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THE CONVERGENCE ANALYSIS OF PD^{α}-TYPE ITERATIVE LEARNING CONTROL FOR RIEMANN-LIOUVILLE FRACTIONAL SYSTEM

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Abstract

The paper is concerned with the convergence of iterative learning control for some fractional equation. Using the Laplace transform and the M-L function, the mild solution is presented. The sufficient conditions of convergence for the open and closed PD^{α}-type iterative learning control are studied. Some examples are given to illustrate our theoretical results.

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1. Introduction

In this paper, we will study the convergence of iterative learning control of the following Riemann-Liouville fractional system:

$$\begin{cases} {}^{RL}D_t^{\alpha} x(t) = Ax(t) + Bu(t), \ t \in J = [0, b], \\ (g_{1-\alpha} * x)(0) = x_0, \\ y(t) = Cx(t), \end{cases} \tag{1.1}$$

where ${}^{RL}D_t^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α, $0 < \alpha < 1$; $A, B, C \in R^{n\times n}$; $u(t)$ is a control vector; and $g_{1-\alpha} = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$, $t > 0$.

The fractional calculus and fractional differential equations have attracted a lot of authors during past years, they made some outstanding works [1-4], because they described many phenomenon in engineering, physics, science, controllability and so on.

Iterative learning control is a process of continuous learning, and it is improve system performance. The important point for this control method is: Adapt to a system that can be continuously repeated during the running process. It is used to track a specific target, and the tracking target tends to remain the same. As far as we know, there are already many results of fractional order iterative control [9-13]. Especially, ILC with initial state learning for fractional order nonlinear systems was discussed in [10]. In [19], high-order D^{α}-type iterative learning control for fractional-order nonlinear time-delay systems was discussed, and the ILC design problem is converted to a stabilization problem for this discrete system, and by introducing a suitable norm and using a generalized Gronwall-Bellman lemma, the sufficiency condition for the robust convergence with respect to the bounded external disturbance of the control input and the tracking errors is obtained. In [20], D$^{\alpha}$-type iterative learning control for fractional-order linear time-delay systems was studied, the authors analyzed the control and learning processes, a discrete system for D$^{\alpha}$-type ILC was established, and ILC design problem was converted to a stabilization problem. Using the (λ, ξ)-norm and a generalized Gronwall-Bellman lemma, the sufficient condition for the robust convergence with respect to the bounded external disturbance of the control input and the tracking errors was obtained. In [23], the authors showed a robust second-order feedback PD$^{\alpha}$-type iterative learning control (ILC) for a class of uncertain fractional-order singular systems. Sufficient conditions for the robust convergence of the proposed PD$^{\alpha}$-type of learning control algorithm, and with respect to the bounded external disturbance and uncertainity, have been established and specified for time domain. In the above work, the authors mainly consider iterative learning control of Caputo fractional system, according to ours knowledge, the work about Riemann-Liouville fractional system are less. In this study, we use PD$^{\alpha}$-type iterative learning algorithm to Riemann-Liouville fractional system, the convergence analysis is discussed by applying λ-norm.

Motivated by the above mentioned works, the rest of this paper is organized as follows: In Section 2, we will show some definitions and preliminaries which will be used in the following parts. In Section 3, we give some results for PD$^{\alpha}$-type ILC for Riemann-Liouville fractional system. In Section 4, some simulation illustrate our proposed control algorithms.

In this paper, the norm for the n-dimensional vector $x = (x_1, x_2, \cdots, x_n)$ is defined as $\|x\| = \max_{1 \le i \le n} |x_i|$, λ-norm is defined as $\| \cdot \|_{\lambda} = \sup_{t \in [0, T]} \{e^{-\lambda t} | \cdot |\}$, $\lambda > 0$.

2. Some Preliminaries for Some Fractional System

In this section, we will show some definitions and preliminaries, for more details, see [1-4].

Definition 2.1. The Riemann-Liouville fractional integral of order α is defined as

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0,$$

where Γ is the gamma function.

For a function $f(t)$ given in the interval $[0, \infty)$, the expression

$${}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t (t-s)^{n-\alpha-1} f(s) dt,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integral part of number α, is called the Riemann-Liouville fractional derivative of order $\alpha > 0$.

Definition 2.2. Caputo's derivative for a function $f : [0, \infty) \to R$ can be written as

$${}^{c}D_t^{\alpha} f(t) = {}^{RL}D_t^{\alpha}\left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\right], \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integral part of real number α.

