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CHARACTERIZATIONS AND TESTING HYPOTHESES FOR $RNBU_{mgf}$ CLASS OF LIFE DISTRIBUTIONS

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Abstract

In this paper, we investigate the probabilistic characteristics for $(RNBU_{mgf})$ class, the closure properties under various reliability operations such as convolution, mixture and the homogeneous Poisson shock model are studied. A new hypothesis test is constructed to test exponentiality against $(RNBU_{mgf})$ based on moment inequality. Pitman asymptotic efficiency (PAE) are studied, the critical values of the test are calculated and tabulated, the power estimates are calculated to assess the performance of the test. Finally, sets of real data are used as examples to elucidate the use of the proposed test statistic for practical problems in the reliability analysis.

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1. Introduction

Stochastic comparisons between probability distributions play a fundamental role in probability, statistics, and some related areas, such as reliability theory, survival analysis, economics, and actuarial science. Certain classes of life distributions and their variations have been introduced in reliability, the applications of these classes of life distributions can be seen in engineering, social, biological science, maintenance, and biometrics. Therefore, statisticians and reliability analysts have shown a growing interest in modelling survival data using classifications of life distributions based on some aspects of ageing.

During the past decades, various classes of life distributions have been proposed in order to model different aspects of aging. The best known of these classes are *IFR*, *IFRA*, *NBU*, *NBUE*, *HNBUE*, and *DMRL*.

The following are the relation between these classes:

$$IFR \subset IFRA \subset NBU \subset NBUE \subset HNBUE,$$

$$IFR \subset DMRL \subset NBUE \subset HNBUE.$$

Consider a device (system or component) with life time T and a continuous life distribution $F(t)$, is put on operation. When the failure occurs the device will be replaced by a sequence of mutually independent devices. The spare devices are independent of the first device and identically distributed with the same life distribution $F(t)$. In the long run, the remaining life distribution of the system under operation at time t is given by stationary renewal distribution as follows:

$$W_F(x) = \frac{1}{\mu}\int_0^x \overline{F}(t)dt, \qquad 0 \le t < \infty,$$

with renewal survival function

$$\overline{W}_F(x) = \frac{1}{\mu}\int_x^\infty \overline{F}(t)dt, \qquad 0 \le t < \infty,$$

where $\mu = \int_0^\infty \overline{F}(u)du.$

For extra details, see Barlow and Proschan [6] and Abouammoh and Ahmed [1, 2].

A non-negative random variable X is said to be renewal new better than used (denoted by $X \in RNBU$) if and only if

$$\overline{W}_F(x+t) \leq \overline{W}_F(x)\overline{W}_F(t), \qquad \forall\ x,\, t \geq 0.$$

Definition 1.1 (Klar and Muller [14]).

Let X and Y be two non-negative random variables with survival functions $\overline{F}$ and $\overline{G}$, respectively. X is said to be smaller than Y in the moment generating function ordering (denoted by $X <_{mgf} Y$) if and only if,

$$\int_0^\infty e^{\lambda x}\overline{F}(x)dx \leq \int_0^\infty e^{\lambda y}\overline{G}(y)dy, \qquad \forall\ \lambda > 0.$$

Definition 1.2. X is renewal new better than used in the moment generating function order (denoted by $X \in RNBU_{mgf}$) if $X_t \leq_{mgf} Y$ for all $t > 0$, where Y is an exponential random variable with the same mean as X. Equivalently, $X \in RNBU_{mgf}$ if and only if,

$$\int_0^\infty e^{\lambda x}\overline{W}_F(x+t)dx \leq \overline{W}_F(t)\int_0^\infty e^{\lambda x}\overline{W}_F(x)dx, \qquad \forall\ \lambda > 0. \tag{1}$$

In literature, many statisticians derived the moment inequalities for the non-parametric families of ageing distributions, among them Ahmad [5], Abu-Youssef [4], El Arishy et al. [9], Mugdadi and Ahmad [19]; Diab et al. [7], Diab [8]; and El-Arishy et al. [10].

The construction of this paper is as follows: In Section 2, we discuss preservation under convolution, mixture, and the homogeneous Poisson shock model for $RNBU_{mgf}$ class of life distribution. We derive the moment inequalities for the $RNBU_{mgf}$ class in Section 3. In Section 4, we present testing exponentiality against $RNBU_{mgf}$ class, the Pitman asymptotic efficiencies (PAE) are calculated for some commonly used distributions in reliability in Section 5. In Section 6, Monte Carlo null distribution critical points are simulated for sample sizes $n = 5(5)50$ and the power estimates of the tests are also calculated. Finally, in Section 7, we discuss some applications to elucidate the usefulness of the proposed tests in reliability analysis.

2. Closure Properties

In this section, we study the closure properties of the renewal new better than used in moment generating function class $(RNBU_{mgf})$ of life distributions under some reliability operations such as convolution, mixture and the shock model in homogeneous case.

2.1. Convolution properties

The aim of this subsection is to discuss preservation under convolution properties of $RNBU_{mgf}$ class.

