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A NEW CHARACTERIZATION OF PRINCIPAL IDEAL DOMAINS

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Abstract

A well-known theorem by Cohn ([3]) from 1968 characterizes the principal ideal domains (PIDs) as atomic Bézout domains, while a theorem by Chinh and Nam ([1]) from 2008 characterizes them as unique factorization domains all of whose maximal ideals are principal. We give a simple new characterization which implies each of these characterizations. For that purpose we introduce a new type of integral domains (we call them PC domains) and using this notion we characterize the PIDs as atomic PC domains. We discuss the importance of PC domains and find their position in a large implication diagram containing various types of integral domains.

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1. Introduction

A well-known theorem by Cohn ([3], Proposition 1.2) from 1968 characterizes the principal ideal domains (PIDs) as atomic Bézout domains, while a theorem by Chinh and Nam ([1], Corollary 1.2) from 2008 characterizes them as unique factorization domains (UFDs) all of whose maximal ideals are principal. In this paper, we give a simple new characterization which implies each of these characterizations. For that purpose we introduce a new type of integral domains (we call them PC domains). They are integral domains whose proper two-generated ideals are contained in proper principal ideals. Using this notion we characterize the PIDs as atomic PC domains. Apart from showing that our new characterization implies the two mentioned characterizations, we discuss the importance of the PC condition, finding its precise position in a large implication diagram (Diagram 2) containing various types of integral domains and comparing its role with respect to the PIDs with the role of the so-called AP condition with respect to the UFDs, namely, AP is the weakest condition on the diagram which, together with atomicity, implies the UFD-ness, while PC is the weakest condition on the diagram which, together with atomicity, implies the PID-ness.

We begin by recalling some definitions and statements. All the notions that we use but not define in this paper can be found in the classical reference books [2] by Cohn, [5] by Gilmer, [6] by Kaplansky, and [7] by Northcott.

In this paper all rings are *integral domains*, i.e., commutative rings with identity in which $xy = 0$ implies $x = 0$ or $y = 0$. A non-zero non-unit element x of an integral domain R is said to be *irreducible* (and called an *atom*) if $x = yz$ with $y, z \in R$ implies that y or z is a unit. A non-zero non-unit element x of an integral domain R is said to be *prime* if $x \mid yz$ with $y, z \in R$ implies $x \mid y$ or $x \mid z$. Every prime element is an atom, but not necessarily vice-versa. Two elements $x, y \in R$ are said to be *associates* if $x = uy$, where u is a unit. We then write $x \sim y$.

An integral domain R is said to be *atomic* if every non-zero non-unit element of R can be written as a (finite) product of atoms. An integral domain R is called a *principal ideal domain* (PID) if every ideal of R is principal. The condition for integral domains that every ideal is principal is called the *PID condition*. An integral domain R is called a *unique factorization domain* (UFD) if it is atomic and for every non-zero, non-unit $x \in R$, every two factorizations of x into atoms are equal up to order and associates. An integral domain R is called an *ACCP domain* if every increasing sequence of principal ideals of R stabilizes. It is well-known that every PID is a UFD, every UFD is an ACCP domain, and every ACCP domain is atomic.

An integral domain R is called a *Bézout domain* if every two-generated ideal of R is principal. (An ideal I of R is said to be *two-generated* if $I = (a, b)$ for some $a, b \in R.$) The condition for integral domains that every two-generated ideal is principal is called the *Bézout condition*. Obviously, every PID is a Bézout domain. The converse is not true.

Proposition 1.1 ([4], pages 306-307)**.** *Bézout condition for integral domains is strictly weaker than the PID condition. More concretely,* $R = \mathbb{Z} + X\mathbb{Q}[X]$ *is a Bézout domain which is not a PID.*

Note that the notation $R = \mathbb{Z} + X\mathbb{Q}[X]$ means that R consists of all the polynomials from $\mathbb{Q}[X]$ whose constant term is from $\mathbb{Z}$.

