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ON THE UNIFORM DISTRIBUTION IN POSITIVE CHARACTERISTIC

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Abstract

Uniform distribution is an important subject in classical Diophantine approximation. There is a close connection between the distribution of real numbers and the estimation of exponential sums via Weyl's criteria. Carlitz gave a definition of uniform distribution in positive characteristic in an elementary way (see [11]), however, we are going to find a geometrical description. In this paper, we present a precise analogue to Weyl's criteria in the case of positive characteristic by using Haar measure. As an application, we show that the uniformly distributed modulo 1 for linear forms and for polynomial functions. In particular, we prove the set $\{m\theta\}$ in the Laurent series field is uniformly distributed modulo 1, where m extends over all the polynomials and θ is a fixed irrational function.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements of characteristic p, $K = \mathbb{F}_q[T]$ be the polynomial ring, $k = \mathbb{F}_q(T)$ be the rational function field, and $k_\infty = \mathbb{F}_q((\frac{1}{T}))$ be the formal Laurent series field. Let v be the normalized exponential valuation of k_∞ with $v(\frac{1}{T}) = 1$ and $v(0) = \infty.$ If α is an element in $k_\infty,$ then α can be uniquely expressed as a Laurent series as follows:

$$\alpha = \sum_{i=n}^{+\infty} a_i \left(\frac{1}{T}\right)^i, \quad n \in \mathbb{Z},\ a_i \in \mathbb{F}_q, \text{ and } a_n \neq 0, \tag{1.1}$$

where $v(\alpha) = n.$ We define the square bracket function $[\alpha]$ by

$$[\alpha] = \sum_{i=n}^{0} a_i \left(\frac{1}{T}\right)^i, \text{ if } n \leq 0, \text{ and } [\alpha] = 0, \text{ if } n > 0, \tag{1.2}$$

which is called the "integral part" of α as usual. We see that $[\alpha] \in K,$ $[\alpha + \beta] = [\alpha] + [\beta],$ $[a\alpha] = a[\alpha]$ for all $a \in \mathbb{F}_q^*,$ and $v(\alpha - [\alpha]) \geq 1.$ In fact, there is a unique polynomial $A = [\alpha],$ such that $v(\alpha - A) \geq 1.$ We write

$\langle\alpha\rangle = \alpha - [\alpha]$, which is called the "fractional part" of α. It is easily seen that $\langle\alpha + \beta\rangle = \langle\alpha\rangle + \langle\beta\rangle$, $\langle a\alpha\rangle = a\langle\alpha\rangle$ for all $a \in \mathbb{F}_q$, and $\langle\alpha + A\rangle = \langle\alpha\rangle$ for all $A \in K$, in particular, we have $\langle\alpha\rangle = 0$ if and only if $\alpha \in K$. The absolute value functions $|\alpha|$ and $\|\alpha\|$ in k_∞ are given by

$$|\alpha| = q^{-v(\alpha)} \text{ and } \|\alpha\| = |\langle\alpha\rangle|. \tag{1.3}$$

It is worth to keep in mind that $|0| = 0, |a| = 1$ for all $a \in \mathbb{F}_q^*$, and $|\alpha\beta| = |\alpha| \cdot |\beta|$ for $\alpha, \beta \in k_\infty$. For the double absolute function, we have $\|\alpha + A\| = \|\alpha\|$ for all $A \in K$, $\|a\alpha\| = \|\alpha\|$ for all $a \in \mathbb{F}_q^*$, $0 \le \|\alpha\| \le \frac{1}{q}$ for all $\alpha \in k_\infty$, and $\|\alpha\| = 0$ if and only if $\alpha \in K$. In particular, we have

$$\|\alpha + \beta\| \le \max\{\|\alpha\|, \|\beta\|\}, \text{ and } \|\alpha\| = \inf_{A \in K}|\alpha - A|. \tag{1.4}$$

Thus, $\|\alpha\|$ is the smallest distance from α to any element of K, and $[\alpha]$ is the nearest polynomial to α.

The valuation ring $\mathfrak{P}_0$ and the valuation ideal $\mathfrak{P}$ of k_∞ are given by

$$\mathfrak{P}_0 = \{\alpha \in k_\infty : |\alpha| \le 1\}, \text{ and } \mathfrak{P} = \{\alpha \in k_\infty : |\alpha| < 1\}. \tag{1.5}$$

If n is an integer, the fractional ideal $\mathfrak{P}_n$ is given by

$$\mathfrak{P}_n = \left(\frac{1}{T}\right)^n \mathfrak{P}_0 = \{\alpha \in k_\infty : |\alpha| \le q^{-n}\}. \tag{1.6}$$

Obviously, $\mathfrak{P}_1 = \mathfrak{P}$ and

$$\cdots \supset \mathfrak{P}_{-2} \supset \mathfrak{P}_{-1} \supset \mathfrak{P}_0 \supset \mathfrak{P}_1 \supset \mathfrak{P}_2 \supset \cdots.$$

The collection $\{\mathfrak{P}_n\}_{n \in \mathbb{Z}}$ is a fundamental system of neighbuorhoods of 0. If $\alpha \in k_\infty$, we denote a ball by $\alpha + \mathfrak{P}_n$

$$\alpha + \mathfrak{P}_n = \{x \in k_\infty : |x - \alpha| \le q^{-n}\}, \tag{1.7}$$

which is usually said to be a ball of center α and radius q^{-n}.

