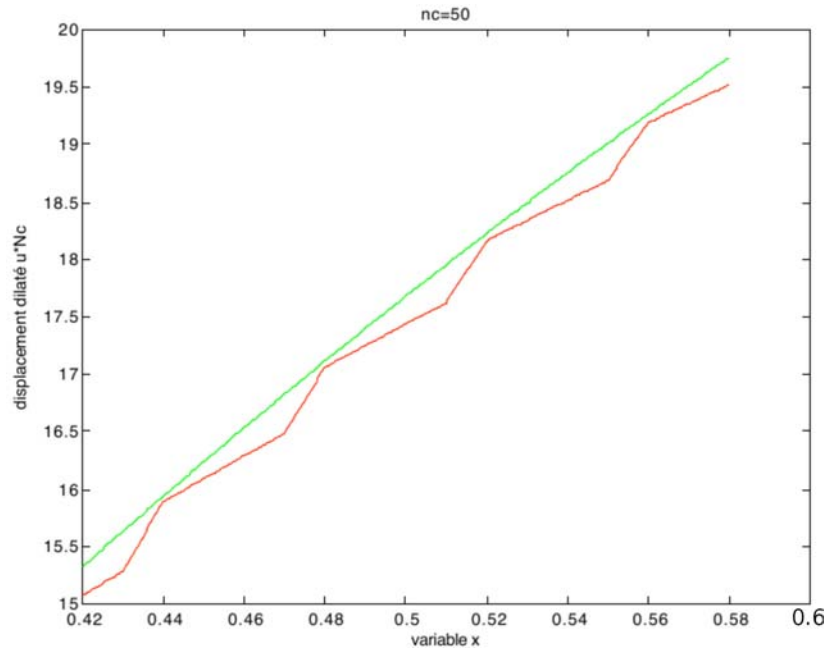


So, we notice « experimentally » that for N_c rather large, the displacement graph is confused with that of a homogeneous bar of stiffness k^G .

2.3.5. Heuristic method

The previous result can be experimentally justified by the « heuristic » method introduced by Sanchez and developed by Suquet. To do this, the graphs in the N_c ratio are magnified around the point $x = 0.5$.

Let us take $f = -\frac{\pi^2}{L^2} \sin\left(\frac{\pi}{L} x\right)$, $F = -\frac{\pi}{L}$.



We can see that the variation in the exact displacement u (in red) over a period of time $[x; x + \ell]$ is practically the same as the limit displacement U (in green), i.e.,

$$u(x + \ell) - u(x) \simeq \ell \frac{du}{dx}. \quad (19)$$

For small ℓ , the variation in normal effort over a period of time $[x; x + \ell]$ is negligible. Therefore, the period over the normal effort N is roughly constant. So the integration of the law of behaviour $\frac{du}{d\xi} = \frac{N}{k\left(\frac{\xi}{\varepsilon}\right)}$ over a

period of time $[x; x + \ell]$ gives

$$u(x + \ell) - u(x) = N \int_x^{x+\ell} \frac{1}{k\left(\frac{\xi}{\varepsilon}\right)} d\xi,$$

which gives the approximate condition of (19), results in

$$N \int_x^{x+\ell} \frac{1}{k\left(\frac{\xi}{\varepsilon}\right)} d\xi = \ell \frac{dU}{dx}.$$

That is the law of macroscopic behaviour:

$$N = k^H \frac{dU}{dx}, \quad (20)$$

where the homogenized stiffness k^H is given by

$$\frac{1}{k^H} = \frac{1}{\ell} \int_x^{x+\ell} \frac{1}{k\left(\frac{\xi}{\varepsilon}\right)} d\xi. \quad (21)$$

By making a variable change and taking into account the periodicity of $k(y)$, the homogenized stiffness k^H is independent of x .

$$\frac{1}{k^H} = \frac{1}{L} \int_0^L \frac{1}{k(y)} dy.$$

In the case considered where the stiffness $k(y)$ is constant in pieces, it is easily verified that:

$$\frac{L}{k^H} = \frac{L_1}{k_1} + \frac{L_2}{k_2}. \quad (22)$$

The stiffness k^H is well equal to the overall stiffness k^G .

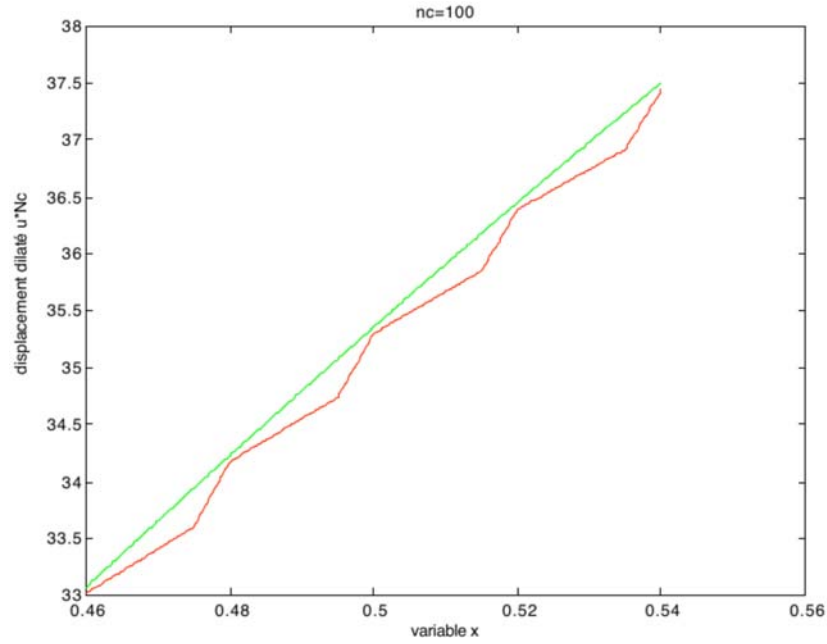
2.3.6. Double scale development

For a large number N_c , the displacement of the heterogeneous bar under a given load is close to that of a homogeneous bar under the same load. When $\varepsilon \rightarrow 0$ we can approximate the displacement of the heterogeneous bar by that of the homogeneous bar. « homogenize » is like stretching $\varepsilon \rightarrow 0$.

The displacements as well as the voltages in these bars depend on the parameter ε and are noted u_ε and N^ε .

The determination of the limits for $\varepsilon \rightarrow 0$ of these quantities is an asymptotic study that can be done by constructing an asymptotic development of u_ε and N^ε with respect to ε .

Let us take $f = -\frac{\pi^2}{L^2} \sin\left(\frac{\pi}{L} x\right)$, $F = -\frac{\pi}{L}$.



The difference between the exact solution u and the limit solution U is a perturbation which is in the order of $\frac{1}{N_c} = \varepsilon$ and this perturbation is, at least locally, periodic of period ℓ . That can be written locally:

$$u(x) \approx U(x) + \varepsilon v(x),$$

where v is a locally periodic function of period ℓ .

This approximation is a beginning of development that is generalized and systematized in the form of

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots, \quad (23)$$

where $u_i(x, y)$ is a function y -periodic of period Y .

This development is called double scale development because it shows two scales of variation.

Indeed, a variation of x in the order of ε causes variations of ε order of $u_n(x, y)$ relative to the first variable but of order 1 relative to the second. Variable x is called a large scale variable and y a small scale variable.