We call the *PIP condition* the condition for integral domains that every prime ideal is principal. We call the *MIP condition* the condition for integral domains that every maximal ideal is principal. The *MIP domains* are the domains which satisfy the MIP condition. Clearly, the PID condition implies the PIP condition and the PIP condition implies the MIP condition. More precise relations between these conditions are given in the next proposition and Corollary 3.5.

Proposition 1.2 ([4], page 283). *The PID condition for integral domains is equivalent to the PIP condition. In other words, if every prime ideal of an integral domain R is principal, then R is a PID.*

The final item that we cover in this introduction is the notion of a *monoid ring* for a commutative monoid M, written additively. The elements of the monoid ring $F[X; M]$, where F is a field and X is a variable, are the *polynomial expressions*, also called *polynomials*,

$$f(X) = a_1 X^{\alpha_1} + \cdots + a_n X^{\alpha_n}, \quad (1)$$

where $n \geq 0$, $a_1, \ldots, a_n \in F$, $\alpha_1, \ldots, \alpha_n \in M$. The polynomials $f(X) = a$, $a \in F$, are called the *constant polynomials*. The addition and the multiplication of the polynomials are naturally defined. We say that M is *cancellative* if for any elements $a, b, c \in M$, $a + b = a + c$ implies $b = c$. The monoid M is *torsion-free* if for any $n \in \mathbb{N}$ and $a, b \in M$, $na = nb$ implies $a = b$. All the monoids that we use in this paper are cancellative and torsion-free, hence the monoid rings $F[X; M]$ are integral domains.

2. A New Condition for Integral Domains

We introduce a new condition for integral domains, that we haven't met in the literature.

Definition 2.1. We call the *principal containment condition* (PC) the condition for integral domains that every proper two-generated ideal is contained in a proper principal ideal. We say that an integral domain is a *PC domain* if it satisfies the PC condition.

Clearly, Bézout condition implies the PC condition, and the MIP condition implies the PC condition.

Proposition 2.2. *There exists a Bézout domain which is not a MIP domain.*

Proof. Consider the monoid ring $R = F[X; \mathbb{Q}_+]$ (F a field), consisting of all the polynomials of the form

$$f(X) = a_0 + a_1 X^{\alpha_1} + \cdots + a_n X^{\alpha_n},$$

with $a_0, a_1, \ldots, a_n \in F$ and $0 < \alpha_1 < \cdots < \alpha_n$ from $\mathbb{Q}_+$. Let $\mathfrak{m}$ be the maximal ideal of R consisting of all the polynomials in R whose constant term is 0. Consider the localization $D = R_{\mathfrak{m}}$. The units of D have the form

$$\frac{a_0 + a_1 X^{\alpha_1} + \cdots + a_m X^{\alpha_m}}{b_0 + b_1 X^{\beta_1} + \cdots + b_n X^{\beta_n}},$$

where the a_i and b_j are from F with a_0, b_0 non-zero. Hence every non-zero element of D has the form uX^{α}, where u is a unit in D and $\alpha \in \mathbb{Q}_+$. The maximal ideal $\mathfrak{m}R_{\mathfrak{m}}$ of D consists of all uX^{α} with $\alpha > 0$ and is not finitely generated. So D does not satisfy the MIP condition. However, for any two elements uX^{α}, vX^{β} of D with $\alpha \leq \beta$ we have $uX^{\alpha} \mid vX^{\beta}$ and so D is Bézout. □

We will show later (see Proposition 3.4) that there also exists an MIP domain which is not a Bézout domain. Thus the notion of a PC domain is strictly weaker than each of the notions MIP and Bézout. Finally in Proposition 3.3, we show that the notion of a PC domain is not "just a union" of the notions of Bézout and MIP domains, i.e., that there is a PC domain which is neither Bézout, nor MIP.