Let k_∞^+ be the additive group of k_∞. Since k_∞^+ is a local compact topological group, there exists a unique Haar measure, up to a positive multiplicative constant. Define by μ the Haar measure on k_∞^+ normalized to have total mass 1 on $\mathfrak{P}$, we thus have

$$\mu(\mathfrak{P}_n) = q^{1-n}, \text{ and } \mu(\alpha + \mathfrak{P}_n) = q^{1-n}, \quad n \in \mathbb{Z}, \alpha \in k_\infty. \tag{1.8}$$

We write $dx = d\mu(x)$, and denote by $L^1(k_\infty)$ the set of all complex valued measurable functions on k_∞ such that

$$\int_{k_\infty} |f(x)| dx < \infty.$$

Let $\hat{k}_\infty^+$ be the dual group of k_∞^+, which is the set of all continuous group homomorphism from k_∞^+ to the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Every element ψ in $\hat{k}_\infty^+$ is said to be an additive character of k_∞. Let ψ_0 be the principal additive character. For each additive character $\psi \neq \psi_0$, there exists an integer n such that ψ is trivial on $\mathfrak{P}_n$. Let

$$n(\psi) := \min\{n : \psi(x) = 1 \text{ for evey } x \in \mathfrak{P}_n\}, \tag{1.9}$$

which is called the conductor of ψ (see [4], (2.6)). We set $n(\psi_0) = \infty$. If $\alpha \in k_\infty$, and $\psi \in \hat{k}_\infty^+$, we define $\psi_\alpha(x) = \psi(\alpha x)$ for all $x \in k_\infty$. Clearly, ψ_α is again an additive character of k_∞, and the conductor of ψ_α is given by (see [4], (2.14))

$$n(\psi_\alpha) = n(\psi) - v(\alpha). \tag{1.10}$$

In the preceding papers [9, 10], we showed that a few basic results on the simultaneous Diophantine approximation in k_∞. In particular, we showed in [10] that the set $\{\langle m\theta\rangle\}_{m\in K}$ are everywhere dense in the valuation ideal $\mathfrak{P}$. For a general background to material on Diophantine approximation in characteristic zero and in positive characteristic, we refer the reader to [1, 2, 3, 7] as well as the survey papers [5, 8]. In this paper, we define and describe the uniform distribution modulo 1 for k_∞, and present a precise analogue to Weyl's criterion in the case of positive characteristic. In particular, we prove the set $\{m\theta\}_{m\in K}$ is uniformly distributed modulo 1.

To state our definitions and results, let $z^{(m)} = (z_{m1}, z_{m2}, \dots, z_{ms})$ $(1 \le m \le M)$ be M s-dimensional vectors in $\mathfrak{P}^s$, let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathfrak{P}^s$ and $r = (r_1, \dots, r_s) \in \mathbb{Z}^s$ with $r_j \ge 1$. We denote by $F_M(\alpha, r)$ a counting numbers function as follows:

$$F_M(\alpha, r) = \#\{z^{(m)} : 1 \le m \le M, \text{ and } z_{mj} \in \alpha_j + \mathfrak{P}_{r_j} \text{ for all } 1 \le j \le s\}. \tag{1.11}$$

The discrepancy of the sequence $\{z^{(m)}\}_{1\le m\le M}$ are defined by

$$\begin{aligned} D_M &= \sup_{\substack{\alpha\in\mathfrak{P}^s \\ r\in\mathbb{Z}^s,\, r_j\ge 1}} \left|M^{-1}F_M(\alpha, r) - \prod_{j=1}^{s}\mu(\alpha_j + \mathfrak{P}_{r_j})\right| \\ &= \sup_{\substack{\alpha\in\mathfrak{P}^s \\ r\in\mathbb{Z}^s,\, r_j\ge 1}} \left|M^{-1}F_M(\alpha, r) - q^{s-\sum_{j=1}^{s} r_j}\right|, \end{aligned} \tag{1.12}$$

Definition 1.1. If $\lim_{M\to\infty} D_M = 0$, we call the sequence $\{z^{(m)}\}_{1\le m<\infty}$ in $\mathfrak{P}^s$ uniformly distributed. Suppose that $z^{(m)} (1 \le m \le M)$ are M vectors in k_∞^s, not necessarily restricted to lie in $\mathfrak{P}^s$. Let $\langle z^{(m)}\rangle = (\langle z_{m1}\rangle, \langle z_{m2}\rangle, \dots, \langle z_{ms}\rangle)$ be the fractional parts vector of $z^{(m)}$. If $\langle z^{(m)}\rangle$ $(1 \le m < \infty)$ is uniformly distributed in $\mathfrak{P}^s$, then we call the sequence $\{z^{(m)}\}_{1\le m<\infty}$ uniformly distributed modulo 1.

The main results of this paper are the following theorems.

Theorem 1.1 (Weyl's criteria)**.** *Let* $z^{(m)}(1 \le m < \infty)$ *be a sequence of vectors in* $\mathfrak{P}^s$. *Then the following statements are equivalent*:

(i) $z^{(m)}(1 \le m < \infty)$ *is uniformly distributed in* $\mathfrak{P}^s$.

(ii) *For all real or complex valued Haar-integrable functions* $f(z) = f(z_1, z_2, \ldots, z_s)$ *on* $\mathfrak{P}^s$, *we have*

$$\lim_{M \to \infty} M^{-1} \sum_{1 \le m \le M} f(z^{(m)}) = \int_{\mathfrak{P}} \cdots \int_{\mathfrak{P}} f(z) dz_1 dz_2 \cdots dz_s. \tag{1.13}$$

(iii) *Let* ψ *be an additive character of* k_∞ *with the conductor* $n(\psi) = 2$. *Then we have*

$$\lim_{M \to \infty} M^{-1} \sum_{1 \le m \le M} \psi(A \cdot z^{(m)}) = 0, \tag{1.14}$$

for all nonzero vectors $A = (A_1, A_2, \ldots, A_s) \in K^s$, *where* $A \cdot z^{(m)}$ *is the inner product of vectors A and* $z^{(m)}$ *given by*

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_s y_s, \text{ where } x = (x_1, \ldots, x_s),\ y = (y_1, \ldots, y_s). \tag{1.15}$$

Corollary 1.1. *Let* $z^{(m)}(1 \le m < \infty)$ *be a sequence of vectors in* k_∞^s, *not necessarily restricted to lie in* $\mathfrak{P}^s$. *Suppose that* ψ *is an additive character on* k_∞ *such that* $n(\psi) = 2$ *and* $\psi(\alpha + A) = \psi(\alpha)$ *for all* $\alpha \in k_\infty$ *and* $A \in K$. *The necessary and sufficient conditions that* $\{z^{(m)}\}$ *be uniformly distributed modulo* 1 *is that*

$$\lim_{M \to \infty} M^{-1} \sum_{1 \le m \le M} \psi(A \cdot z^{(m)}) = 0, \tag{1.16}$$

for all nonzero vectors $A = (A_1, \ldots, A_s) \in K^s$.

