

## ON EQUIVALENT LIE FOLIATIONS

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### Abstract

In this paper we study in which condition two Lie foliations are equivalent. In particular, our foliations are defined by the orbit of a group action on a compact manifold.

Every manifold in this paper is compact and our Lie group  $G$  is connected and simply connected.

### 1. Introduction

Let  $G$  be a simply connected Lie group and  $M$  be a compact manifold of dimension  $n$ . A Lie  $G$ -foliation is the given of a set  $\mathcal{F}$  of couples  $(U; f)$ , where  $U$  is an open subset of  $M$  and  $f : U \rightarrow G$  a submersion, having the following properties:

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2010 Mathematics Subject Classification: Primary 53B05; Secondary 53B20.

Keywords and phrases: foliation, nilpotent, Lie group, action group, manifold, compact.

Received September 27, 2018

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(i) The open sets  $U$  cover  $M$ .

(ii) for all  $(U; f)$  and  $(W; h) \in \mathcal{F}$ , there exists  $g \in G$  such that, for all  $x \in U \cap W$ , we have  $f(x) = h(x)g$ .

In particular, level surfaces of the submersions  $f$ , for  $(U; f) \in \mathcal{F}$ , recollect to form a Lie foliation on  $M$ . To avoid ambiguities, it is further assumed that  $\mathcal{F}$  is maximal in the following sense: if  $U$  is an open of  $M$  and  $f : U \rightarrow G$  a submersion, if, for any  $(W; h)$  in  $\mathcal{F}$ , there exists  $g$  in  $G$  with  $f = h.g$  on  $U \cap W$ , we have  $(U; f) \in \mathcal{F}$ . In this case  $G$  is called the transverse group of the  $G$ -foliation.

In the general case the structure transverse of  $G$ -foliation is given by Fedida's theorem [3]:

Let  $\mathcal{F}$  be a  $G$ -foliation on a compact manifold  $M$ . Let  $\tilde{M}$  be the universal covering of  $M$  and  $\tilde{\mathcal{F}}$  the recovery of  $\mathcal{F}$  on  $\tilde{M}$ . Then there exists a morphism  $h : \pi_1(M) \rightarrow G$  and one locally trivial fibration  $D : \tilde{M} \rightarrow G$  whose fibers are the leaves of  $\tilde{\mathcal{F}}$  and such that for any  $\gamma \in \pi_1(M)$ , the following diagram is commutative:

$$\begin{array}{ccc}
 \tilde{M} & \xrightarrow{\gamma} & \tilde{M} \\
 D \downarrow & & \downarrow D \\
 G & \xrightarrow{h(\gamma)} & G
 \end{array}$$

where the first line represents the transformation of  $\gamma \in \pi_1(M)$  on  $\tilde{M}$ . The group  $\Gamma = h\{\pi_1(M)\}$  (which is a subgroup of  $G$ ) is called the holonomy group of  $\mathcal{F}$  and the fibration  $D : \tilde{M} \rightarrow G$  is called the developing application of the foliation  $\mathcal{F}$ . This structure theorem makes it possible to build Lie foliation on a manifold.

If  $\mathfrak{g}$  is the Lie algebra of  $G$ . A *Maurer-Cartan* form with values in  $\mathfrak{g}$  is a differential 1-form  $\omega \in \Omega^1(M, \mathfrak{g})$  with trivial formal, i.e.,

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

Here  $[\omega, \omega]$  is the differential 2-form on  $M$  with values in  $\mathfrak{g}$  given by  $[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$ . If  $\omega$  is non singular, i.e., if  $\omega_x : T_x(M) \rightarrow \mathfrak{g}$  is surjective at any point  $x \in M$ , then the dimension of  $\mathfrak{g}$  is finite and  $\ker(\omega)$  is subbundle of  $T(M)$  of codimension  $\dim \mathfrak{g}$ . Vanishing of the formal curvature implies that the subbundle  $\ker(\omega)$  is involutive and hence defines a foliation  $\mathcal{F}$  on  $M$ , with  $\text{codim } \mathcal{F} = \dim \mathfrak{g}$  and

$$T(\mathcal{F}) = \ker(\omega).$$

A Lie foliation is a foliation defined in this way by a non-singular Maurer-Cartan form. Lie  $G$ -foliations form a very special class of foliations and satisfy various strong properties. For example, any left invariant Riemannian metric of  $G$  gives rise to a metric on the normal bundle of  $\mathcal{F}$ , invariant by the holonomy pseudogroup; that is,  $\mathcal{F}$  is a Riemannian foliation. Each leaf of  $\mathcal{F}$  has trivial holonomy and they are mutually Lipschitz diffeomorphic. Conversely by the work of Molino [14], the study of Riemannian foliations reduces to that of Lie foliations. See [15] for detailed accounts. Classical examples of Lie  $G$  foliations are:

**Example 1.1.** Let  $H$  be a Lie group and a surjective homomorphism  $f : H \rightarrow G$ , and let  $\Gamma$  be a uniform lattice of  $H$ . Then the right action of  $N = \ker(f)$  on the quotient space  $H/\Gamma$ , gives rise to a Lie  $G$ -foliation, with leaves diffeomorphic to  $N/\Gamma \cap N$ .