Let \bar{x} and $\bar{x} + \varepsilon L$ be two points on the bar, according to the development (23), we have

$$\begin{aligned} u_\varepsilon(\bar{x}) &= u_0(\bar{x}) + \varepsilon u_1\left(\bar{x}, \frac{\bar{x}}{\varepsilon}\right) + \varepsilon^2(\dots), \\ u_\varepsilon(\bar{x} + \varepsilon L) &= u_0(\bar{x} + \varepsilon L) + \varepsilon u_1\left(\bar{x} + \varepsilon L, \frac{\bar{x}}{\varepsilon} + L\right) + \varepsilon^2(\dots). \end{aligned}$$

Let us consider the periodicity of the functions u_n and using Taylor developments:

$$u_\varepsilon(\bar{x} + \varepsilon L) = u_0(\bar{x}) + \varepsilon \left(u_1\left(\bar{x}, \frac{\bar{x}}{\varepsilon}\right) + \frac{du_0}{dx}(\bar{x})L \right) + \varepsilon^2(\dots),$$

and by difference:

$$u_\varepsilon(\bar{x} + \varepsilon L) - u_0(\bar{x}) = \varepsilon \frac{du_0}{dx}(\bar{x})L + \varepsilon^2(\dots).$$

We find, at order ε^2 near equality:

$$u_\varepsilon(\bar{x} + \varepsilon L) - u_0(\bar{x}) = \varepsilon \frac{du_0}{dx}(\bar{x})L,$$

that was used in the heuristic method.

By derivation of (23), it comes

$$\begin{aligned} \frac{du_\varepsilon}{dx}(x) &= \frac{1}{\varepsilon} \frac{du_0}{dy}\left(x, \frac{x}{\varepsilon}\right) + \frac{du_0}{dx}\left(x, \frac{x}{\varepsilon}\right) + \frac{du_1}{dy}\left(x, \frac{x}{\varepsilon}\right) \\ &\quad + \varepsilon \left(\frac{du_1}{dx}\left(x, \frac{x}{\varepsilon}\right) + \frac{du_2}{dy}\left(x, \frac{x}{\varepsilon}\right) \right) + \dots, \end{aligned}$$

which reported in the law of behaviour (16), shows that the normal force N_ε develops in

$$N_\varepsilon(x) = \frac{1}{\varepsilon} N^{(-1)}\left(x, \frac{x}{\varepsilon}\right) + N^{(0)}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon N^{(1)}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 N\left(x, \frac{x}{\varepsilon}\right) + \dots,$$

where the $N^{(n)}(x, y)$ are y -periodic of period Y and verify by identification terms of the same power of ε :

$$N^{(-1)} = k(y) \frac{du_0}{dy}, \quad (24a)$$

$$N^{(0)} = k(y) \left(\frac{du_0}{dx} + \frac{du_1}{dy} \right), \quad (24b)$$

$$N^{(1)} = k(y) \left(\frac{du_1}{dx} + \frac{du_2}{dy} \right), \quad (24c)$$

.....

By derivation, we obtain by omitting the variables:

$$\begin{aligned} \frac{dN_\varepsilon}{dx} &= \frac{1}{\varepsilon^2} \frac{dN^{(-1)}}{dy} + \frac{1}{\varepsilon} \left(\frac{dN^{(-1)}}{dx} + \frac{dN^{(0)}}{dy} \right) \\ &\quad + \frac{dN^{(0)}}{dx} + \frac{dN^{(1)}}{dy} + \varepsilon \left(\frac{dN^{(1)}}{dx} + \frac{dN^{(2)}}{dy} \right) + \dots. \end{aligned}$$

The transfer of this development into the equation of equilibrium (15) gives after identification of terms of the same powers of ε

$$\frac{dN^{(-1)}}{dy} = 0, \quad (25a)$$

$$\frac{dN^{(-1)}}{dx} + \frac{dN^{(0)}}{dy} = 0, \quad (25b)$$

$$\frac{dN^{(0)}}{dx} + \frac{dN^{(1)}}{dy} + f = 0, \quad (25c)$$

.....

From Equation (25a) we deduce that $N^{(-1)}$ does not depend on y (which is obviously compatible with the periodicity in y). Equation (24a) is then integrated in

$$u_0(x, y) = N^{(-1)}(x) \int_0^y \frac{1}{k(\zeta)} d\zeta + \tilde{u}_0(x),$$

where $\tilde{u}_0(x)$ is an integration constant for the differential equation in y (25a), i.e., a function of the large-scale variable x alone.

The periodicity condition $u_0(x, L) = u_0(x, 0)$ is equivalent to:

$$N^{(-1)} \int_0^L \frac{1}{k(\zeta)} d\zeta = 0,$$

k being strictly positive, we have

$$N^{(-1)} = 0,$$

and

$$u_0(x, y) = \tilde{u}_0(x),$$

i.e., u_0 depends only on the large-scale variable x .

To obtain the «macroscopic» equations satisfied by u_0 , we use Equations (25) and (24).

$N^{(-1)}$ being zero, Equation (25b) implies that $N^{(0)}$ does not depend on y . The integration on $[0; L]$ of Equation (24b) rewritten:

$$\frac{du_0}{dx} + \frac{du_1}{dy} = \frac{1}{k(y)} N^{(0)}, \quad (26)$$

then gives, taking into account the periodicity of u_1 and the independence of u_0 with respect to y :

$$L \frac{du_0}{dx} = N^{(0)} \int_0^L \frac{1}{k(\zeta)} d\zeta.$$

That we can write

$$N^{(0)} = k^H \frac{du_0}{dx}, \quad (27)$$

with in the case considered where k is constant in pieces

$$\frac{L}{k^H} = \frac{L_1}{k_1} + \frac{L_2}{k_2}.$$

This law of macroscopic behaviour is exactly the law (20) found by the heuristic method.

To determine the equilibrium equation verified by $N^{(0)}$, we integrate Equation (25c) on $[0; L]$, which gives, considering the periodicity of $N^{(1)}$:

$$\frac{dN^{(0)}}{dx} + \langle f \rangle = 0, \quad (28)$$

where the average $\langle f \rangle$ of f is defined by

$$\langle f \rangle = \frac{1}{L} \int_0^L f(y) dy. \quad (29)$$

The asymptotic method thus made it possible to determine the macroscopic equivalent model of the finely heterogeneous bar constituted of the homogenized constitutive law (27) and the macroscopic equilibrium equation (28).

2.3.7. Convergence

The results of the asymptotic double scale developments of the previous paragraph remain formal until such time as convergence has not been proven. Several convergence methods have been applied to the homogenization of periodic media, G -convergence, Γ -convergence, H -convergence, the so-called « Tartar » method, two-scale convergence, bursting method. The two-scale convergence method will be studied in the next section.

3. Two-Scale Convergence

This section is devoted to the concept of two-scale convergence and some of its properties. We highlight how some results regarding two-scale convergence relate to traditional results of weak and strong convergence.

Weak convergence does not keep information on the local behaviour of functions. This loss of information in weak limit causes some “unpleasant” properties. The study of the function $\sin\left(\frac{2\pi x}{\varepsilon}\right)$ is a particular example.

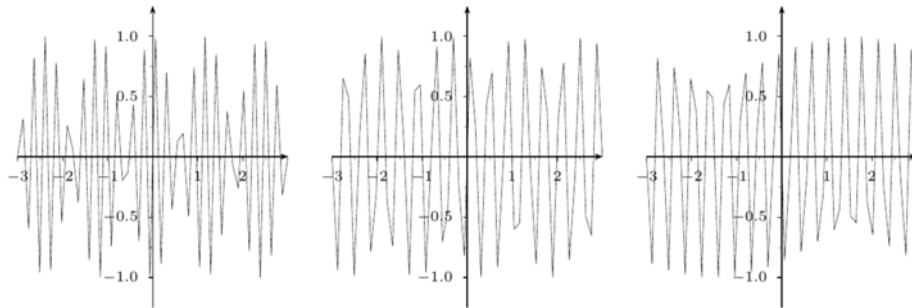


Figure 3. Graph of $\sin\left(\frac{2\pi x}{\varepsilon}\right)$ with $\varepsilon = \frac{1}{20}, \frac{1}{120}, \frac{1}{520}$.

If we take $u_\varepsilon(x) = v_\varepsilon(x) = \sin\left(\frac{2\pi x}{\varepsilon}\right)$, u_ε and v_ε converge weakly in $L^2(0, \pi)$ to 0. But the product $u_\varepsilon v_\varepsilon$ converges to $\frac{1}{2}$. The previous example shows that the evolution of a problem is linked to small perturbations. These perturbations cannot be neglected because they can lead to completely divergent results in reality.