In Section 3 below, we shall introduce an additive character ψ such that $n(\psi) = 2$ and $\psi(\alpha + A) = \psi(\alpha)$ for all $\alpha \in k_\infty$, $A \in K$, of which may be regarded as an analogue to the exponential function $e^{2\pi ix}$ in the complex number field. Therefore, the summation on the left-hand side of (1.14) may be regarded as the exponential sums in k_∞.

To apply the above Weyl's criteria, we next show that some of classical examples of uniformly distributed modulo 1. First, we consider the simplest case as follows.

Theorem 1.2. *Suppose that* $\theta \in k_\infty$ *and* $\theta \notin k$. *Then the sequence* $\{m\theta\}_{m\in K}$ *in* k_∞ *is uniformly distributed modulo* 1.

There is two nature generalizations of Theorem 1.2, and the first one is uniformly distributed modulo 1 for linear forms. Let $L(x) = L(x_1, x_2, \ldots, x_s) = a_1x_1 + a_2x_2 + \cdots + a_sx_s$, where $\alpha_i \in k_\infty$. $L(x)$ is said to be a linear form over k_∞ in variables $x_1, x_2, \ldots, x_s$. We show that

Theorem 1.3. *Let* $L_i(x)(1 \le i \le n)$ *be* n *linear forms in the* s *variables* $x_1, x_2, \ldots, x_s$. *Suppose that the only set of polynomial vectors* $A = (A_1, A_2, \ldots, A_n) \in K^n$ *such that*

$$A_1L_1(x) + A_2L_2(x) + \cdots + A_nL_n(x),$$

has polynomial coefficients in $x_1, x_2, \ldots, x_s$ *is* $A = 0$. *Then the sequence of vectors* $z^{(m)} = (L_1(m), L_2(m), \ldots, L_n(m))$ *for* $m = (m_1, \ldots, m_s) \in K^s$ *is uniformly distributed modulo* 1 *as* $|m| := \min_{1\le i\le n}|m_i| \to \infty$.

Let $s = 1$. As a straightforward consequence, we have

Corollary 1.2. *Let* $\theta_1, \theta_2, \ldots, \theta_n$ *be* n *elements in* k_∞ *such that* $\{1, \theta_1, \ldots, \theta_n\}$ *are linearly independent over* k. *Then the sequence of vectors* $z^{(m)} = (m\theta_1, m\theta_2, \ldots, m\theta_n)$ *for* $m \in K$ *is uniformly distributed modulo* 1.

Next, we show that another example of uniformly distributed modulo 1 for polynomial functions.

Theorem 1.4. *Let* $f(x) = a_n x^n + \cdots + a_1 x + a_0$ *be a polynomial over* k_∞ *of degree less than p with the variable x. Suppose that* $f(x)$ *has at least one irrational function coefficient* a_j *with* $j \geq 1$ *(i.e.,* $a_j \notin k$*). Then, the sequence* $\{f(m)\}_{m \in K}$ *is uniformly distributed modulo* 1.

Clearly, if $n = 1$, Theorem 1.4 becomes Theorem 1.2. All of the results we present here are very famous in the real number field, for details we refer the reader to ([3], Chapter 4). Throughout this paper, the notation $\alpha \equiv \beta \pmod 1$ means that there is a polynomial A such that $\alpha = A + \beta$, where $\alpha, \beta \in k_\infty$. In particular, we have $\alpha = \langle \alpha \rangle \pmod 1$ for all $\alpha \in k_\infty$. This result has been proved by Carlitz in a very elementary method (see [11]) and also other scholars gave a proof in similar method. Now we will give a proof based on Fourier transformation.

2. Proof of the Weyl's Criteria

To prove Theorem 1.1, we need some basic techniques of harmonic analysis on k_∞. The reader should consult ([4], Chapter 2) and ([6], Chapter 2) for details. Let f be a complex valued function and $f \in L^1(k_\infty)$, the Fourier transform $\hat{f}$ takes the form

$$\hat{f}(x) = \int_{k_\infty} f(t)\psi(xt)\,dt, \quad x \in k_\infty, \tag{2.1}$$

where ψ is an additive character on k_∞. If $\hat{f} \in L^1(k_\infty)$, then its Fourier transform via the conjugate of ψ takes the form

$$\hat{\hat{f}}(t) = \int_{k_\infty} \hat{f}(x)\psi(-tx)\,dx, \quad t \in k_\infty. \tag{2.2}$$

Lemma 2.1. *If* $\hat{f} \in L^1(k_\infty)$ *for continuous* $f \in L^1(k_\infty)$, *then*

$$\hat{\hat{f}}(t) = q^{2-n(\psi)} f(t), \tag{2.3}$$

for all $t \in k_\infty$, *where* $n(\psi)$ *is the conductor of* ψ.