Definition 2.3. The definition of the two-parameter function of the Mittag-Leffler type is described by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0, z \in C,$$

if $\beta = 1$. we get the Mittag-Leffler function of one-parameter

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Lemma 2.4. *From the Definition* 2.4 *in* [14], *the function of the Mittag-Leffler type* $E_{\alpha,\alpha}(At^{\alpha})$ *is exponentially bounded, thus there are positive constant* $C_1, C_2, M,$ *and* $e_{\alpha}(t) = e^{\|A\|^{\frac{1}{\alpha}}t} < M$, *such that* $\|E_{\alpha,1}(At^{\alpha})\| \leq C_1 e_{\alpha}(t) \leq C_1 M$, $\|E_{\alpha,\alpha}(At^{\alpha})\| \leq C_2 e_{\alpha}(t) \leq C_2 M$.

Now, according to the papers ([15], [16]), we shall give the following lemma:

Lemma 2.5. *The mild solution of Equation* (1.1) *is given by*

$$x(t) = t^{\alpha-1}E_{\alpha,\alpha}(A, t)x_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})Bu(s)ds. \tag{1.2}$$

3. Open and Closed-Loop Case

Consider the following fractional system: $k = 0, 1, 2, 3, \cdots$

$$\begin{cases} {}^{RL}D_t^{\alpha}x_k(t) = Ax_k(t) + Bu_k(t), \quad t \in J = [0, b], \\ y_k(t) = Cx_k(t). \end{cases} \tag{1.3}$$

For the Equation (1.3), we use the following open and closed-loop PD$^{\alpha}$-type ILC algorithm, $t \in [0, b]$:

$$u_{k+1}(t) = u_k(t) + L_1 e_k(t) + L_2 e_{k+1}^{(\alpha)}(t), \tag{1.4}$$

where L_1, L_2 are the parameters which will be determined, $e_k(t) = y_d(t) - y_k(t)$; $y_d(t)$ be the target track function.

We make the following assumptions:

$$\text{H(1)}: \kappa_1 = \|I + CB\| - \frac{\|L_2CA\|\|B\|C_1Mbe^{\lambda b}}{\alpha} > 0,$$

$$\text{H(2)}: \kappa_2 = I + \frac{\|L_1C\|\|B\|C_1Mbe^{\lambda b}}{\alpha}, \frac{\kappa_2}{\kappa_1} < 1.$$

Theorem 3.1. *Assume that the open and closed-loop* PD^{α}*-type ILC algorithm* (1.4) *is used*, H(1) *and* H(2) *hold*, $y_k(\cdot)$ *be the output of the Equation* (1.3), *then* $\lim_{k\to\infty}\|e_k\|_\lambda = 0,\ t \in J$.

Proof. Firstly, we set

$$\delta x_k(t) = x_d(t) - x_k(t),$$

and

$$\delta u_k(t) = u_d(t) - u_k(t),$$

from (1.3) and ILC algorithm (1.4), we get

$$\delta x_k^{(\alpha)}(t) = {}^{RL}D_t^{\alpha}\delta x_k(t) = A\delta x_k(t) + B\delta u_k(t),$$

$$\begin{aligned} e_{k+1}^{(\alpha)}(t) = y_d^{(\alpha)}(t) - y_{k+1}^{(\alpha)}(t) &= C(x_d^{(\alpha)}(t) - x_{k+1}^{(\alpha)}(t)) = C\delta x_{k+1}^{(\alpha)}(t) \\ &= CA\delta x_{k+1}(t) + CB\delta u_{k+1}(t), \end{aligned} \tag{1.5}$$

$$\delta u_{k+1}(t) = \delta u_k(t) - L_1 e_k(t) - L_2 e_{k+1}^{(\alpha)}(t). \tag{1.6}$$

Substituting (1.5) into (1.6) gives

$$\begin{aligned} \delta u_{k+1}(t) &= \delta u_k(t) - L_1(y_d(t) - y_k(t)) - L_2(CA\delta x_{k+1}(t) + CB\delta u_{k+1}(t)) \\ &= \delta u_k(t) - L_1 C\delta x_k(t) - L_2(CA\delta x_{k+1}(t) + CB\delta u_{k+1}(t)), \end{aligned}$$

and

$$(I + CB)\delta u_{k+1}(t) = \delta u_k(t) - L_1 C\delta x_k(t) - L_2 CA\delta x_{k+1}(t)), \tag{1.7}$$

take λ-norm of (1.7),

$$\|(I + CB)\|\,\|\delta u_{k+1}\|_\lambda \le \|\delta u_k\|_\lambda + \|L_1 C\|\,\|\delta x_k\|_\lambda + \|L_2 CA\|\,\|\delta x_{k+1}\|_\lambda.$$