The two-scale convergence allows to capture the oscillations of a function in reasoning with that of the type $\varphi\left(x, \frac{x}{\varepsilon}\right)$, a periodic test function. In some cases, therefore, two-scale convergence makes it possible to overcome the loss of information on local behaviour, it is an intermediate convergence between weak convergence and strong convergence. The test function $\varphi\left(x, \frac{x}{\varepsilon}\right)$ being a function of one variable becomes by passing to the limit two-scale, a function $\varphi(x, y)$ dependent on two variables. The variable $y = \frac{x}{\varepsilon} \in Y$ is the *fast variable* also called *microscopic variable*, while $x \in \Omega$ is the *slow variable* also called *macroscopic variable*.

In this section, we will give first the essential definitions of two-scale convergence in L^P . Secondly, we will give some homogenization problems solved by two-scale convergence. And we will finish by an application of two-scale convergence to a homogenization problem.

3.1. Definition and essential properties of two-scale convergence in L^P

In this part, we give the definition of two-scale convergence as well as some properties and examples related to two-scale convergence.

The idea of two-scale convergence consists in moving from a sequence of one variable functions to two variables.

We pose

$$\|w\|_p = \left(\int_Y |w(y)|^p dy \right)^{\frac{1}{p}}, \quad w \in L^p_{\#}(Y).$$

We define a norm in $L^p_{\#}(Y)$ which in fact is a Banach space (a Hilbert if $p = 2$).

3.1.1. Definition of two-scale convergence

For all function $\psi \in L^p(\Omega; C_{\#}(Y))$, we designate by ψ^ε the function of $L^p(\Omega)$ defined by

$$\psi^\varepsilon(x) = \psi\left(x, \frac{x}{\varepsilon}\right), \quad x \in \Omega, \quad \varepsilon > 0.$$

We give a fundamental result.

Lemma 2 ([55], p. 1). *When $\varepsilon \rightarrow 0$, we have $\psi^\varepsilon \rightarrow \tilde{\psi}$ in $L^p(\Omega)$ -weak, where $\tilde{\psi}(x) = \int_Y \psi(x, y) dy$ is a $L^p(\Omega)$ function.*

In other words:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \psi^\varepsilon(x) \varphi(x) dx = \int_{\Omega} \tilde{\psi}(x) \varphi(x) dx = \int_{\Omega} \int_Y \psi(x, y) \varphi(x) dy dx, \quad \forall \varphi \in L^q(\Omega),$$

$$\text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Let's define now the notion of two-scale convergence.

Definition 1. A sequence (generalize) $(u_\varepsilon)_{\varepsilon > 0}$ of functions in $L^p(\Omega)$ is said two-scale convergent in $L^p(\Omega)$ to a function $u_0 \in L^p(\Omega \times Y)$, if for $\varepsilon \rightarrow 0$, we have

$$\int_{\Omega} u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx \rightarrow \int_{\Omega} \int_Y u_0(x, y) \psi(x, y) dy dx, \quad \forall \psi \in L^q(\Omega; C_{\#}(Y)).$$

(30)

Remark 3. Nguetseng has also extended two-scale convergence beyond periodicity, see [56].

Remark 4 ([53], p. 17). u_ε means u_{ε_n} with $\varepsilon_n \xrightarrow[n \rightarrow +\infty]{} 0$. The n index will be omitted routinely.

Remark 5 ([49], p. 17). There is a confusion in literature about the choice of the space of the test function. Some authors take the test function in $\mathcal{D}(\Omega; C_{\sharp}^{\infty}(Y))$ instead of $L^P(\Omega)$.

In the Example 4 ([49], p. 11), the authors give an example of a sequence (u_ε) in $L^P(\Omega)$ satisfying (30) for $\psi \in \mathcal{D}(\Omega; C_{\sharp}^{\infty}(Y))$ but which is neither bounded nor weakly convergent in $L^P(\Omega)$.

In the Example 5 ([49], p. 11), they replace $\mathcal{D}(\Omega; C_{\sharp}^{\infty}(Y))$ by $C(\Omega; C_{\sharp}^{\infty}(Y))$ and they give an example of test function $\psi \in C(\Omega; C_{\sharp}^{\infty}(Y))$, for which the sequence (u_ε) does not converge weakly in $L^P(\Omega)$ to the function $v(x)$ defined by $\int_Y u(x, y) dy$. This sequence is not bounded also.

Then, when (u_ε) is bounded in $L^P(\Omega)$ we can take $\psi(x, y)$ in $\mathcal{D}(\Omega; C_{\sharp}^{\infty}(Y))$.

Proposition 1 ([49], p. 12). *Let (u_ε) be a bounded sequence in $L^P(\Omega)$ such that*

$$\int_{\Omega} u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx \rightarrow \int_{\Omega} \int_Y u(x, y) \psi(x, y) dy dx$$

for every $\psi \in \mathcal{D}(\Omega; C_{\sharp}^{\infty}(Y))$. Then u_ε two-scale converges to u .

Proposition 2 ([49], p. 6). *Let's be ψ a continuous function from $\mathbb{R}_x^N \times \mathbb{R}_y^N$ to \mathbb{R} . We suppose again that for all $x = (x_1, x_2, \dots, x_N)$, the partial function $y \rightarrow \psi(x, y)$ from \mathbb{R}_y^N to \mathbb{R} is Y -periodic. Then, we suppose that Ω is bounded, and we pose*

$$u_\varepsilon(x) = \psi\left(x, \frac{x}{\varepsilon}\right), \quad x \in \Omega, \quad (31)$$

for all $\varepsilon > 0$. The sequence $(u_\varepsilon)_{\varepsilon > 0}$ is weakly two-scale convergent in $L^p(\Omega)$ to the function $u_0 \in L^p(\Omega; L^p_\#(Y))$ defined by

$$u_0(x, y) = \psi(x, y), \quad x \in \Omega, y \in \mathbb{R}_y^N. \quad (32)$$

Proof. Let's show that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y u_0(x, y) \varphi(x, y) dy dx \quad \forall \varphi \in L^q(\Omega, C_\#(Y)),$$

i.e.,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \psi\left(x, \frac{x}{\varepsilon}\right) \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y \psi(x, y) \varphi(x, y) dy dx.$$

Let's pose that $\eta^\varepsilon(x) = \psi\left(x, \frac{x}{\varepsilon}\right) \varphi\left(x, \frac{x}{\varepsilon}\right)$.

Obviously we have $u_\varepsilon \in L^p(\Omega)$, because u_ε is continuous and Ω is bounded.

We pose $v_\varepsilon(x) = \varphi\left(x, \frac{x}{\varepsilon}\right)$.

$v_\varepsilon \in L^p(\Omega)$ because Ω is bounded. Then $\eta^\varepsilon(x) \in L^p(\Omega)$.

According to the Lemma 2, we have $\eta^\varepsilon(x) \rightarrow \eta(x)$ in $L^p(\Omega)$ -weak where $\eta(x) = \int_Y \psi(x, y) \varphi(x, y) dy$. In other words, we have:

$$\int_{\Omega} \eta^\varepsilon(x) \alpha(x) dx \rightarrow \int_{\Omega} \int_Y \psi(x, y) \varphi(x, y) \alpha(x) dy dx \quad \forall \alpha \in L^q(\Omega).$$

In particular $\alpha(x) = 1$ (Ω being bounded) leads to

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \psi\left(x, \frac{x}{\varepsilon}\right) \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y \psi(x, y) \varphi(x, y) dy dx.$$

Proposition 3 ([G. Nguetseng]). *Let's be ψ the function from $\mathbb{R}_x^N \times \mathbb{R}_y^N$ to \mathbb{R} given by*

$$\psi(x, y) = \varphi(x)u(y), \quad x \in \mathbb{R}^N, y \in \mathbb{R}^N,$$

where φ is continuous with compact support in \mathbb{R}_x^N , and $u \in C_{\#}^0(Y)$.