Consider now the following two-part diagram:

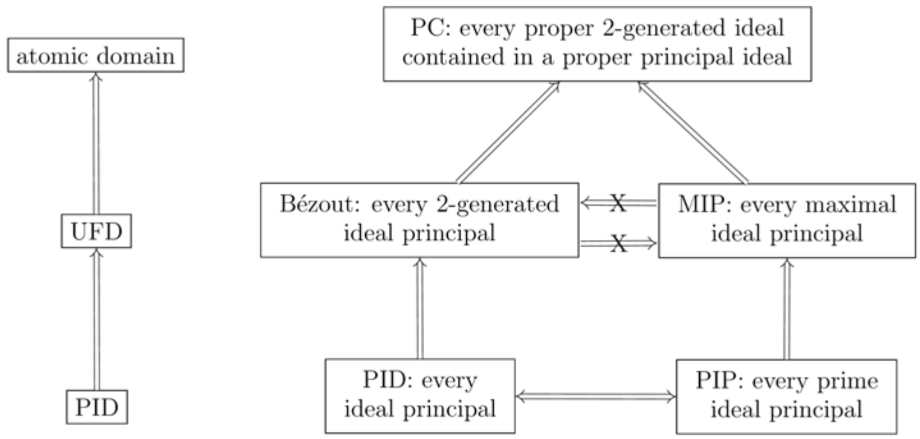

Diagram 1.

There is one equivalence in the diagram, the rest are implications (and all of them are strict) and non-implications. The higher the condition is (in each of the two parts of the diagram), the weaker it is. One can try to characterize PIDs by combining one condition from the left part of the diagram with one condition from the right part of the diagram.

The next two theorems are characterizations of PIDs of that type. The first one (*Cohn's Theorem*) is Theorem 2.3 that was first stated in ([3], Proposition 1.2). (Cohn remarks in [3] that it is easy to prove that Bézout's domains which satisfy ACCP are PIDs, however, ACCP is not equivalent to atomicity, as it was later shown.) The proof can be seen in Cohn's book ([2], 10.5, Theorem 3). The second one is Theorem 2.4, proved in 2008 by Chinh and Nam in [1].

Theorem 2.3 (Cohn's Theorem)**.** *If R is an atomic Bézout domain*, *then R is a PID.*

Theorem 2.4 ([1], Corollary 1.2)**.** *If R is a UFD in which every maximal ideal is principal*, *then R is a PID.*

Our next theorem improves both of the above theorems. It weakens one of the conditions in Cohn's theorem and both conditions in the Chinh and Nam's theorem.

Theorem 2.5. *Let R be an atomic domain which satisfies the PC condition. Then R is a PID.*

Proof. By Proposition 1.2, it is enough to show that every prime ideal is principal. Let P be a nonzero prime ideal of R. Let $x \neq 0$ be an element of P. Since R is atomic, we can write $x = p_1 p_2 \cdots p_n$, where the p_i's are atoms. As $p_1 p_2 \cdots p_n \in P$ and P is prime, at least one of the p_i's, say p_1, is in P. We claim that $P = (p_1)$. Let y be an element of P. Since R satisfies PC, $(p_1, y) \subset (c)$ for some proper principal ideal (c). From $p_1 = ct$ for some $t \in R$, we have $t \sim 1$ (as p_1 is an atom and $c \not\sim 1$). Now from $y = cr$ for some $r \in R$, we get $y = p_1 t^{-1} r$, hence $y \in (p_1)$. Thus $P = (p_1)$. □

3. Merging the Diagrams and Making Them More Detailed

In this section, we will merge the diagrams from the previous section and make them more detailed. That will illustrate the importance of the notion of a PC domain that we introduced in the previous section. We first need to give some definitions.

An integral domain is called a *GCD domain* if every two elements of it have a greatest common divisor (see [6], page 32). An element c of an integral domain D is called *primal* if for any $a, b \in D$ we have: $c \mid ab \Rightarrow c = c_1 c_2$, where $c_1 \mid a$ and $c_2 \mid b$. This notion was introduced in [3], where a new version of the definition of Schreier domains is also given: an integral domain D is *Schreier* if it is integrally closed and each of its elements is primal. The notion of pre-Schreier domains is introduced in [8]: an integral domain is *pre-Schreier* if each of its elements is primal.