By Lemma 2.5, one can find

$$\begin{aligned} \delta x_{k+1}(t) &= x_d(t) - x_{k+1}(t) \\ &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha})B\delta u_{k+1}(s)ds, \end{aligned} \tag{1.8}$$

take λ-norm on the both side of (1.8),

$$\|\delta x_{k+1}\|_{\lambda} \leq \frac{\|B\| C_2 M b e^{\lambda b}}{\alpha} \|\delta u_{k+1}\|_{\lambda},$$

$$\|\delta x_{k}\|_{\lambda} \leq \frac{\|B\| C_2 M b e^{\lambda b}}{\alpha} \|\delta u_{k}\|_{\lambda}. \tag{1.9}$$

Combining (1.9) and (1.10), one can get

$$\|(I + CB)\| \|\delta u_{k+1}\|_{\lambda} \leq \|\delta u_{k}\|_{\lambda} + \frac{\|L_1 C\| \|B\| C_2 M b e^{\lambda b}}{\alpha} \|\delta u_{k}\|_{\lambda} + \frac{\|L_2 CA\| \|B\| C_2 M b e^{\lambda b}}{\alpha} \|\delta u_{k+1}\|_{\lambda},$$

$$(\|I + CB\| - \frac{\|L_2 CA\| \|B\| C_2 M b e^{\lambda b}}{\alpha})\|\delta u_{k+1}\|_{\lambda} \leq (I + \frac{\|L_1 C\| \|B\| C_2 M b e^{\lambda b}}{\alpha})\|\delta u_{k}\|_{\lambda}.$$

We may assume that

$$\kappa_1 = \|I + CB\| - \frac{\|L_2 CA\| \|B\| C_2 M b e^{\lambda b}}{\alpha},$$

$$\kappa_2 = I + \frac{\|L_1 C\| \|B\| C_2 M b e^{\lambda b}}{\alpha},$$

then

$$\kappa_1 \|\delta u_{k+1}\|_{\lambda} \leq \kappa_2 \|\delta u_{k}\|_{\lambda},$$

by the condition H(1); H(2),

$$\lim_{k \to \infty} \|\delta u_k\|_{\lambda} = 0.$$

On the other hand, the $(k+1)$-th iterative error is

$$e_{k+1}(t) = Cx_d(t) - Cx_{k+1}(t)$$

$$= C \int_0^t (t-s)^{\alpha - 1} E_{\alpha, \alpha}(A(t-s)^{\alpha}) B \delta u_{k+1}(s) ds,$$

$$\|e_{k+1}\|_{\lambda} \leq \frac{\|C\| \|B\| C_2 M b e^{\lambda b}}{\alpha} \|\delta u_{k+1}\|_{\lambda}, \tag{1.10}$$

taking $k \to \infty$ on two side of (1.10), it yields

$$\lim_{k\to\infty} \|e_k\|_\lambda = 0,$$

this complete the proof.

Theorem 3.1 Implied that the tracking error $e_k(t)$ depend on C and $x_k(t)$, it is also observed for (1.10) that the boundness of the parameters C, B, C_2, M implies the boundness of the $\|e_k\|_\lambda$.

4. Simulations

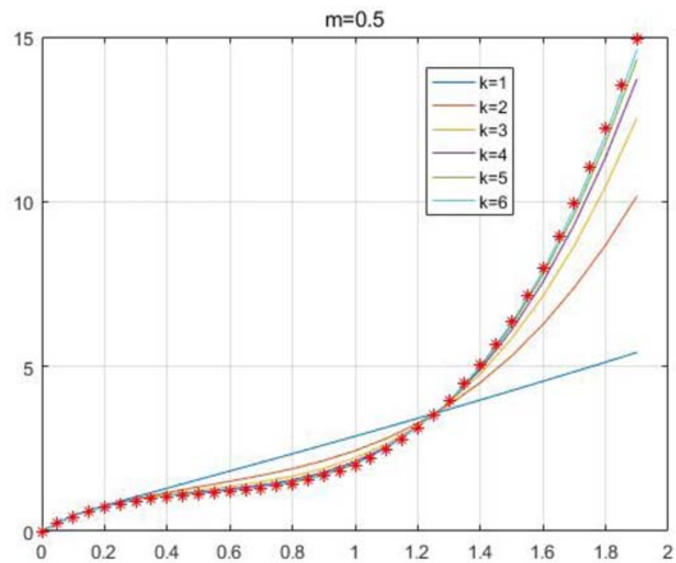
Consider the following PD^α-type ILC system:

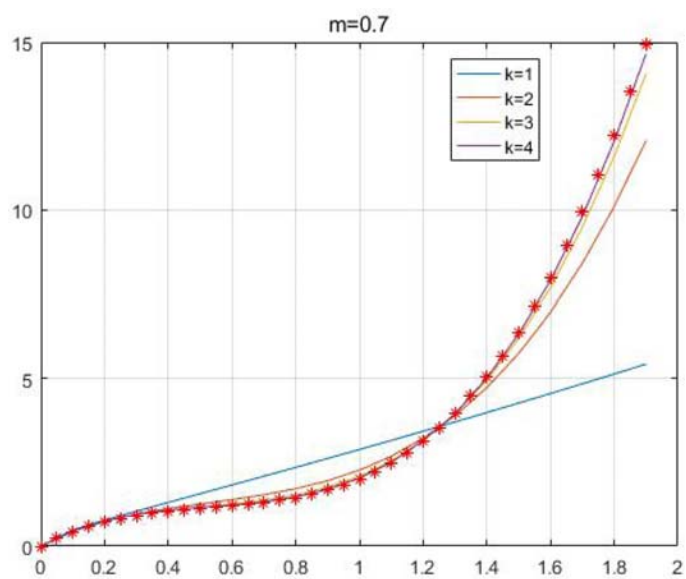
$$\begin{cases} {}^{RL}D_t^{0.5} x_k(t) = x_k(t) + 0.6u_k(t), \quad t \in J = [0, 1.8], \\ x(0) = 0.04, \\ y_k(t) = 0.5x_k(t), \end{cases} \tag{1.11}$$

with the iterative learning control $u_{k+1}(t) = u_k(t) + 0.001e_k(t) + 0.001$ $e_{k+1}^{(0.5)}(t)$.

We make the initial control $u_0(\cdot) = 0$, the target trajectory equation $y_d(t) = 5t^3 - 8t^2 + 5t$, $t \in (0, 1.8)$, and $\alpha = 0.5$, $A = 1$, $B = 0.6$, $C = 0.5$, $\lambda = 1$, $L_1 = 0.001$, $L_2 = 0.001$ and $C_1 = 1$, $C_2 = 1$, $\kappa_1 \approx 1.18$, $\kappa_2 \approx 1.12$, one can find all conditions of Theorem 3.1 are satisfied. We can use correction method, if $e(k) > 0$, $u(k) = u(k) - m \times e(k)$, or if $e(k) < 0$, $u(k) = u(k) + m \times e(k)$, k is the number of iteration, m is the parameters, we choose $m = 0.5, 0.7, 1, 1.2$ and the output of the system are shown in the following figures, ***denote the desired trajectory, — denote the output of the system, the tracking error is shown, which imply the number of iteration and the tracking error.

From Figures 1, 2 and Table 1, we find the tracking error tend to zero within 6 iteration, so the output of the system can track the desired trajectory almost perfectly. By comparing four case, when $m = 1$, the iteration number is only 2 and the tracking error is 0.001, thus the tracking performance is best and improved over the iteration domain.