We define u_{ε} like in (31), where Ω isn't supposed bounded.

Then, the sequence $(u_{\varepsilon})_{\varepsilon > 0}$ weakly two-scale converges in $L^p(\Omega)$ to the function u_0 given by (32).

Proof. Let's show that $\int_{\Omega} \varphi(x)u(x/\varepsilon)h(x, x/\varepsilon)dx \rightarrow \int_{\Omega} \int_Y \varphi(x)u(y)h(x, y)dydx$, $\forall h \in L^q(\Omega \times Y)\mathcal{D}(\Omega \times Y)$ is dense in $L^q(\Omega \times Y)$ so it comes down to showing that

$$\int_{\Omega} \varphi(x)u(x/\varepsilon)h(x, x/\varepsilon)dx \rightarrow \int_{\Omega} \int_Y \varphi(x)u(y)h(x, y)dydx, \quad \forall h \in \mathcal{D}(\Omega \times Y),$$

$\mathcal{D}(\Omega) \otimes \mathcal{D}(Y)$ is dense in $\mathcal{D}(\Omega \times Y)$ so it comes down to showing that

$$\int_{\Omega} \varphi(x)u(x/\varepsilon)h_1(x)h_2(x/\varepsilon)dx \rightarrow \int_{\Omega} \int_Y \varphi(x)u(y)h_1(x)h_2(y)dydx, \quad \forall h_1 \in \mathcal{D}(\Omega),$$

$$\forall h_2 \in \mathcal{D}(Y)$$

$$\int_{\Omega} \varphi(x)u(x/\varepsilon)h_1(x)h_2(x/\varepsilon)dx = \int_{\Omega} \varphi(x)u(x/\varepsilon)h_2(x/\varepsilon)h_1(x)dx.$$

Let's pose $v(x / \varepsilon) = u(x / \varepsilon)h_2(x / \varepsilon)$ we have $\int_{\Omega} \varphi(x)u(x / \varepsilon)h_2(x / \varepsilon)h_1(x)dx$
 $= \int_{\Omega} \varphi(x)v(x / \varepsilon)h_1(x)dx$. φ is continuous with compact support and
 $v(x / \varepsilon) \in C_{\sharp}^1(Y)$ then $\varphi(x)v(x / \varepsilon)$ is continuous with compact support.

According to the Lemma 2, $\int_{\Omega} \varphi(x)v(x / \varepsilon)h_1(x)dx \rightarrow \int_{\Omega} \int_Y \varphi(x)v(y)h_1$
 $(x)dydx$.

We conclude that

$\int_{\Omega} u_{\varepsilon}(x)h(x, x / \varepsilon)dx \rightarrow \int_{\Omega} \int_Y u_0(x, y)h(x, y)dydx, \forall h \in L^q(\Omega \times Y)$, where
 $u_0(x, y) = \psi(x, y) = \varphi(x)v(y)$.

Proposition 4. *Let's be ψ the function from $\mathbb{R}_x^N \times \mathbb{R}_y^N$ to \mathbb{R} given by*

$$\psi(x, y) = \varphi(x)u(y), \quad x \in \Omega \subset \mathbb{R}^N, y \in \mathbb{R}^N,$$

where φ is bounded with compact support in \mathbb{R}_x^N , and $u \in C_{\sharp}^1(Y)$.

We define u_{ε} like in (31), where Ω isn't supposed bounded.

Then, the sequence $(u_{\varepsilon})_{\varepsilon>0}$ weakly two-scale converges in $L^p(\Omega \times Y)$
to the function u_0 given by (32).

Proof. Let's show that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_{\varepsilon}(x)h(x, x / \varepsilon)dx = \int_{\Omega} \int_Y u_0(x, y)h(x, y)dydx \quad h \in L^q(\Omega \times Y).$$

We have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_{\varepsilon}(x)h(x, x / \varepsilon)dx = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi(x)u(x / \varepsilon)h(x, x / \varepsilon)dx.$$

φ is bounded with compact support and u is continuous $\Rightarrow \varphi \otimes u$ is bounded then $\varphi \otimes u \in L^\infty(\Omega)$. As $\varphi \otimes u$ is with compact support then $\varphi \otimes u \in L^r(\Omega) \forall r$.

It is sufficient that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ so that $u_\varepsilon(x) = \varphi(x)u(x/\varepsilon) \in L^p(\Omega)$.

According to the Lemma 2,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi(x)u(x/\varepsilon)h(x, x/\varepsilon)dx &= \int_{\Omega} \int_Y \varphi(x)u(y)h(x, y)dydx \\ &= \int_{\Omega} \int_Y u_0(x, y)h(x, y)dydx. \end{aligned}$$

Now we will give some easy examples of sequences which two-scale converges.

3.1.2. Examples

The notation is the same like the above.

Example 1. Let's take $N = 1$, $1 \leq p < +\infty$, $\Omega = [0, 1[$ and $Y = [0; 2\pi]$.

We pose $\psi(x, y) = \sin(2\pi y)$, $u_\varepsilon(x) = \sin(2\pi x/\varepsilon)$, u_ε is continuous on Ω and Ω is bounded then $u_\varepsilon \in L^p(\Omega)$.

$y \rightarrow \psi(x, y) = \sin(2\pi y)$ is Y -periodic.

According to the Proposition 2, we deduce that $\sin(2\pi x/\varepsilon)$ converges two-scale in $L^p(\Omega)$ to $\sin(2\pi y) \in L^p(\Omega; L^p_{\#}(Y))$.

Example 2. Let's take $N = 1$, $1 \leq p < +\infty$, $\Omega = \mathbb{R}$ and $Y = [0; 1]$.

We pose $\psi(x, y) = \varphi(x)u(y)$ with $\varphi(x) = 2(x)$ if $x \in [0, 1/2[$, $-2x + 2$ if $x \in [1/2; 1[$ and 0 elsewhere, we take $u(y) = \sin(2\pi y)$.

φ is continuous with compact support in \mathbb{R}_x .

$u(y)$ is continuous and Y -periodic.

We define $u_\varepsilon(x) = \psi(x, x/\varepsilon)$, $\forall x \in \Omega$.

According to the Proposition 3, we deduce that $\varphi(x)\sin(2\pi x/\varepsilon)$ converges two-scale in $L^p(\Omega)$ to $\varphi(x)\sin(2\pi y) \in L^p(\Omega; L^p_\#(Y))$.

Example 3. If we take $u_\varepsilon(x) = \frac{1}{2}\sin\left(\frac{x}{\varepsilon}\right) + \frac{1}{2}\sin(x) + 1$ with $b_1(x) = \frac{1}{2}$, $a(x/\varepsilon) = \sin\left(\frac{x}{\varepsilon}\right)$ and $b_2(x) = \frac{1}{2}\sin(x) + 1$. $a(x/\varepsilon)$ converges weakly to 0, then $u_\varepsilon(x)$ converges weakly to $\frac{1}{2}\sin(x) + 1$ and it two-scale converges (strongly) to $\frac{1}{2}\sin(y) + \frac{1}{2}\sin(x) + 1$.

Remark 6. To better understand this example, see properties (4) and (5) on the next page.

3.1.3. General properties

Theorem 1 ([49], p. 6). *Let $f(x, y) \in L^1(\Omega, C_\#(Y))$. Then $f\left(x, \frac{x}{\varepsilon}\right)$ is a measurable function on Ω such that*

$$\left\|f\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^1(\Omega)} \leq \|f(x, y)\|_{L^1(\Omega, C_\#(Y))} = \int_{\Omega} \sup_{y \in Y} |f(x, y)| dx,$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} f\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y f(x, y) dy dx.$$

Proof. For the proof, see ([49], pp. 6-7).

We now expose a result of compactness for two-scale convergence.