Clearly every Schreier domains is pre-Schreier, but not conversely. A new proof of the well-known result that every GCD domain is Schreier was given in [3]. The converse is not true. Also, every Bézout domain is GCD, but not conversely (see [3]). An integral domain is called an *AP domain* if each of its atoms is prime, i.e., if the notions of an atom and of a prime element in it coincide. Every pre-Schreier domain is an AP domain, but not vice-versa (see [8]). It is well-known that an integral domain is a UFD if and only if it is atomic and AP.

Let us say a few words about the importance of the notion of PC domains. An old result of Skolem from 1939 states that an integral domain is a UFD if and only if it is atomic and GCD. However, weaker conditions were found which, together with atomicity, imply the UFD condition, namely, an integral domain is UFD if and only if it is atomic and AP (or pre-Schreier, or Schreier, or GCD). An analogous situation is with the conditions which, together with atomicity, imply the PID condition (see Diagram 2). Cohn's 1968 theorem ([3]) states an integral domain is PID if and only if it is atomic and Bézout. The result of Chinh and Nam ([1]) states that an integral domain is a PID if and only if it is UFD and MIP, which is, as a consequence of our Theorem 2.5, equivalent with atomic and MIP. *Our notion of PC domains provides a condition which is weaker than each of the conditions Bézout and MIP*, *however*, *it is still strong enough to be*, *together with atomicity*, *equivalent with the PID condition*. That is the main value of this notion.

We will now start justifying Diagram 2.

Proposition 3.1. *Every PC domain is an AP domain*.

Proof. Let R be a PC domain and let a be an atom of R. Suppose $a \mid xy$ for some $x, y \in R,$ but $a \nmid x$ and $a \nmid y.$ Then x, y are not units. The ideal (a, x) is proper, otherwise $ra + sx = 1$ for some $r, s \in R,$ hence $rya + sxy = y,$ hence $rya + sta = y$ for some $t \in R,$ hence $a \mid y,$ a contradiction. Since R is PC, there is a proper ideal (b) containing $(a, x).$ But then $a \in (b),$ so $b \mid a,$ hence (since a is an atom and b is a non-unit) $b \sim a.$ Also $x \in (b),$ so $b \mid x,$ hence $a \mid x$ (as $b \sim a),$ a contradiction. □