Assume  $G$  is compact, let  $B$  be a closed manifold, and let  $\phi : \pi_1(B) \rightarrow G$  be a group homomorphism. The suspension gives a fibration  $G \rightarrow M \rightarrow B$  together with a Lie  $G$ -foliations transverse to the fibers; the leaf is the  $\ker \phi$  covering of  $B$ .

We study in this paper when two Lie foliations are topologically equivalent, or, more generally, whether there is a continuous map  $f$  from  $M_1$  to  $M_2$  whose restriction to each leaf of  $\mathcal{F}_1$  is a covering map onto a leaf of  $\mathcal{F}_2$  (with  $\mathcal{F}_i$  Lie foliation of  $M_i$ ,  $i = 1, 2$ ). If the foliations are obtain by an action, Benardete studied the question and gave some results.

## 2. Homogeneous Foliations

An *automorphism* of a foliated manifold  $(M, \mathcal{F})$  is a diffeomorphism  $\phi : M \rightarrow M$  which preserves the foliation, i.e., for which the image of any leaf is a leaf, or equivalently,  $d\phi(T(\mathcal{F})) = T(\mathcal{F})$ . The automorphisms of  $(M, \mathcal{F})$  form a group which we shall denote by  $Aut(M, \mathcal{F})$ .

A foliation  $\mathcal{F}$  of a manifold  $M$  is called *homogeneous* if the group  $Aut(M, \mathcal{F})$  acts transitively on  $M$ , or in other words, if for any two points  $x, y \in M$ , there exists a diffeomorphism  $\phi : M \rightarrow M$  which preserves the foliation and maps  $x$  to  $y$ .

**Example 2.1.** Let  $\mathcal{F}$  be a foliation of a manifold  $M$  and let  $x$  and  $y$  be two points of  $M$  lying on the same leaf of  $\mathcal{F}$ . Then  $\mathcal{F}$  is a homogeneous foliation of  $M$ .

In particular, all the leaves of a homogeneous foliation are diffeomorphic. In this following example we have a homogeneous foliation.

**Example 2.2.** Let  $M$  be a differential manifold dimension  $n + m$  and  $G$  be a connected Lie group of dimension  $m$ . An action of  $G$  on  $M$  is the map  $\phi : G \times M \rightarrow M$  given by

- (1)  $\phi(e, x) = x$  for all  $x \in M$  (where  $e$  is unit element of  $G$ ).
- (2)  $\phi(g', \phi(g, x)) = \phi(gg', x)$ , for all  $x \in M$  and  $g, g' \in G$ .

Suppose for all  $x \in M$ , the dimension of the isotropy group  $G_x = \{g \in G : \phi(g, x) = x\}$  is not depend to  $x$ . Then the action  $\phi$  define a foliation  $\mathcal{F}$  of dimension  $m - \dim G_x$ ; these leaves are the orbits  $\{\phi(g, x) : g \in G\}$ . In particular, it is the case where the isotropy group is discret. As an integrable subbundle of  $T(M)$ , this foliation can simply be described in terms of Lie algebra  $\mathfrak{g}$  of  $G$ , namely, as the image of derivative of the action, which is a map of vector bundles  $\mathfrak{g} \times M \rightarrow T(M)$  of constant rank.

In the case where  $M = H/\Gamma$  with  $H$  a Lie group,  $\Gamma$  a discret subgroup and  $G$  a connected Lie subgroup of  $H$ ; then the foliation  $\mathcal{F}$  obtained by the action of  $G$  on  $M$  is a homogeneous foliation.

**Proposition 2.3.** *The homogeneous foliation thus defined is a Lie foliation whose transverse group is  $G$ .*

**Proof.** Let  $H$  be a simply connected Lie group,  $\Gamma$  be a  $H$  lattice and  $G$  be a connected Lie subgroup of  $H$ . By definition the action of  $G$  on the manifold  $H/\Gamma$  defines a  $\mathcal{F}$  which is homogeneous on  $H/\Gamma$ .

Since the map  $\rho : H \rightarrow G$  is a surjective morphism, then  $\mathcal{F}$  is a Lie foliation whose transverse group is  $G$ . In fact the images by  $\rho$  of the classes  $H$  modulo  $\ker \rho$  are the  $G$ -foliation on  $H/\Gamma$  which coincides with  $\mathcal{F}$ . Its developing map is  $\rho$  and its holonomy group is  $\Gamma$ .