Theorem 2 ([49], p. 13). *From any bounded sequence u_ε of $L^p(\Omega)$, we can extract a subsequence that converges two-scale to a limit $u_0(x, y) \in L^p(\Omega \times Y)$.*

Proof. For the proof, see ([49], p. 13-14).

Simple cases of two-scale convergence (see [5], pp. 15-16) for properties (1) to (3) and ([53], p. 107-108) for properties (4) to (7).

(1) Any sequence u_ε which converges strongly in $L^2(\Omega)$ to a limit $u(x)$, two-scale converges to the same limit $u(x)$.

(2) For any smooth function $u_0(x, y)$, being Y -periodic in y , the associated sequence $u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right)$ two-scale converges to $u_0(x, y)$.

(3) Any sequence u_ε which admits an asymptotic expansion of type $u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots$; where the functions $u_i(x, y)$ are smooth and Y -periodic in y , converges two-scale to the first term of the expansion, namely $u_0(x, y)$ since the Theorem 1 implies that

$$\int_{\Omega} u_i\left(x, \frac{x}{\varepsilon}\right) \psi\left(x, \frac{x}{\varepsilon}\right) dx \rightarrow \int_{\Omega} \int_Y u_0(x, y) \psi(x, y) dy dx \text{ for all } \psi \in L^p(\Omega, C_{\#}^i(Y)).$$

(4) Let $a(y)$ be a Y -periodic bounded function with zero mean value $\int_Y a(y) dy = 0$ and let $b_1(x), b_2(x) \in L^p(\Omega)$. Then, the sequence (u_ε) defined by $u_\varepsilon(x) = b_1(x)a(x/\varepsilon) + b_2(x)$ converges weakly to $b_2(x)$ and it two-scale converges (strongly) to $b_1(x)a(y) + b_2(x)$. We see that the weak limit is the function b_2 only.

(5) Let us consider the same functions $a(y), b_1(x), b_2(x)$, but another sequence (v_ε) defined by $v_\varepsilon(x) = b_1(x)a(x/\varepsilon^2) + b_2(x)$. Then the two-scale and weak limits coincide, which means that the two-scale limit is constant in the variable y . In this case, the information on oscillations is not kept. Similarly, taking a sequence given by $w_\varepsilon(x) = b_1(x)a(cx\varepsilon) + b_2$ with c irrational, the two-scale limit equals to b_2 only. It is the consequence that diminishing the periods in sequences is not in resonance with the periods in the test function.

(6) Let $u(y)$ be a Y -periodic function, we consider the functions $u_0(x, y) = u(y)$ and $\tilde{u}_0(y) = u(y - 1/2)$. We define the sequence (u_ε) by $u_\varepsilon(x) = \tilde{u}_0(x/\varepsilon)$.

Then $\|u_\varepsilon\|_{L^p(\Omega)} \rightarrow \|u_0\|_{L^p(\Omega \times Y)}$, but (u_ε) two-scale converges to \tilde{u}_0 .

(7) Let us consider sequences (u_ε) from (4) and (v_ε) from (5). Denoting the two-scale limit by u_0 and the weak limit by u , we have $\lim \|u_\varepsilon\|_{L^p(\Omega)} = \|u_0\|_{L^p(\Omega \times Y)} > \|u\|_{L^p(\Omega)}$; $\lim \|v_\varepsilon\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega \times Y)} > \|u\|_{L^p(\Omega)}$ and finally the sum $u_\varepsilon + v_\varepsilon$ satisfies sharp inequalities.

Proposition 5 ([49], pp. 9-10). *Let $u(x, y)$ be a smooth function defined on $\Omega \times \mathbb{R}^N$, Y -periodic in y . Then the limit two-scale of $u_\varepsilon(x) = u\left(x, \frac{x}{\varepsilon^2}\right)$ is equal to the weak L^p limit, namely $\int_Z u(x, z) dz$ where Z is the unit cube in \mathbb{R}^N .*

Proof. Indeed, let $f(x, y, z)$ be a smooth function which is Y -periodic in y and Z -periodic in z . Then it is well known that

$$f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \rightarrow \int_Y \int_Z f(x, y, z) dz dy \quad \text{weakly in } L^p.$$

This implies that

$$\int_\Omega u\left(x, \frac{x}{\varepsilon^2}\right) \phi\left(x, \frac{x}{\varepsilon}\right) dx \rightarrow \int_\Omega \int_Y \int_Z u(x, z) dz \phi(x, y) dy dx, \quad \forall \phi \in L^q(\Omega, C_\#(Y)).$$

In other words, the two-scale limit v of u_ε is $v(x) = \int_Z u(x, z) dz$.

As we can see, the previous compactness theorem generalizes the theorem in $L^2(\Omega)$ (from any bounded sequence in $L^2(\Omega)$ one can extract a subsequence which converges weakly in $L^2(\Omega)$). This is all the more

not surprising since the definition of two-scale convergence is written under conditions of weak convergence. It is a convergence for any test function and not a standard convergence.

Two-scale convergence also allows us to justify convergence of the sequence of gradients ∇u_ε in the sense of the following proposal.

Proposition 6 ([41], p. 34). *Let u_ε be a bounded sequence in $H^1(\Omega)$. Then u_ε converges two-scale, to a function $u_0(x) \in H^1(\Omega)$ and there is a function $u_1(x, y) \in L^2(\Omega; H^1_\#(Y))$ such that ∇u_ε two-scale converges to $\nabla_x u_0(x) + \nabla_y u_1(x, y)$:*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y u_0(x) \psi(x, y) dy dx, \quad \forall \psi \in L^2(\Omega, C_\#(Y)),$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla u_\varepsilon(x) \Psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y (\nabla_x u_0(x) + \nabla_y u_1(x, y)) \Psi(x, y) dy dx,$$

$$\forall \Psi \in (L^2(\Omega; C_\#(Y)))^N.$$

Remark 7

- Two-scale convergence can also extend to periodic surface cases ([8] and [54]). If we note by Γ_ε a collection of periodically repeated surfaces in the Ω domain following the ε period (the edges of the inclusions present in Ω for the example in Figure 1), and by Γ the corresponding reference surface such as $\forall x \in \Gamma_\varepsilon \Rightarrow y = \frac{x}{\varepsilon} \in \Gamma$.

- The Proposition 6, can be expanded to the L^p space (see [49], p. 22).

Proposition 7 ([41], p. 34). *For all sequence u_ε on $L^2(\Gamma_\varepsilon)$ which checks*

$$\varepsilon \int_{\Gamma_\varepsilon} |u_\varepsilon|^2 dx \leq C, \quad C > 0, \quad (33)$$

there is a subsequence, always noted u_ε , and a function $u_0(x, y) \in L^2(\Omega; L^2_\#(\Gamma))$ such that u_ε two-scale converges to u_0 in the meaning

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_\Gamma u_0(x, y) \psi(x, y) dy dx, \quad (34)$$

for any test function Y -periodic $\psi(x, y) \in L^2(\Omega; C_\#(\Gamma))$.

Remark 8 ([41], p. 34). If u_ε is bounded in $L^2(\Omega)$ and if ∇u_ε is bounded in $L^2(\Omega)$ as well, using Lemma 4.2.4 in [9], we deduce that u_ε verifies (33) and two-scale converges to a limit v_0 which is nothing but the trace of u_0 on Γ , u_0 being the two-scale limit of u_ε . v_0 is noted by u_0 . If u_ε is bounded in $H^1(\Omega)$ we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx &= \int_\Omega \int_Y u_0(x) \psi(x, y) dy dx, \\ \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Gamma_\varepsilon} u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) d\gamma(x) &= \int_\Omega \int_\Gamma u_0(x) \psi(x, y) d\gamma(y) dx, \\ \lim_{\varepsilon \rightarrow 0} \int_\Omega \nabla u_\varepsilon(x) \cdot \Psi\left(x, \frac{x}{\varepsilon}\right) dx &= \int_\Omega \int_Y (\nabla_x u_0(x) + \nabla_y u_1(x, y)) \\ &\quad \cdot \Psi(x, y) dy dx, \end{aligned} \quad (35)$$

for all function $\psi(x, y) \in L^2(\Omega; C_\#(Y))$ and any function $\Psi(x, y) \in (L^2(\Omega; C_\#(Y)))^N$.