Proof. From the general theory (see [6], or page 26 of [4]), we know that $\hat{\hat{f}} = c_\psi f$ for some constant c_ψ which is independent of *f*. To determine this constant, we take *f* to be the characteristic function of $\mathfrak{P}$. It is easily seen that

$$\hat{f}(x) = \int_{k_\infty} f(t)\psi(xt)dt = \int_{\mathfrak{P}} \psi(xt)dt.$$

Since $n(\psi_x) = n(\psi) - v(x)$ (see (1.10)), it follows that $\hat{f}$ is the characteristic function of $\mathfrak{P}_{n(\psi)-1}$. Hence by (2.2), we have

$$\hat{\hat{f}}(t) = \int_{k_\infty} \hat{f}(x)\psi(-tx)dx = \int_{\mathfrak{P}_{n(\psi)-1}} \psi(-tx)dx.$$

One easily computes that

$$\hat{\hat{f}}(t) = \mu(\mathfrak{P}_{n(\psi)-1}) f(t) = q^{2-n(\psi)} f(t).$$

The lemma follows at once. □

Let *f* be a continuous function on $\mathfrak{P}$. We denote by $c_f(A)$ the Fourier coefficients of *f* on $\mathfrak{P}$ as follows:

$$c_f(A) = \int_{\mathfrak{P}} f(t)\psi(At)dt, \quad A \in K. \tag{2.4}$$

Since $k_\infty^+ = K \times \mathfrak{P}$, we have the following Fourier series expansion.

Lemma 2.2. *Let* $n(\psi) = 2,$ *and suppose that f is a continuous complex valued function on* $\mathfrak{P}$ *such that*

$$\sum_{A \in K} |c_f(A)| < \infty. \tag{2.5}$$

Then for all $t \in \mathfrak{P},$ *we have*

$$f(t) = \sum_{A \in K} c_f(A)\psi(-At). \tag{2.6}$$

Proof. Extend f to all of k_∞ by defining $f(t) = 0$ for $t \in k_\infty - \mathfrak{P}.$ Since $\mathfrak{P}$ is open in $k_\infty,$ thus f is continuous on $k_\infty,$ and

$$\hat{f}(t) = \int_{\mathfrak{P}} f(t)\psi(xt)dt.$$

We note that if $x \in \mathfrak{P}_i,$ $y \in \mathfrak{P}_j,$ then $xy \in \mathfrak{P}_{i+j},$ in particular, we have $xy \in \mathfrak{P}_2,$ if $x \in \mathfrak{P}$ and $y \in \mathfrak{P}.$ Since $n(\psi) = 2,$ it follows that $\psi(xy) = 1$ for all $x \in \mathfrak{P}$ and $y \in \mathfrak{P}.$ We see that for $x = b + y$ with $y \in \mathfrak{P},$ then

$$\hat{f}(x) = \hat{f}(b + y) = \int_{\mathfrak{P}} f(t)\psi(t(b + y))dt = \int_{\mathfrak{P}} f(t)\psi(tb)dt = \hat{f}(b).$$

Hence, $\hat{f}$ is periodic on k_∞ with $\mathfrak{P}$ as a group of periods. By the definition of f on $k_\infty,$ we have

$$\hat{f}(A) = \int_{k_\infty} f(x)\psi(Ax)dx = \int_{\mathfrak{P}} f(x)\psi(Ax)dx = c_f(A).$$

It follows that

$$\begin{aligned} \int_{k_\infty} |\hat{f}(x)|dt &= \sum_{A \in K} \int_{A+\mathfrak{P}} |\hat{f}(x)|dx \\ &= \sum_{A \in K} |\hat{f}(A)| = \sum_{A \in K} |c_f(A)| < \infty, \end{aligned}$$

by the assumption. Thus $\hat{f} \in L^1(k_\infty)$. If $t \in \mathfrak{P}$, by Lemma 2.1, we have

$$\begin{aligned} f(t) &= \int_{k_\infty} \hat{f}(x)\psi(-tx)dx \\ &= \sum_{A \in K} \hat{f}(A) \int_{A+\mathfrak{P}} \psi(-tx)dx \\ &= \sum_{A \in K} \hat{f}(A)\psi(-tA) \int_{\mathfrak{P}} \psi(-tx)dx \\ &= \sum_{A \in K} \hat{f}(A)\psi(-At). \end{aligned}$$

We complete the proof of Lemma 2.2. □

Proof of Theorem 1.1. To simplify the notations, we assume that $s = 1$, since there are no great additional complications when $s > 1$. Our vectors $z^{(m)}$ $(1 \leq m < \infty)$ are thus substantially elements in $\mathfrak{P}$, which we shall denote by z_m $(1 \leq m < \infty)$. To prove Theorem 1.1, it is enough to prove the cycle of implications about z_m

$$\text{(A)} \to \text{(B)} \to \text{(C)} \to \text{(D)} \to \text{(A)},$$

where $z_m \in \mathfrak{P}$ for $1 \leq m < \infty$, and (A), (B), (C), and (D) are statements about z_m as follows.

Statement (A). z_m $(1 \leq m < \infty)$ is uniformly distributed in $\mathfrak{P}$.

Statement (B). Suppose that $\alpha \in \mathfrak{P}$ and r is a positive integer given, then

$$M^{-1}F_M(\alpha, r) \to q^{1-r}, \text{ as } M \to \infty,$$

where $M \geq 1$ is a positive integer, $F_M(\alpha, r)$ (as before) is the number of solutions of

$$z_m \in \alpha + \mathfrak{P}_r, \quad 1 \leq m \leq M.$$

Uniformity with respect to α and r is not assumed.

Statement (C).

$$M^{-1} \sum_{1 \leq m \leq M} f(z_m) \to \int_{\mathfrak{P}} f(z) dz, \text{ as } M \to \infty,$$

for all functions $f(z)$ Haar-integrable in $\mathfrak{P}$.

Statement (D). Let ψ be any additive character with $n(\psi) = 2$, then

$$M^{-1} \sum_{1 \leq m \leq M} \psi(Az_m) \to 0, \text{ as } M \to \infty,$$

for all polynomials $A \neq 0$. Again, no uniformity with respect to A is assumed.

Proof that (A) implies (B). It is trivial, since (B) is an ostensibly weaker form of (A).