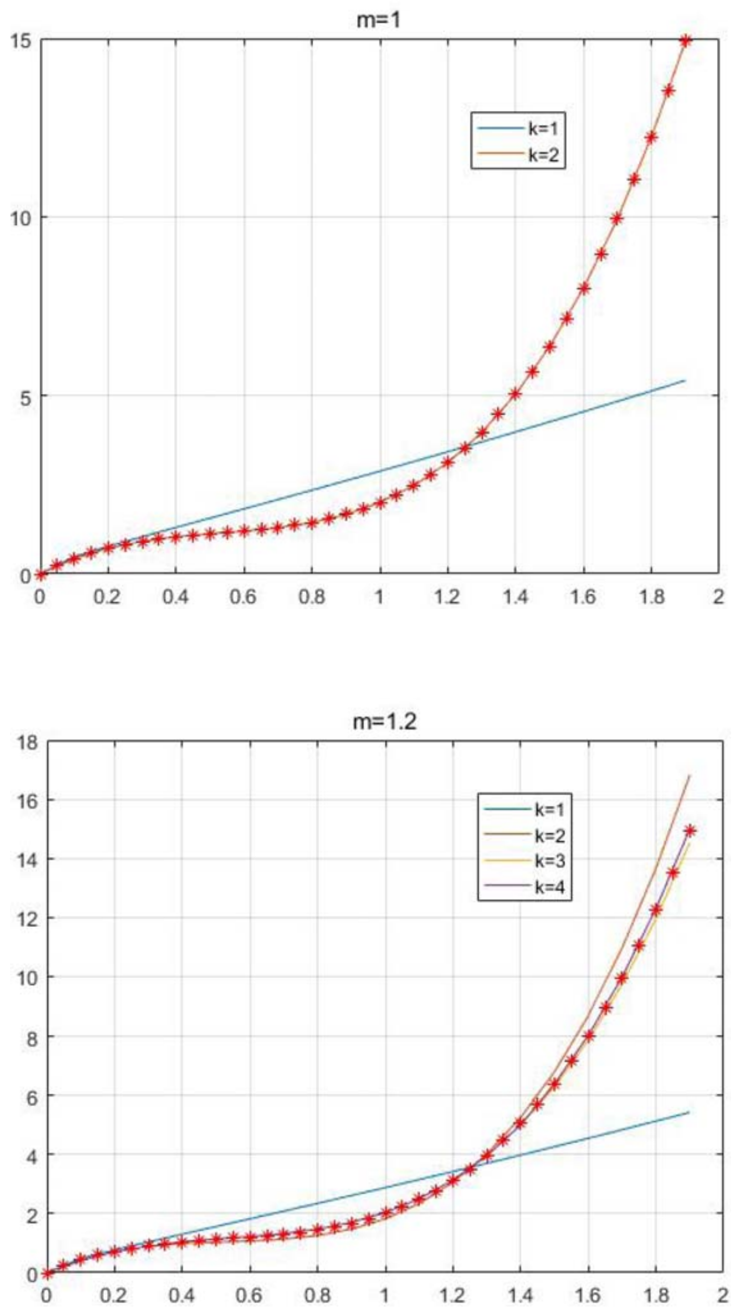
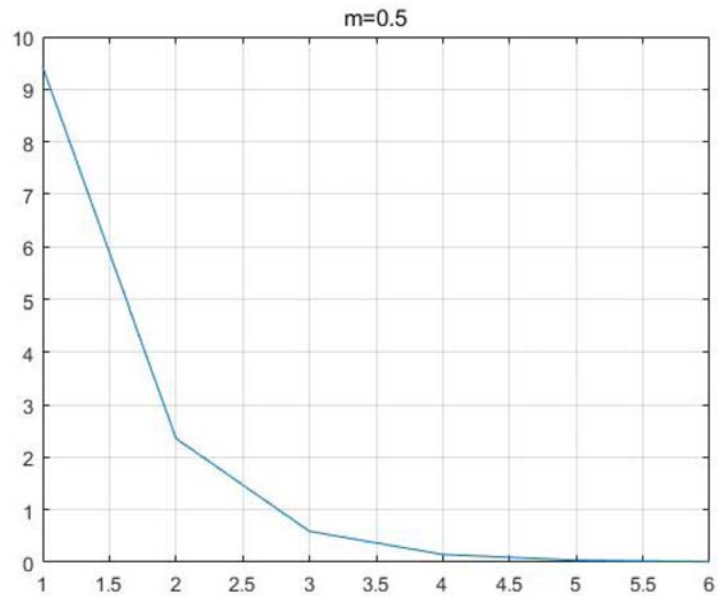
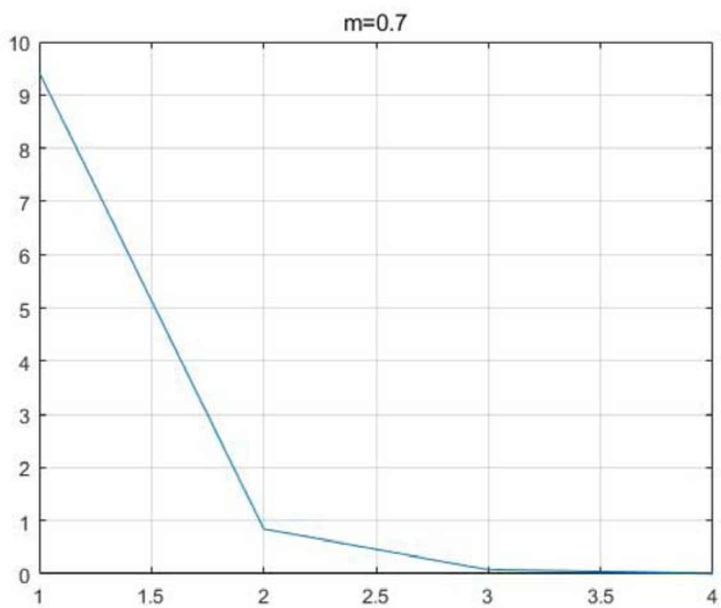