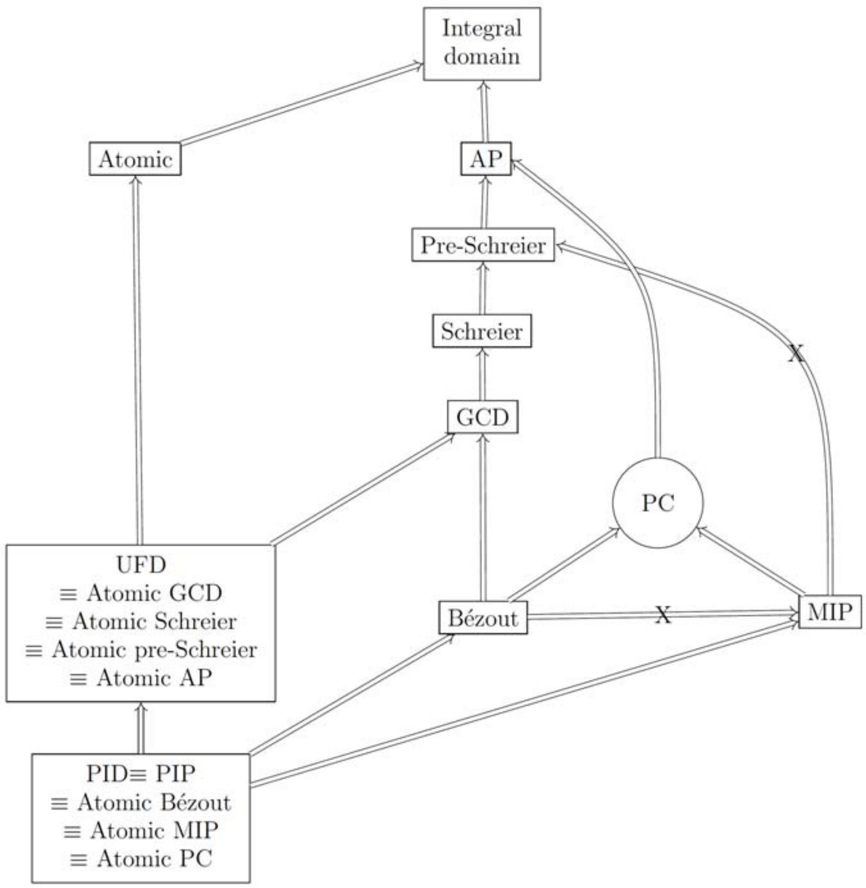

Diagram 2.

Proposition 3.2. *There exists an AP domain which is not a PC domain.*

Proof. Consider the additive monoid $M = \mathbb{N}_0 \times \mathbb{N}_0$ and the associated monoid domain $R = F[X;\, M],$ where F is a field. The polynomials $f \in R$ whose constant term is 0 form a maximal ideal, say $\mathfrak{m},$ of R. Let $D = R_{\mathfrak{m}}$ be the localization of R at $\mathfrak{m}.$ The elements of D have the form

$$x = \frac{X^{(r,s)} \cdot (a_0 + a_1 X^{(m_1, n_1)} \cdots + a_k X^{(m_k, n_k)})}{1 + b_1 X^{(p_1, q_1)} + \cdots + b_l X^{(p_l, q_l)}}, \tag{2}$$

where $k, l \geq 0, a_i, b_j \in F$ $(0 \leq i \leq k, 1 \leq j \leq l)$, $a_0 \neq 0$, and $(m_1, n_1), \ldots, (m_k, n_k)$ (pairwise distinct), $(p_1, q_1), \ldots, (p_l, q_l)$ (pairwise distinct), (r, s) are elements of $\mathbb{N}_0 \times \mathbb{N}_0$. Hence $x \sim X^{(r,s)}$ and so the only atoms of D are $X^{(0,1)}$ and $X^{(1,0)}$, and they are both prime. Thus D is an AP domain. The ideal $(X^{(0,1)}, X^{(1,0)})$ is proper, but it is not contained in a proper principal ideal as no $X^{(r,s)}$ can divide both $X^{(1,0)}$ and $X^{(0,1)}$ unless it is a unit. Thus D is not a PC domain. □

Proposition 3.3. *There exists a PC domain which is neither pre-Schreier* (*hence not Bézout*), *nor MIP.*

Proof. Let i be an irrational number such that $0 < i < 1$. Let q be a rational number such that $19 < q < 20$. Consider the additive submonoid

$$M = \left(\left[0, 5 + \frac{i}{2}\right] \cap \mathbb{Q}\right) \cup \left(5 + \frac{i}{2}, \infty\right)$$

of $\mathbb{R}_+$. Since $5 < 5 + \frac{i}{2} < 5.5$, we have

$$8 < q - 10 - i < 10,$$

so that $q - 10 - i \in M$. Let r be a rational number from $(10, 10 + i)$. Then $8 < q - r < 10$. We claim that it is impossible to find four numbers $\alpha, \beta, \alpha', \beta' \in M$ such that the following relations hold (at the same time):

$$\alpha + \beta = 10 + i, \tag{3}$$

$$\alpha + \alpha' = r, \tag{4}$$

$$\beta + \beta' = q - r. \tag{5}$$

Suppose to the contrary. Then by (4) at least one of the elements α, α' is $\leq \frac{r}{2}$, hence $< 5 + \frac{i}{2}$, hence rational. Since $\alpha + \alpha'$ is rational, the other element is rational too. Thus α is rational. In the same way β is rational. However, by the Equation (3) $\alpha + \beta$ is irrational, a contradiction.