**Example 2.4.** Let  $G$  be a simply connected Lie group,  $H$  be a connected subgroup of  $G$  and  $\Gamma$  be a lattice in  $G$ . The orbits of  $H$  define a foliation  $\mathcal{F}$  on  $G/\Gamma$  which is a homogeneous Lie  $G$ -foliation.

Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  defined respectively by the action of  $G_1$  on the manifold  $M_1 = H_1/\Gamma_1$  and the action of  $G_2$  on  $M_2 = H_2/\Gamma_2$ . If  $H_i$  is a simply connected solvable Lie group,  $\Gamma$  a lattice of  $H_i$  and  $G_i$  a connected Lie subgroup of  $H_i$ , then Benardete-Witte have showed

**Theorem 2.5** ([9]). *If the maximal torus of  $H_i$  is included in  $\overline{Ad_{H_i}\Gamma_i}$ , then  $\mathcal{F}_1$  is topologically equivalent to  $\mathcal{F}_2$  there exist an isomorphism  $\sigma : H_1 \rightarrow H_2$  such that  $\sigma(G_1) = G_2$  and  $\sigma(\Gamma_1)$  is conjugate to  $\Gamma_2$ .*

We want to generalise this result for any two nilpotent Lie foliations.

### 3. Equivalence of Nilpotent Lie Foliations

The nilpotent Lie foliations have been classified in inverse image by Haefliger. The following theorem characterizes the holonomy groups of such foliations. In this section, we recall some results of rigidity of lattices before stating the equivalence theorem of Lie foliations.

**Proposition 3.1** ([11]). *Let  $G_1$  and  $G_2$  be two simply connected nilpotent Lie groups, and  $K_1$  be a uniform subgroup of  $G_1$  : then any continuous homomorphism  $h : K_1 \rightarrow G_2$  extends to a unique continuous homomorphism  $\bar{h} : G_1 \rightarrow G_2$ .*

**Proposition 3.2** ([11]). *Let  $\Gamma$  be a nilpotent Lie group, of finite type, and without torsion. Then there exist a simply connected nilpotent Lie group  $\tilde{G}$  and an injective homomorphism  $J : \Gamma \rightarrow \tilde{G}$  such that  $J(\Gamma)$  is a discrete uniform subgroup of  $\tilde{G}$ .*

**Theorem 3.3** ([6]). *In a nilpotent Lie group  $G$  any finite type uniform subgroup is feasible as a holonomy group of Lie  $G$ -foliation on a compact manifold.*

It is shown that a Lie foliation defined by the Lie group  $\mathbb{R}^n$  is close to a foliation with closed leaves and the compact support is fiber on  $\mathbb{R}^n/\Gamma$ , where  $\Gamma$  is a discrete subgroup of  $\mathbb{R}^n$ : For this, we deform the holonomy group  $\Gamma$  to get a discrete subgroup of  $\mathbb{R}^n$ . The reciprocal classification is known thanks to the work of Mal'tsev and Haefliger, more precisely we have

**Theorem 3.4** ([6]). *Any nilpotent Lie foliation on a compact manifold is a reverse image of a homogeneous foliation.*

Using the results of Haefliger and Benardete we show that

**Theorem 3.5.** *Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be two nilpotent Lie foliations on a compact manifold whose respective transverse groups are  $G_1$  and  $G_2$ . Then  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are topologically equivalent.*

**Proof.** Let  $\Gamma_i$  be the holonomy group of the nilpotent Lie  $G_i$ -foliation  $\mathcal{F}_i$  (with  $i = 1, 2$ ),  $\Gamma_i$  is a uniform subgroup and finite type of  $G_i$ . We know, according to [11],  $\Gamma_i$  is a cocompact lattice in a nilpotent Lie group  $H_i$ , the canonical injection  $\rho : \Gamma_i \rightarrow G_i$  extend to a continuous homomorphism  $\rho' : H_i \rightarrow G_i$  so we have  $\rho = \rho' \circ j$ , where  $j : \Gamma_i \rightarrow H_i$ . We obtain an exact small sequence

$$1 \rightarrow F_i \rightarrow H_i \rightarrow G_i \rightarrow 1,$$

which define a homogeneous Lie  $G_i$ -foliation on the manifold  $H_i/\Gamma_i$ . We call  $\mathcal{F}'_i$  this foliation.

Since the foliations  $\mathcal{F}'_1$  and  $\mathcal{F}'_2$  are homogeneous then they are defined by the action of  $G_i$  on  $H_i/\Gamma_i$ . So since [9],  $\mathcal{F}'_1$  is topologically equivalence to  $\mathcal{F}'_2$ .

$\mathcal{F}'_1$  is inverse image of  $\mathcal{F}_1$  and  $\mathcal{F}'_2$  is inverse image of  $\mathcal{F}_2$ , hence the result.

**Remark 3.6.** In the case where our transverses groups are solvable and not nilpotent the theorem may not work because Meigniez has shown that there are solvable Lie foliation which are not the inverse image of any homogeneous Lie foliation.

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