Proof that (B) implies (C). By considering the real and imaginary parts of $f(z)$ separately, we may suppose without loss of generality that $f(z)$ is a real valued function and, by adding an appropriate constant to $f(z)$, that $f(z) \geq 0$. Since f is Haar-integrable in $\mathfrak{P}$, for each $r \in \mathbb{Z}$, $r \geq 1$, we have

$$\int_{\mathfrak{P}} f(z) dz = \sum_{\alpha \in \mathfrak{P}/\mathfrak{P}_r} \int_{\alpha + \mathfrak{P}_r} f(t) dt.$$

Denote γ_α and Γ_α by

$$\gamma_\alpha = \min_{t \in \alpha + \mathfrak{P}_r} f(t), \text{ and } \Gamma_\alpha = \max_{t \in \alpha + \mathfrak{P}_r} f(t).$$

For each $\epsilon > 0$, if r is sufficiently large, then we have

$$\int_{\mathfrak{P}} f(z) dz - \epsilon \leq q^{1-r} \sum_{\alpha \in \mathfrak{P}/\mathfrak{P}_r} \gamma_\alpha \leq q^{1-r} \sum_{\alpha \in \mathfrak{P}/\mathfrak{P}_r} \Gamma_\alpha \leq \int_{\mathfrak{P}} f(z) dz + \epsilon. \quad (2.7)$$

It is easy to see that

$$M^{-1}\sum_{1\le m\le M} f(z_m) = \sum_{\alpha\in\mathfrak{P}/\mathfrak{P}_r} M^{-1} \sum_{\substack{1\le m\le M\\ z_m\in\alpha+\mathfrak{P}_r}} f(z_m).$$

If M is sufficiently large, by statement (B) we have

$$(1-\epsilon)q^{1-r}\sum_{\alpha\in\mathfrak{P}/\mathfrak{P}_r}\gamma_\alpha \le M^{-1}\sum_{1\le m\le M} f(z_m) \le (1+\epsilon)q^{1-r}\sum_{\alpha\in\mathfrak{P}/\mathfrak{P}_r}\Gamma_\alpha. \tag{2.8}$$

If follows that

$$M^{-1}\sum_{1\le m\le M} f(z_m) \le (1+\epsilon)q^{1-r}\sum_{\alpha\in\mathfrak{P}/\mathfrak{P}_r}\Gamma_\alpha \le (1+\epsilon)\left(\int_{\mathfrak{P}} f(z)dz + \epsilon\right).$$

On the other hand, we have similarly

$$M^{-1}\sum_{1\le m\le M} f(z_m) \ge (1-\epsilon)q^{1-r}\sum_{\alpha\in\mathfrak{P}/\mathfrak{P}_r}\gamma_\alpha \ge (1-\epsilon)\left(\int_{\mathfrak{P}} f(z)dz - \epsilon\right).$$

Therefore, we have

$$M^{-1}\sum_{1\le m\le M} f(z_m) \to \int_{\mathfrak{P}} f(z)dz, \text{ as } M\to\infty.$$

Proof that (C) implies (D). Since $z_m \in \mathfrak{P}$ $(1\le m<\infty)$, we write $\psi(Az_m) = \psi_A(z_m)$, where $n(\psi_A) = n(\psi) - v(A) \ge 2$ if A is a polynomial and $A \ne 0$. We see that ψ_A is a nontrivial additive character on $\mathfrak{P}$ for all $A \in K$ and $A \ne 0$. It follows that

$$\int_{\mathfrak{P}} \psi(At)dt = \begin{cases} 1, \text{ if } A = 0,\\ 0, \text{ if } A\in K,\ A\ne 0.\end{cases}$$

Since ψ_A is continuous on $\mathfrak{P}$, by (C), we have

$$M^{-1}\sum_{1\le m\le M}\psi(Az_m) \to \int_{\mathfrak{P}}\psi(Az)dz = 0,$$

for all nonzero polynomials A.

Proof that (D) implies (A). Suppose $\alpha \in \mathfrak{P}$ and $r \in \mathbb{Z}$ with $r \geq 1.$ Let $\chi_{\alpha, r}$ be the characteristic function of $\alpha + \mathfrak{P}_r.$ By the definition of $F_M(\alpha, r),$ we see that

$$M^{-1} F_M(\alpha, r) = M^{-1} \sum_{1 \leq m \leq M} \chi_{\alpha, r}(z_m). \tag{2.9}$$

To make use of Lemma 2.2, we must compute the Fourier coefficients of $\chi_{\alpha, r}.$ Let $A \in K,$ it is easily seen that

$$\begin{aligned} c_{\chi_{\alpha, r}}(A) &= \int_{\mathfrak{P}} \chi_{\alpha, r}(t) \psi(At) dt \\ &= \int_{\alpha + \mathfrak{P}_r} \psi(At) dt = \psi(A\alpha) \int_{\mathfrak{P}_r} \psi(Ax) dx. \end{aligned}$$

Since $n(\psi_A) = 2 - v(A),$ we have

$$c_{\chi_{\alpha, r}}(A) = \begin{cases} \psi(A\alpha) q^{1-r}, & \text{if } |A| \leq q^{r-2}, \\ 0, & \text{if } |A| > q^{r-2}. \end{cases}$$

Then

$$\sum_{A \in K} |c_{\chi_{\alpha, r}}(A)| < \infty.$$

By Lemma 2.2, the Fourier expansion of $\chi_{\alpha, r}$ on $\mathfrak{P}$ is

$$\begin{aligned} \chi_{\alpha, r}(z_m) &= \sum_{A \in K} c_{\chi_{\alpha, r}}(A) \psi(-Az_m) \\ &= q^{1-r} \sum_{\substack{A \in K \\ |A| \leq q^{r-2}}} \psi(A\alpha) \psi(-Az_m). \end{aligned} \tag{2.10}$$