Figure 1. The desired trajectory and the result of iteration.


m=0.5
10
9
8
7
6
5
4
3
2
1
0
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6


m=0.7
10
9
8
7
6
5
4
3
2
1
0
1
1.5
2
2.5
3
3.5
4

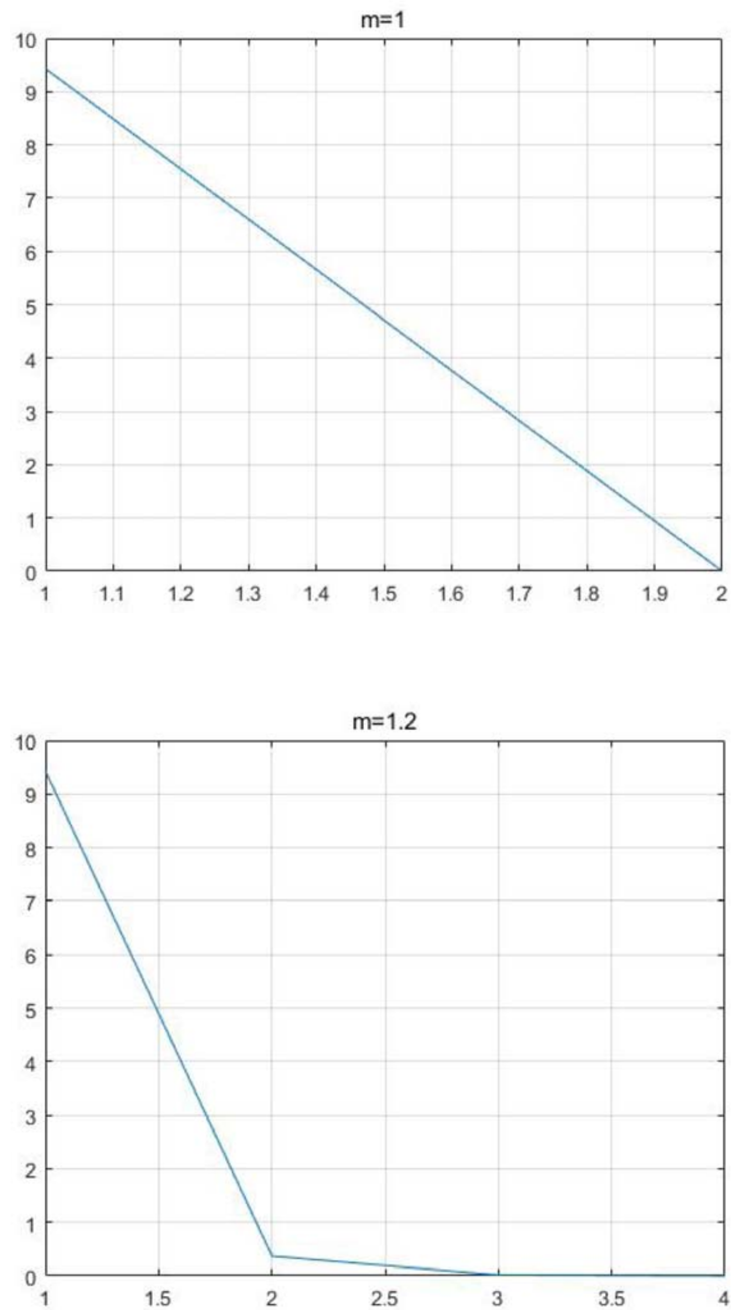


Figure 2. The tracking error.

Table 1. The number of iterations and the tracking error

m	The number of iterations	The tracking error	Run time (second)
0.5	6	0.002	43.32
0.7	4	0.0015	29.62
1	2	0.001	15
1.2	4	0.002	29.38

5. Conclusion

The paper is concerned with the convergence of iterative learning control for some fractional systems. Here, we use Laplace transform and the M-L function to get the mild solution for fractional systems. By the λ-norm, we get the sufficient conditions of convergence for the open and closed PD^{α}-type iterative learning control. Our results are new and bring inspiration for subsequent research.

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