Proposition 8 ([49], p. 22). *Let u_ε be a bounded sequence of $W^{1,p}(\Omega)$. Then u_ε converges two-scale, to a function $u_0(x) \in W^{1,p}(\Omega)$ and there is a function $u_1(x, y) \in L^p(\Omega; W^{1,p}_\#(\Omega))$ such that ∇u_ε two-scale converges to $\nabla_x u_0(x) + \nabla_y u_1(x, y)$.*

Remark 9 ([51], p. 6). For a sequence (u_ε) in $L^2(\Omega)$ space, we have the following relations for the convergences:

Strong convergence \Rightarrow Two-scale convergence \Rightarrow Weak convergence.

Weak convergence $\not\Rightarrow$ Two-scale convergence $\not\Rightarrow$ Strong convergence.

The definition of two-scale convergence can take into account time dependence, [50].

Definition 2 ([27], p. 22). Let (u_ε) be a sequence of functions in $L^2((0, T) \times \Omega)$. This sequence two-scale converges to a limit $u \in L^2((0, T) \times \Omega \times Y)$ if we have

$$\forall \psi(t, x, y) \in L^2((0, T) \times \Omega; C_{\#}^1(Y)),$$

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} u_\varepsilon(t, x) \psi\left(t, x, \frac{x}{\varepsilon}\right) dx dt = \int_0^T \int_{\Omega} \int_Y u(t, x, y) \psi(t, x, y) dy dx dt.$$

(36)

The definition quickly expands to a vector or tensor functions valued.

From a similar result in [3], the results are extended to time-dependent functions resulting in the following compactness as in [59].

Lemma 3 ([27], p. 22). (a) *Each bounded sequence in $L^2((0, T) \times \Omega)$ contains a subsequence which two-scale converges to a limit $u \in L^2((0, T) \times \Omega \times Y)$.*

(b) *Let (u_ε) be a bounded sequence in $L^2(0, T; H^1(\Omega))$. Then, there exists $u_0 \in L^2(0, T; H^1(\Omega))$ and $u_1 \in L^2((0, T) \times \Omega; H_{\#}^1(Y))$ such that up to a subsequence, u_ε and ∇u_ε two-scale converges to u_0 and $\nabla_x u_0(t, x) + \nabla_y u_1(t, x, y)$, respectively.*

If in addition $\partial_t u_\varepsilon$ is bounded in $L^2((0, T) \times \Omega)$, then $\partial_t u_0 \in L^2((0, T) \times \Omega)$ and $\partial_t u_\varepsilon$ two-scale converges to $\partial_t u_0$.

Two-scale convergence is defined in $H^{-1}(0, T)$ as follows:

Definition 3 ([27], p. 37). Let $(\phi_\varepsilon)_{\varepsilon>0}$ be a sequence in $H^{-1}(0, T; M)$.

We say that ϕ_ε two-scale converges to ϕ in $H^{-1}(0, T; L_0^2(\Omega \times Y))$ when:

$$\begin{aligned} \forall \psi(t, x, y) \in H_0^1((0, T); L^2(\Omega; C_{\sharp}^1(Y))), \\ \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \phi_\varepsilon(t, x) \psi\left(t, x, \frac{x}{\varepsilon}\right) dx dt = \int_0^T \int_{\Omega} \int_Y \phi(t, x, y) \psi(t, x, y) dy dx dt. \end{aligned} \tag{37}$$

Since the time variable is only a parameter in the two-scale convergence defined by (37) and $H^{-1}(0, T)$ is separable, the weak two-scale convergence defined by (37) has the same compactness property as in the square-integrable in time case (Lemma 3).

3.2. Alternative approaches to the two-scale convergence ([53], pp. 99-100)

In [13], the authors dealt with a homogenization technique which was used for the description of porous media. It is suitable for an alternative approach to two-scale convergence. The alternative approach is based on the so-called periodic unfolding also called two-scale transform. This idea is based on the so-called transform which changes a sequence of one-variable functions $\{u_\varepsilon(x)\}$ into a sequence of two-variable functions $\{\mathcal{T}_\varepsilon u(x, y)\}$.

For each ε let us consider small non-overlapping cubes $C_\varepsilon^k = \varepsilon Y + \varepsilon k$, $k \in \mathbb{Z}^N$. Here, for sake of simplicity, we restrict ourselves to the domains Ω that can be decomposed into these cubes, i.e., $\bar{\Omega} = \bigcup_k \bar{C}_\varepsilon^k$. The sequence $\{\mathcal{T}_\varepsilon u(x, y)\}$ is defined by the relation

$$\mathcal{T}_\varepsilon u(x, y) = u_\varepsilon \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon y \right), \quad x \in \Omega, y \in Y, \quad (38)$$

where $[x] = k$ is the vector of the greatest integers k_i less than or equal to x_i . On each cube $C_\varepsilon^k \times Y$ the function $\mathcal{T}_\varepsilon u$ is constant in the variable x and as a function of y it is the function $u_\varepsilon(x)$ on C_ε^k transformed onto the unit cube Y . The alternative definition reads:

Definition 4. We say that a sequence (u_ε) in $L^p(\Omega)$ two-scale converges to a function $u_0(x, y) \in L^p(\Omega \times Y)$, if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \int_Y [\mathcal{T}_\varepsilon u(x, y) - u_0(x, y)] \psi(x, y) dy dx = 0, \quad (39)$$

for any test function $\psi \in \mathcal{D}(\Omega; C_{\sharp}^\infty(Y))$. Moreover, if $\mathcal{T}_\varepsilon u \rightarrow u_0$ in $L^p(\Omega \times Y)$ strongly, we say that (u_ε) two-scale converges strongly.

We can refer to the article [29], where the approach mentioned above is also used and is called periodic unfolding. Many authors have applied this approach to the case of periodic multi-scale problems.

The other alternative approach is based on the so-called inverse two-scale transform which changes a sequence $\{\bar{\psi}_\varepsilon(x)\}$ of one-variable functions into a sequence of two-variable functions $\psi(x, y)$. The functions $\bar{\psi}_\varepsilon$ are built as follows. Similarly, as in the previous transform, we consider non-overlapping cubes C_ε^k that cover the Ω domain (here the

Ω domain does not need a union of C_ε^k cubes, i.e., $\bar{\Omega} \subseteq \bigcup_k \bar{C}_\varepsilon^k$). Outside the domain we pose $\psi(x, y) = 0$. Let us average the extended function $\psi(x, y)$ with respect to the first variable:

$$\bar{\psi}_\varepsilon(x) = \frac{1}{\varepsilon^N} \int_{C_\varepsilon^k} \psi\left(\xi, \frac{x}{\varepsilon}\right) d\xi, \quad x \in C_\varepsilon^k. \quad (40)$$

Definition 5. Let (u_ε) be a sequence in $L^p(\Omega)$. We say that (u_ε) two-scale converges to a function $u_0 \in L^p(\Omega \times Y)$, if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) \bar{\psi}_\varepsilon(x) dx = \int_{\Omega} \int_Y u_0(x, y) \psi(x, y) dy dx,$$

for each test function $\psi(x, y) \in \mathcal{D}(\Omega; C_{\#}^\infty(Y))$. Moreover, if

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon(x) - \bar{u}_0^\varepsilon(x)\|_{L^p(\Omega)} = 0, \text{ where } \bar{u}_0^\varepsilon(x) = \frac{1}{\varepsilon^N} \int_{C_\varepsilon^k} u_0\left(\xi, \frac{x}{\varepsilon}\right) d\xi, \quad x \in C_\varepsilon^k,$$

we say that (u_ε) two-scale converges strongly.