It follows by (2.9) that

$$D_M = \sup_{\alpha\in\mathfrak{P}, r\geq 1}\left| q^{1-r}\sum_{\substack{A\in K\\ |A|\leq q^{r-2}}}\psi(A\alpha)M^{-1}\sum_{1\leq m\leq M}\psi(-Az_m) - q^{1-r}\right|$$

$$= \sup_{\alpha\in\mathfrak{P}, r\geq 1}\left| q^{1-r}\sum_{\substack{A\in K\\ 0<|A|\leq q^{r-2}}}\psi(A\alpha)M^{-1}\sum_{1\leq m\leq M}\psi(-Az_m)\right|$$

$$\leq q\max_{\substack{A\in K\\ A\neq 0}} M^{-1}\left|\sum_{1\leq m\leq M}\psi(Az_m)\right|.$$

By statement (D), we have $D_M \to 0$ as $M \to \infty$. This is the proof of Theorem 1.1. □

Proof of Corollary 1.1. Without loss of generality, we may assume that $s = 1$. Let z_m $(1 \leq m < \infty)$ be elements in k_∞, not necessarily restricted to lie in $\mathfrak{P}$, we have $z_m \equiv \langle z_m\rangle\,(\text{mod}\,1)$. Suppose that ψ is an additive character with $n(\psi) = 2$ and $\psi(\alpha) = \psi(\beta)$ if $\alpha \equiv \beta\,(\text{mod}\,1)$. Then $\psi(A) = 1$ for all $A \in K$, and

$$\sum_{1\leq m\leq M}\psi(Az_m) = \sum_{1\leq m\leq M}\psi(A\langle z_m\rangle).$$

The assertion of Corollary 1.1 follows by Theorem 1.1 immediately. □

3. Uniform Distribution for Linear Forms

In this section, we first introduce an additive character ψ with $n(\psi) = 2$, and $\psi(\alpha) = \psi(\beta)$ if $\alpha = \beta\,(\text{mod}\,1)$, so that the assertion of Corollary 1.1 makes sense. If $\alpha \in k_\infty$, then α can be uniquely written as a Laurent series as follows:

$$\alpha = \sum_{i=s}^{+\infty} a_i\left(\frac{1}{T}\right)^i,\ s \in \mathbb{Z},\ a_i \in \mathbb{F}_q, \text{ and } a_s \neq 0. \tag{3.1}$$

For every positive integer $n \geq 1$, we set $\tau_n(\alpha) = a_n$, which is a map from k_∞ to $\mathbb{F}_q$ such that $\tau_n(a\alpha) = a\tau_n(\alpha)$ for all $a \in \mathbb{F}_q$, $\tau_n(\alpha + \beta) = \tau_n(\alpha) + \tau_n(\beta)$, and $\tau_n(A) = 0$ for all $A \in K (n \geq 1)$. In particular, we have

$$\tau_n(T^i \alpha) = \tau_{n+i}(\alpha) \text{ for all } i \geq 0, \tag{3.2}$$

and

$$\tau_n(\alpha) \neq 0, \text{ if } n = v(\alpha). \tag{3.3}$$

Let λ be a fixed primitive p-th root of 1, we thus introduce an additive character $\psi^{(n)}$ as follows:

$$\psi^{(n)}(\alpha) = \lambda^{\mathrm{tr}(\tau_n(\alpha))} = \lambda^{\mathrm{tr}(a_n)}, \tag{3.4}$$

where tr is the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$. Obviously, $\psi^{(n)}$ is an additive character on k_∞ with the conductor $n(\psi^{(n)}) = n + 1$ for each positive integer n. In particular, we have $n(\psi^{(1)}) = 2$, and

$$\psi^{(1)}(\alpha + A) = \psi^{(1)}(\alpha), \text{ for all } \alpha \in k_\infty, \text{ and } A \in K. \tag{3.5}$$

In the sequel of this paper, we fix $\psi = \psi^{(1)}$. ψ should play a role of exponential function in k_∞.

Suppose that $A.\ H \in K$ and $A \neq 0$, $H \neq 0$, $\theta \in k_\infty$, we denote by $S_A(\theta, H)$ the exponential sums as follows:

$$S_A(\theta, H) = \sum_{\substack{m \in K \\ |m| < |H|}} \psi(mA\theta), \tag{3.6}$$

where $\psi = \psi^{(1)}$ given by (3.4).

Lemma 3.1. (i) *If* $\theta \in k$ *is a rational function, then there is a polynomial* $A \neq 0$ *such that*

$$S_A(\theta, H) = |H|, \text{ for all } H \in K \text{ with } H \neq 0. \tag{3.7}$$

(ii) *If* $\theta \notin k$ *is an irrational function in* k_∞ *and* $A \in K$ *with* $A \neq 0$, *then we have*

$$S_A(\theta, H) = 0, \text{ if } |H| \geq \|A\theta\|^{-1}, \tag{3.8}$$

where $\|\alpha\| = |\langle\alpha\rangle|$ $(\alpha \in k_\infty)$ *is the smallest distance from* α *to any element of K* (*see* (1.4) *above*).

Proof. (i) is trivial. Since θ is a rational function, there is a polynomial $A \neq 0$ such that $A\theta \in K$. We thus have $\psi(mA\theta) = 1$ for all $m \in K$ and $S_A(\theta, H) = |H|$ for all $H \in K$ with $H \neq 0$.