Let now $R = F[X; M]$, where F is a field. Then the polynomials $f \in R$ whose constant term is 0 form a maximal ideal, say $\mathfrak{m}$, of R. Let $D = R_{\mathfrak{m}}$, the localization of R at $\mathfrak{m}$. The elements of D have the form

$$x = \frac{X^{\gamma}(a_0 + a_1X^{\gamma_1} + \cdots + a_mX^{\gamma_m})}{1 + b_1X^{\delta_1} + \cdots + b_mX^{\delta_m}},$$

where $m, n \geq 0, a_i, b_j \in F$ $(0 \leq i \leq m, 0 \leq j \leq n)$, and $\gamma, \gamma_1, \ldots, \gamma_m$, $\delta_1, \ldots, \delta_n$ are elements of M with $0 < \gamma_1 < \cdots < \gamma_m$, $0 < \delta_1 < \cdots < \delta_n$. We can write $x = X^{\gamma}u$, where u is a unit in D, $\gamma \in M$. The element x is a unit if and only if $\gamma = 0$. Since $q - 10 - i \in M$, we have

$$X^{10+i} | X^q = X^r X^{q-r}. \tag{6}$$

We show that it is not possible to find two elements $y, z \in D$ such that $y | X^r$, $z | X^{q-r}$, and $yz = X^{10+i}$. Suppose to the contrary. Then we can assume $y = X^{\alpha}$ and $z = X^{\beta}$ for some $\alpha, \beta \in M$, such that there are $\alpha', \beta' \in M$ satisfying the relations (3), (4), and (5). However, we showed above that that is not possible. Hence D is not pre-Schreier. In particular, D is not Bézout.

Note that the maximal ideal $\mathfrak{m}R_{\mathfrak{m}}$ of D is not finitely generated since for any $X^{\gamma_1}, \ldots, X^{\gamma_t}$, with $\gamma_i > 0$ $(i = 1, \ldots, t)$ elements of M, there is a $\gamma \in M$ such that $0 < \gamma < \min\{\gamma_1, \ldots, \gamma_t\}$, so that $X^{\gamma} \notin (X^{\gamma_1}, \ldots, X^{\gamma_t})$. Thus D is not MIP.

However, D is a PC domain since for any $X^{\gamma_1}, X^{\gamma_2} \in D$ (with $\gamma_1, \gamma_2 > 0$ elements of M) there is a sufficiently small positive rational number $\gamma \in M$ such that $X^{\gamma} \mid X^{\gamma_1}$ and $X^{\gamma} \mid X^{\gamma_2}$. Hence $D \supset (X^{\gamma}) \supseteq (X^{\gamma_1}, X^{\gamma_2})$.

□

Proposition 3.4. *There exists an MIP domain which is not pre-Schreier (hence not Bézout).*

Proof. Let the numbers $i, q, r,$ and the monoid M be like in Proposition 3.3. Consider the submonoid

$$N = (\mathbb{Z} \times (M \setminus \{0\})) \cup \mathbb{N}_0,$$

of the additive monoid $\mathbb{Z} \times \mathbb{R}_+$. Let $R = F[X; N]$, where F is a field. The polynomials $f \in R$ whose constant term is 0 form a maximal ideal, say $\mathfrak{m}$, of R. Let $D = R_{\mathfrak{m}}$ be the localization of R at $\mathfrak{m}$. The elements of D have the form

$$x = \frac{a_0 X^{(k_0, \alpha_0)} + \cdots + a_m X^{(k_m, \alpha_m)}}{1 + b_1 X^{(l_1, \beta_1)} + \cdots + b_n X^{(l_n, \beta_n)}}, \tag{7}$$