Remark 10. The two-scale transformation used in the Definition 4 allows us to define two-scale convergence and strong two-scale convergence more naturally with the help of weak convergence with regular test functions. On the other hand, the Definition 5 is similar to the classic definition byNguetseng and Allaire, but differs from (30) in the choice of test function on the left side.

Why do we look for alternative approaches to the original one? In the definition we want to test the convergence with functions from a space as small as possible, thus, smooth functions are convenient. On the other hand, in applications the largest class is desirable. In Definition 1 we cannot take the test functions $\psi(x, y)$ from the whole space $L^q(\Omega \times Y)$, since it is not correctly defined on the zero-measure set $\{[x, y] : y = x/\varepsilon\}$ and thus the measurability of the composed function $\psi(x, x/\varepsilon)$ is not

guaranteed. Moreover, the test functions can satisfy some convergences, which is not always obvious. The class of suitable test functions is discussed at the top of the following section. Such functions are called admissible. Further, we will see that the two mentioned alternative definitions avoid described problems.

The main result of convergence on two-scale is convergence into a norm.

Theorem 3 ([49], p. 10). *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(\Omega)$ which two-scale converges to $u_0 \in L^p(\Omega \times Y)$. Then*

$$u_\varepsilon \rightarrow v(x) = \int_Y u_0(x, y) dy \quad \text{weakly in } L^p(\Omega), \quad (41)$$

and (u_ε) is bounded.

Theorem 4 ([49], p. 17). *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(\Omega)$ which two-scale converges to $u_0 \in L^p(\Omega \times Y)$. Then*

$$\liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^p(\Omega)} \geq \|u_0\|_{L^p(\Omega \times Y)} \geq \|u\|_{L^p(\Omega)}, \quad (42)$$

where $u(x) = \int_Y u_0(x, y) dy$.

Remark 11. We note that the result (42) of the Theorem 4 can be compared with the well-known fact that if $u_\varepsilon \rightarrow u$ weakly, then

$$\liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^p(\Omega)} \geq \|u\|_{L^p(\Omega)}.$$

So far, we have given the main results on the bounded sequences of $L^p(\Omega)$. For bounded sequences of $H^1(\Omega)$, the results may be more accurate.

Theorem 5 ([10], pp. 23). *Let $(u_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $H^1(\Omega)$. Then, there exists $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega, H^1_{\sharp}(Y))$, such that, up to the extraction of a subsequence, one has*

u_ε converges to u_0 weakly in $H^1(\Omega)$,

u_ε converges to u_0 strongly in $L^2(\Omega)$,

u_ε two-scale converges to u_0 ,

∇u_ε two-scale converges to $\nabla_x u_0(x) + \nabla_y u_1(x, y)$.

Proposition 9 ([10], pp. 26). *Let $(u_\varepsilon)_{\varepsilon>0}$ be a bounded sequence of $L^2(\Omega)$ such that $(\varepsilon \nabla u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(\Omega)$. There exists $u_0(x, y) \in L^2(\Omega \times Y)$, such that, up to the extraction of a subsequence, one has*

u_ε two-scale converges to u_0 ,

$\varepsilon \nabla u_\varepsilon$ two-scale converges to $\nabla_y u_0(x, y)$.

Theorem 6 ([49], p. 20). *Any function $u \in L^p(\Omega \times Y)$ is attained as a two-scale limit.*

3.3. Other properties of two-scale convergence ([53], p. 107-108)

The sequences from the previous two properties (4) and (5) point to an interesting fact. An extracted subsequence of weakly converging sequence converges to the same limit. In the case of two-scale convergence, we must consider convergence also with respect to subsequences of periods. Otherwise the limits may differ (e.g., the sequence (v_ε) from the properties above can be considered as extracted subsequence from (u_ε)).

Definition 6. Let $u \in L^P(\Omega \times Y)$, we define T_ε^* by

$$(T_\varepsilon^* u)(x, y) = \begin{cases} u\left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y\right), & \text{for } x \in \Omega_\varepsilon, \\ u(x), & \text{for } x \in \Lambda_\varepsilon, \text{ where } \Lambda_\varepsilon = \Omega - \Omega_\varepsilon. \end{cases}$$

In the case where $\Omega = \mathbb{R}^N$ or if the domain Ω can be written as interior of a union of complete cells we can take $\Omega_\varepsilon = \Omega$ and $\Lambda_\varepsilon = \emptyset$.

Property 1 ([39], p. 78). Let $f, g \in L^P(\Omega)$ and $\psi \in L^\infty(Y_\sharp)$, such that $\int_Y \psi(y) dy = 0$.

Then the sequence $u_\varepsilon(x) = f(x)\psi\left(\frac{x}{\varepsilon}\right) + g(x)$ is bounded in $L^P(\Omega)$.

Since its unfolding $T_\varepsilon^* u_\varepsilon$ yields $(T_\varepsilon^* u_\varepsilon)(x, y) = f(x)\psi(y) + g(x)$ in $\Omega_\varepsilon \times Y$. It strongly two-scale converges in $L^P(\Omega)$ to the limit $u_0(x, y) = f(x)\psi(y) + g(x)$. The sequence u_ε converges to $g(x)$ in $L^P(\Omega)$ weakly, but not strongly, unless that $f(x) \equiv 0$ or $\psi(y) \equiv 0$. The property shows that the local oscillations of u_ε , which are lost in the usual weak limit $L^P(\Omega)$, are conserved in the strong two-scale limit.

Remark 12. In the previous property (4), the sequence was strongly two-scale converging. It was caused by the fact that the period ε of $\psi\left(\frac{x}{\varepsilon}\right)$ was “in resonance” with the scale $E = \{\varepsilon\}$. Modifying the sequence to $u_\varepsilon(x) = f(x)\psi\left(\frac{2x}{\varepsilon}\right) + g(x)$ with the same scale E , the sequence also strongly two-scale converge but the limit is $u_0(x, y) = f(x)\psi(2y) + g(x)$. The weak $L^P(\Omega)$ limit is unchanged.

If the period of the function ψ is not “in resonance” with the E scale, i.e., their ratio is irrational, e.g., $u_\varepsilon(x) = f(x)\psi\left(\frac{x}{\sqrt{2\varepsilon}}\right) + g(x)$, then the sequence u_ε does not converge two-scale strongly but only weakly. Its limit $u_0(x, y) = g(x)$ is independent of y , i.e., the local oscillations are again lost.

4. Solving Some Problems by Two-Scale Convergence Method

We will give in this section some problems solved with two-scale convergence and applied it to an example.

4.1. Some problems solved by the two-scale convergence method

Two-scale convergence has been used to study different problems. We give an overview of these problems.

Two-scale convergence applied to the Vlasov-Poisson equation and Vlasov equation

Jiann-Sheng Jiang and Chi-Kun Lin use two-scale convergence in their article “Weak turbulence plasma induced by two-scale homogenization”, see [46]. They showed that the homogenization limit, in the sense of two-scale limit, of the distribution function satisfies the linear Vlasov-Poisson equations. Moreover, the limit distribution function can be decomposed into the mean and the fluctuation parts. They also investigate the Landau damping from the point of view of homogenization through the two-scale limit.

They use the two-scale convergence on the quasi-linear theory of the plasma turbulence described by the Vlasov-Poisson system.

In the same topic Aurore Back and Emmanuel Frenod use it in their article “Geometric two-scale convergence on manifold and applications to the Vlasov equation”, see [17]. They adapt the two-scale convergence on manifold using the Birkhoff’s theorem and they do asymptotic analysis using geometric objects (differential forms).

Two-scale convergence applied to fluid flow in porous media

Gregoire Allaire applied the two-scale convergence to fluid flow in porous media, see ([4], pp. 13-19). He considered the steady Stokes equations in a porous medium Ω_ε with a Dirichlet boundary condition. Denoting by u_ε and p_ε the velocity and pressure of the fluid, and f the density of forces acting on the fluid. The system of equations is

$$\begin{cases} \nabla p_\varepsilon - \varepsilon^2 \Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (43)$$

Two-scale convergence applied to behaviour of the warping and the torsion problems

Polisevski and Mascarenhas in [61] use the two-scale convergence method to study the homogenized behaviour of the warping and the torsion problems in a two-dimensional domain with a quasi-periodic perforation, as well as the relationship between the two homogenized problems. The Neumann problem was also analyzed.