To prove (ii), we note that by (3.2)

$$\psi^{(n)}(T^i\theta) = \lambda^{tr(\tau_n(T^i\theta))} = \lambda^{tr(\tau_{n+i}(\theta))} = \psi^{i+n}(\theta). \tag{3.9}$$

Let $\deg(H) = h \geq 1$, and $m = b_{h-1}T^{h-1} + b_{h-2}T^{h-2} + \cdots + b_1T + b_0$. Then, we may rewrite $S_A(\theta, H)$ as follows

$$S_A(\theta, H) = \prod_{i=0}^{h-1}\Big(\sum_{b_i \in \mathbb{F}_q} \psi(b_iT^iA\theta)\Big). \tag{3.10}$$

Since $|H| \geq \|A\theta\|^{-1}$ by assumption, we thus have $h \geq v(\langle A\theta\rangle) \geq 1$. Then there exists one of i such that $0 \leq i \leq h-1$, and $i = v(\langle A\theta\rangle) - 1$. Let $i + 1 = v(\langle A\theta\rangle)$, we have by (3.9)

$$\begin{aligned}\sum_{b_i \in \mathbb{F}_q} \psi(b_iT^iA\theta) &= \sum_{b_i \in \mathbb{F}_q} \psi(T^ib_i\langle A\theta\rangle) = \sum_{b_i \in \mathbb{F}_q} \psi^{1+i}(b_i\langle A\theta\rangle) \\ &= \sum_{b_i \in \mathbb{F}_q} \lambda^{tr(\tau_{1+i}(b_i\langle A\theta\rangle))} = \sum_{b_i \in \mathbb{F}_q} \lambda^{tr(b_i\tau_{1+i}(\langle A\theta\rangle))} = 0,\end{aligned}$$

because of $\tau_{1+i}(\langle A\theta\rangle) \neq 0$ by (3.3). It follows that $S_A(\theta, H) = 0$. We complete the proof of Lemma 3.1. □

As a straightforward consequence of Corollary 1.1 and Lemma 3.1, we have the following corollary, which contains the assertion of Theorem 1.2.

Corollary 3.1. *If* $\theta \in k_\infty$ *and* $\theta \notin k$. *then the sequence* $\{m\theta\}_{m \in K}$ *is uniformly distributed modulo* 1. *If* $\theta \in k$, *then it is not uniformly distributed modulo* 1.

There is a high dimensional version of Lemma 3.1. Let $L_1(x), \ldots, L_n(x)$ be n linear forms in variables $x = (x_1, x_2, \ldots, x_n)$ given by

$$\begin{bmatrix} L_1(x) \\ \vdots \\ L_n(x) \end{bmatrix} = B \begin{bmatrix} x_1 \\ \vdots \\ x_s \end{bmatrix}, \tag{3.11}$$

where $B = (a_{ij})_{n \times s}$ is an $n \times s$ matrix over k_∞. We denote

$$S_A(B, H) = \sum_{\substack{m_1 \in K \\ |m_1| < |H_1|}} \cdots \sum_{\substack{m_s \in K \\ |m_s| < |H_s|}} \psi(A_1 L_1(m) + \cdots + A_n L_n(m)), \tag{3.12}$$

where $H = (H_1, \ldots, H_s) \in K^s$, $A = (A_1, A_2, \ldots, A_n) \in K^n$, and $m = (m_1, \ldots, m_s) \in K^s$. We set $|H| = \min\{|H_i| : 1 \leq i \leq s\}$.

Lemma 3.2. *If* $L_1(x), \ldots, L_n(x)$ *are* n *linear forms given by* (3.11), *and the only set of polynomial vector* $A = (A_1, \ldots, A_n) \in K^n$ *such that* $A_1 L_1(x) + \cdots + A_n L_n(x)$ *has polynomial coefficients in* $x_1, x_2, \ldots, x_s$ *is* $A = 0$. *Then, we have*

$$S_A(B, H) = 0,$$

for sufficiently large $|H|$, *where* $A = (A_1, \ldots, A_n) \in K^n$ *is a nonzero polynomial vector.*

Proof. If $A = (A_1, \ldots, A_n) \in K^n$ and $A \neq 0$ is given, we may write

$$A_1L_1(x) + \cdots A_nL_n(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

where $c_i \in k_\infty$ depends only on $A = (A_1, \ldots, A_n)$ and the matrix B. Since $A \neq 0$, there is at least a $c_i \notin k$. By (3.12), we have

$$S_A(B, H) = \prod_{j=1}^{s}\Bigg(\sum_{\substack{m_j \in K \\ |m_j|<|H_j|}} \psi(m_jc_j)\Bigg).$$

Suppose that $c_i \notin k$, we thus have by Lemma 3.1,

$$\sum_{\substack{m_i \in K \\ |m_i|<|H_i|}} \psi(m_ic_i) = 0, \text{ as } |H_i| > \|c_i\|^{-1}.$$

If follows that $S_A(B, H) = 0$ as $|H|$ is large enough. We complete the proof of Lemma 3.2. □

Now Theorem 1.3 follows immediately from Theorem 1.1 and the above lemma.

4. Uniform Distribution for Polynomial Functions

In this section, we give a proof of Theorem 1.4. To do this we need the following lemmas.

Lemma 4.1. *Let* $u(m)$ *be a complex valued function on* K, *and* $N, M \in K$ *be two polynomials such that* $1 \leq |N| \leq |M|$. *Then*

$$|N| \,\Big|\sum_{\substack{m \in K \\ |m|<|M|}} u(m)\Big|^2 \leq |M| \left(\sum_{\substack{m \in K \\ |m|<|M|}} |u(m)|^2 + \sum_{\substack{h \in K \\ 0<|h|<|N|}} \sum_{\substack{m \in K \\ |m|<|M|}} u(m)\overline{u(m+h)} \right). \tag{4.1}$$

Proof. Let $g \in K$ be a fixed polynomial such that $|g| < |N| < |M|$, it is easily seen that

$$\sum_{\substack{m \in K \\ |m| < |M|}} u(m) = \sum_{\substack{t \in K \\ |t| < |M|}} u(t - g). \tag{4.2}$$

If follows that

$$|N| \sum_{\substack{m \in K \\ |m| < |M|}} u(m) = \sum_{\substack{t \in K \\ |t| < |M|}} \sum_{\substack{g \in K \\ |g| < |N|}} u(t - g).$$