where $m, n \geq 0, a_i, b_j \in F$ $(0 \leq i \leq m, 1 \leq j \leq n)$, and $(k_0, \alpha_0), \ldots, (k_m, \alpha_m)$, $(l_1, \beta_1), \ldots, (l_n, \beta_n)$ are elements of N. We assume that $\alpha_0 \leq \cdots \leq \alpha_m$ and the (k_i, α_i) are pairwise distinct, as well as that $0 < \beta_1 \leq \cdots \leq \beta_n$ and the (l_j, β_j) are pairwise distinct. Let ν be the largest element of $\{0, 1, \ldots, m\}$ such that $\alpha_0 = \cdots = \alpha_\nu$. Then we denote

$$x^* = a_0 X^{(k_0, \alpha_0)} + \cdots + a_\nu X^{(k_\nu, \alpha_0)}.$$

Note that for any $x, y \in D$, we have

$$(xy)^* = x^* y^*. \tag{8}$$

Suppose also that $k_0 < k_1 < \cdots < k_\nu$. We consider two cases.

Case 1: $\alpha_0 = 0$. Then we factor out $X^{(k_0, 0)}$ from the numerator in (7) and have

$$x = (X^{(1,0)})^{k_0} \cdot \frac{a_0 + a_1 X^{(k_1 - k_0,\, 0)} + \cdots + a_\nu X^{(k_\nu - k_0, \alpha_0)} + \cdots + a_m X^{(k_m, \alpha_m)}}{1 + b_1 X^{(l_1, \beta_1)} + \cdots + b_n X^{(l_n, \beta_n)}},$$

so that either

$$x = u \quad (\text{if } k_0 = 0), \tag{9}$$

or

$$x = (X^{(1,0)})^{k_0} u \quad (\text{if } k_0 \geq 1), \tag{10}$$

where u is a unit in D.

Case 2: $\alpha_0 > 0$. Then we factor out any $X^{(k,0)} (k \in \mathbb{N}_0)$ from the numerator in (7) and we have

$$x = (X^{(1,0)})^{k} \cdot \frac{a_0 X^{(k_0 - k, \alpha_0)} + \cdots + a_m X^{(k_m - k, \alpha_m)}}{1 + b_1 X^{(l_1, \beta_1)} + \cdots + b_n X^{(l_n, \beta_n)}}. \tag{11}$$

Denote $\mathfrak{n} = (X^{(1,0)})$, the ideal of D generated by $X^{(1,0)}$. It follows from (9), (10), and (11) that $\mathfrak{n} = \mathfrak{m}R_{\mathfrak{m}}$, the maximal ideal of D, and that in the Case 1, x is an element of $\mathfrak{n}^{k_0} \setminus \mathfrak{n}^{k_0+1} (k_0 \geq 0)$, and in the Case 2, x is an element of $\mathfrak{n}^{\omega} = \bigcap_{k=1}^{\infty} \mathfrak{n}^k$. Since the maximal ideal is principal, D is an MIP domain.

We now show that D is not pre-Schreier. By (6) from Proposition 3.3,

$$X^{(0,10+i)} \mid X^{(0,q)} = X^{(0,r)} X^{(0,q-r)}. \tag{12}$$

We show that it is not possible to find two elements $y, z \in D$ such that

$$y \mid X^{(0,r)},$$

$$z \mid X^{(0,q-r)},$$

$$yz = X^{(0,10+i)}. \tag{13}$$

Suppose to the contrary. Then

$$yy' = X^{(0,r)}, \tag{14}$$

$$zz' = X^{(0,q-r)}, \tag{15}$$

for some $y', z' \in D$. Let $\alpha, \beta, \alpha', \beta'$ be the second coordinate of the exponents that appear in y^*, z^*, y'^*, and z'^*, respectively. Then from (13), (14), and (15), using (8), we get

$$\alpha + \beta = 10 + i,$$

$$\alpha + \alpha' = r,$$

$$\beta + \beta' = q - r.$$

However, this is not possible as we have seen in the proof of Proposition 3.3. □

Corollary 3.5. *The MIP condition is strictly weaker than the PIP condition.*

Proof. Otherwise every MIP domain would be a PID, hence pre-Schreier, contradicting the previous proposition. □

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