Two-scale convergence applied to the stochastic

Wright et al. in [23] studied stochastic homogenization of partial differential equations and generalise the method of two-scale convergence. The two-scale convergence has been extended from the periodic setting to the more general stochastic setting. The compactness and convergence results are important tools in the resolution of homogenization problems for partial differential equation. These results can be applied to the monotone operators and nonlocal. For the two-scale convergence of almost periodic function, see [26].

Two-scale convergence applied to the viscous fluid

Saint Jean-Paulin and Polisevski in [60] used the two-scale convergence in the study of viscous fluid. The so-called dual-porosity model for immiscible incompressible two-phase flow in a natural fractured reservoir has been studied in the means of two-scale convergence method by Bourgeat et al. in [22].

Two-scale convergence on BV-spaces

The two-scale convergence method has been used by Amar in [11] to study some classes of homogenization problems on BV-spaces for periodic energies with linear growth.

Two-scale convergence applied to other problems

The two-scale limit has many others applications. It can also be applied to fluid flow problems, see [2, 12, 20, 21, 34, 38, 62], and to several other types of homogenized problems. For magnetic field problems, see [40, 64] and for parabolic problems, see [25, 44, 65]. For degenerated parabolic differential equations, see [32, 33, 52]. For spectral problems in fluid-solid structures, see [7]. Other applications of two-scale convergence can be found in [1, 48, 50] and especially the books [31, 35].

Other methods of analyzing multi-scale problems exist. One of these methods is the Representative Elementary Volume method (VER or RVE) [28, 42, 43].

The basic idea is to calculate means of disturbances in a volume representative of the heterogeneity of the system, then use these means to describe macroscopically the problem initially described in a microscopic way.

The method of mathematical homogenization is based on the same principle. Nevertheless, the averages it provides are obtained in a mathematically more rigorous and justified manner. Moreover, it allows to approach the local values of the disturbances thanks to corrective terms. The physical homogenization method is limited to the definition of an average value of these disturbances.

4.2. Applications to 2nd order elliptic linear equations ([10], pp. 27)

We now turn to the homogenization of the model problem and generalize the method to classical second order elliptic PDEs. As we shall see, the main strategy consists in the following methodology:

- find a sequence $(u_\varepsilon)_\varepsilon$ bounded in H^1 ;
- extract a subsequence and apply the Theorem 5;
- identify the problems solved by u_0 and u_1 ;
- show that the whole sequence converges;
- give sufficient conditions to get the strong convergence and prove the multiscale expansion. Let's consider the problem (2), namely,

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right) & = f(x) \text{ in } \Omega, \\ u_\varepsilon & = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set. Note that under the classic tensor assumptions of A , namely, the existence of $0 < c < C$ such that

$$\forall \xi \in \mathbb{R}^N, c|\xi|^2 \leq (A\xi, \xi) \leq C|\xi|^2, \quad (44)$$

and the fact that $f \in L^2(\Omega)$, the problem (2) admits a unique solution $u_\varepsilon \in H_0^1(\Omega)$, which moreover satisfies

$$\begin{aligned} -\int_{\Omega} \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right) \cdot u_\varepsilon(x) &= -\int_{\partial\Omega} \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right) \cdot u_\varepsilon(x) \cdot n(x) d\sigma \\ &\quad + \int_{\Omega} \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right) \cdot \nabla u_\varepsilon(x) dx \\ &= \int_{\Omega} \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right) \cdot \nabla u_\varepsilon(x) dx \\ &= \int_{\Omega} \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x), \nabla u_\varepsilon(x)\right) dx. \end{aligned}$$

The following variational formulation:

$$\begin{cases} \int_{\Omega} \left(A \left(\frac{x}{\varepsilon} \right) \nabla u_{\varepsilon}(x), \nabla u_{\varepsilon}(x) \right) dx & = \int_{\Omega} f(x) u_{\varepsilon}(x) dx \text{ in } \Omega, \\ u_{\varepsilon} & = 0 \text{ on } \partial\Omega \end{cases} \quad (45)$$

lead to

$$\begin{aligned} \int_{\Omega} \left(A \left(\frac{x}{\varepsilon} \right) \nabla u_{\varepsilon}(x), \nabla u_{\varepsilon}(x) \right) dx &= \int_{\Omega} f(x) u_{\varepsilon}(x) dx \\ &\leq \|f\|_{L^2} \|u_{\varepsilon}\|_{L^2} \\ &\leq C_P \|f\|_{L^2} \|\nabla u_{\varepsilon}\|_{L^2}, \end{aligned}$$

where C_P represents Poincaré's constant on Ω . Using the coercivity hypothesis (44), we easily deduce

$$\left(A \left(\frac{x}{\varepsilon} \right) \nabla u_{\varepsilon}(x), \nabla u_{\varepsilon}(x) \right)_{0,\Omega} = \int_{\Omega} \left(A \left(\frac{x}{\varepsilon} \right) \nabla u_{\varepsilon}(x), \nabla u_{\varepsilon}(x) \right) dx,$$

or

$$c \|\nabla u_{\varepsilon}\|_{L^2}^2 \leq \left(A \left(\frac{x}{\varepsilon} \right) \nabla u_{\varepsilon}(x), \nabla u_{\varepsilon}(x) \right)_{0,\Omega} \leq C_P \|f\|_{L^2} \|\nabla u_{\varepsilon}\|_{L^2}.$$

Then

$$\begin{aligned} \|\nabla u_{\varepsilon}\|_{L^2} &\leq \frac{C_P}{c} \|f\|_{L^2} \\ &\lesssim \|f\|_{L^2}. \end{aligned}$$

We have the following theorem.

Theorem 7 ([49], p. 25). *The solution sequence of the Equation (45) weakly converges to u_0 in $W_0^{1,2}(\Omega)$ and the sequence ∇u_{ε} two-scale converges to $\nabla u_0(x) + \nabla_y u_1(x, y)$, where (u_0, u_1) is the single solution in $W_0^{1,2}(\Omega) \times L^2(\Omega, W_{\#}^{1,2}(Y))$ of the homogenized equation:*

$$\int_{\Omega} \int_Y (A(y)(\nabla u_0(x) + \nabla_y u_1(x, y)), \nabla v(x) + \nabla_y v_1(x, y)) dy dx = \int_{\Omega} f v dx,$$

for all $v(x) \in W_0^{1,2}(\Omega)$ and $v_1(x, y) \in L^2(\Omega, W_{\sharp}^{1,2}(Y))$. (46)

5. Conclusion

The theory of homogenization uses the method of asymptotic analysis to describe the behaviour of a physics system. A new method of convergence of homogenization processes is proposed to justify the theory of homogenization: two-scale convergence. In this article we surveyed some phenomena in the two-scale convergence. By the original definitions from Nguetseng we give again some other definitions. The two-scale convergence has been generalized to the multiscale convergence, see [6] and Corollary 1.16 of [3]. We also see that the two-scale convergence can be defined by a new way: the alternative approaches. This principle is to transform a sequence of one variable function to a sequence of two variable functions. We give some examples to help the reader to more understand the notion of two-scale convergence. We see also that the choice of the space is very important to be able to apply the definition of two-scale convergence.

Acknowledgement

We thank Professor Gabriel Nguetseng for the help that he gave us to more understand the notion of two-scale convergence. The help he gave us with the Lemma 2, the Proposition 2 and Proposition 3 have been very fruitful and this has helped us to give the Examples 1 and 2. We thank Professor Nakoulima for the different discussions we had together. The time spent working with him allowed us to better understand the subject. And his contribution was very important for Examples 1 and 2. Our thanks again to Professor Gregoire Allaire for some directions he give us.

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