By Schwartz's inequality, we have

$$|N|^2 | \sum_{\substack{m \in K \\ |m| < |M|}} u(m)|^2 \leq |M| \sum_{\substack{t \in K \\ |t| < |M|}} | \sum_{\substack{g \in K \\ |g| < |N|}} u(t - g)|^2$$

$$= |M| \sum_{\substack{t \in K \\ |t| < |M|}} \sum_{\substack{g_1, g_2 \in K \\ |g_1| < |N|, |g_2| < |N|}} u(t - g_1)\overline{u(t - g_2)} \tag{4.3}$$

$$= |M|\,|N| \sum_{\substack{m \in K \\ |m| < |M|}} |u(m)|^2 + |M| \sum_{\substack{t \in K \\ |t| < |M|}} \sum_{g_1 \neq g_2} u(t - g_1)\overline{u(t - g_2)}.$$

Let $m = t - g_1$ and $h = g_1 - g_2$ in the above inequality. Then the right-hand side of (4.3) becomes

$$|N|\,|M| \left(\sum_{\substack{m \in K \\ |m| < |M|}} |u(m)|^2 + \sum_{\substack{h \in K \\ 0 < |h| < |M|}} \sum_{\substack{m \in K \\ |m| < |M|}} u(m)\overline{u(m + h)} \right).$$

The lemma follows immediately. □

Lemma 4.2. *Let* $z_m \in k_\infty$ *for* $m \in K$, *and suppose that*

$$|M|^{-1} \sum_{\substack{m \in K \\ |m| < |M|}} \psi(z_{m+h} - z_m) \to 0 \ (|M| \to \infty),$$

for each $h \in K$, $h \neq 0$, *not necessarily uniformly in h. Then*

$$|M|^{-1} \sum_{\substack{m \in K \\ |m| < |M|}} \psi(z_m) \to 0 \ (|M| \to \infty).$$

Proof. Put $u(m) = \psi(z_m)$ in Lemma 4.1, then for all $N, M \in K$, $0 < |N| \leq |M|$ we have

$$|N| \, |\sum_{\substack{m \in K \\ |m| < |M|}} \psi(z_m)|^2 \leq |M|^2 + |M| \sum_{\substack{h \in K \\ 0 < |h| < |N|}} \sum_{\substack{m \in K \\ |m| < |M|}} \psi(z_m) \overline{\psi(z_{m+h})}.$$

It follows that

$$|M|^{-2} | \sum_{\substack{m \in K \\ |m| < |M|}} \psi(z_m)|^2 \leq \frac{1}{|N|} + \frac{1}{|N||M|} \sum_{\substack{h \in K \\ 0 < |h| < |N|}} \left| \sum_{\substack{m \in K \\ |m| < |M|}} \psi(z_{m+h} - z_m) \right|. \tag{4.4}$$

If, now, N is fixed and let $|M| \to \infty$, the right-hand side of (4.4) tends to $\frac{1}{|N|}$, which is arbitrarily small by appropriate initial choice of N. Hence the left-hand side of (4.4) must tends to 0 as $|M| \to \infty$. We complete the proof of Lemma 4.2. □

Lemma 4.3. *A sufficient condition for the sequence* $\{z_m\}_{m \in K}$ *in* k_∞ *to be uniformly distributed modulo* 1 *is that the sequence* $\{z_{m+h} - z_m\}_{m \in K}$ *is uniformly distributed modulo* 1 *for each polynomial* $h \neq 0$.

Proof. By the hypothesis and Corollary 1.1, we have

$$|M|^{-1} \sum_{\substack{m \in K \\ |m| < |M|}} \psi(A(z_{m+h} - z_m)) \to 0, \text{ as } |M| \to \infty,$$

for all polynomials A and h with $A \neq 0$, $h \neq 0$. By Lemma 4.2 applied to $\psi(Az_m)$ we deduce

$$|M|^{-1} \sum_{\substack{m \in K \\ |m| < |M|}} \psi(Az_m) \to 0, \text{ as } |M| \to \infty,$$

for all $A \in K$, $A \neq 0$. The sequence $\{z_m\}$ is thus uniformly distributed modulo 1 by Corollary 1.1 again. □

Proof of Theorem 1.4. Since $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in k_\infty[x]$ has at least one irrational function coefficient a_j with $j \geq 1$, we suppose first that the leading coefficient a_n is an irrational function (i.e., $a_n \notin k$). When $n = 1$, the result has been proved in Theorem 1.2, we thus may assume that $n > 1$, and the result has been proved for $n - 1$. For any fixed polynomial $h \neq 0$, we denote that $z_m = f(m)$ for all $m \in K$, and thus

$$z_{m+h} - z_m = f(m + h) - f(m),$$

which is a polynomial in m of degree $n - 1$ with the irrational function leading coefficient $ha_n C_n^1$. With $\deg f(x) < p$. We always have $ha_n C_n^1$ is not zero. Hence the result for n follows from that for $n - 1$ and Lemma 4.3.

If, however, a_n is a rational function and there is some s $(1 \leq s < n)$ such that a_s is an irrational function but $a_{s+1}, \ldots, a_n$ are rational functions. Let $N \in K$, $N \neq 0$, such that $Na_{s+1}, \ldots, Na_n$ are polynomials. It is clearly enough to show that

$$\xi_m = f(Nm + h), \quad m \in K,$$

is uniformly distributed modulo 1 for each $h \in K$ and $|h| < |N|$. But

$$\begin{aligned}\xi_m &= a_0 + a_1(Nm + h) + \cdots + a_n(Nm + h)^n \\ &\equiv a_0 + a_1(Nm + h) + \cdots + a_s(Nm + h)^s + a_{s+1}h^{s+1} + \cdots + a_n h^n \pmod 1 \\ &= b_0 + b_1 m + \cdots + b_s m^s,\end{aligned}$$

where $b_0, b_1, \ldots, b_s$ are independent of m. In particular, $b_s = N^s a_s$ is an irrational function. This is the first case, so the theorem is proved generally. □

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