Theorem 2.1. *The $RNBU_{mgf}$ class is preserved under convolution.*

Proof. Suppose that F_1 and F_2 are two independent $RNBU_{mgf}$ lifetime distributions and their convolution is given by

$$\overline{F}(x) = \int_0^\infty \overline{F}_1(x-y)dF_2(y),$$

then

$$\mu\int_0^\infty \int_{x+t}^\infty e^{\lambda x}\overline{F}(u)dudx = \mu\int_0^\infty \int_{x+t}^\infty e^{\lambda x}\int_0^\infty \overline{F}_1(u-y)dF_2(y)dudx$$

$$= \mu\int_0^\infty \int_0^\infty \int_{x+t}^\infty e^{\lambda x}\overline{F}_1(u-y)dudxdF_2(y).$$

Since $\overline{F}_1$ is $RNBU_{mgf}$, then

$$\mu\int_0^\infty\int_{x+t}^\infty e^{\lambda x}\overline{F}(u)dudx \leq \int_0^\infty\left[\int_t^\infty \overline{F}_1(u-y)du\right.$$

$$\left.\cdot\int_0^\infty\int_x^\infty e^{\lambda x}\overline{F}_1(u-y)dudx\right]dF_2(y)$$

$$\leq \int_t^\infty\int_0^\infty \overline{F}_1(u-y)dF_2(y)du$$

$$\cdot\int_0^\infty\int_x^\infty e^{\lambda x}\int_0^\infty \overline{F}_1(u-y)dF_2(y)dudx$$

$$\leq \int_t^\infty \overline{F}(u)du\int_0^\infty\int_x^\infty e^{\lambda x}\overline{F}(u)dudx,$$

which complete the proof.

2.2. Mixture properties

The following theorem is stated and proved to show that the $RNBU_{mgf}$ class is preserved under mixture.

Theorem 2.2. *The* $RNBU_{mgf}$ *class is preserved under mixture.*

Proof. Suppose that $F(x)$ is the mixture of F_α, where each F_α is $RNBU_{mgf}$ since

$$\overline{F}(x) = \int_0^\infty \overline{F}_\alpha(x)dG(\alpha),$$

then

$$\mu\int_0^\infty\int_{x+t}^\infty e^{\lambda x}\overline{F}(u)dudx = \mu\int_0^\infty\int_{x+t}^\infty e^{\lambda x}\int_0^\infty \overline{F}_\alpha(u)dG(\alpha)dudx$$

$$= \mu\int_0^\infty\int_0^\infty\int_{x+t}^\infty e^{\lambda x}\overline{F}_\alpha(u)dudxdG(\alpha),$$

since F_α is $RNBU_{mgf}$, then

$$\mu\int_0^\infty\int_{x+t}^\infty e^{\lambda x}\overline{F}(u)dudx \le \int_0^\infty\left[\int_t^\infty \overline{F}_\alpha(u)du\int_0^\infty\int_x^\infty e^{\lambda x}\overline{F}_\alpha(u)dudx\right]dG(\alpha).$$

Upon using Chebyshev inequality for similarity ordered functions, we get

$$\mu\int_0^\infty\int_{x+t}^\infty e^{\lambda x}\overline{F}(u)dudx \le \int_t^\infty\int_0^\infty \overline{F}_\alpha(u)dG(\alpha)du$$

$$\cdot\int_0^\infty\int_x^\infty e^{\lambda x}\int_0^\infty \overline{F}_\alpha(u)dG(\alpha)dudx$$

$$\le \int_t^\infty \overline{F}(u)du.\int_0^\infty\int_x^\infty e^{\lambda x}\overline{F}(u)dudx,$$

which complete the proof.

2.3. Homogeneous Poisson shock model

An important application of ageing notion is shock models. Suppose that a device is subject to shocks occurring randomly in time according to a Poisson process with constant intensity s. Suppose further that the device has probability $\overline{P}_k$ of surviving the first k shocks, where $1 = \overline{P}_0 \ge \overline{P}_1 \ge \dots$, then the survival function of the device is given by,

$$\overline{H}(t) = \sum_{k=0}^\infty \overline{P}_k \frac{(st)^k}{k!} e^{-st}. \tag{2}$$

This shock model has been studied by Marshall et al. [18] for different ageing properties such as *IFR*, *IFRA*, *NBU*, and *NBUE*. Klefsjo [15] for *HNBUE* and Mahmoud et al. [17] for $NBURFR - t_0$.

Definition 2.1. A discrete distribution P_k, $k = 0, 1, \ldots, \infty$ with survival function $\overline{P}_k = 1 - P_k$ is said to have discrete $RNBU_{mgf}$ if,

$$\frac{m}{s}\sum_{j=0}^{\infty}\sum_{r=j+l}^{\infty} Z^j \overline{P}_r \leq \sum_{k=l}^{\infty}\overline{P}_k \sum_{j=0}^{\infty}\sum_{r=j}^{\infty} Z^j \overline{P}_r. \tag{3}$$

Theorem 2.3. *If* P_k *is discrete* $RNBU_{mgf}$, *then* $\overline{H}(t)$ *given by* (2) *is* $RNBU_{mgf}$.

Proof. Using Equation (2), we get

$$\mu_H = \int_0^{\infty}\overline{H}(u)du = \frac{1}{s}\sum_{k=0}^{\infty}\overline{P}_k\int_0^{\infty}\frac{(su)^k}{k!}e^{-su}d(su) = \frac{1}{s}\sum_{k=0}^{\infty}\overline{P}_k = \frac{m}{s},$$

where m is the mean of the discrete distribution $\overline{P}_k$.

Now, we need to show that

$$\mu_H\int_0^{\infty}\int_{x+t}^{\infty}e^{\lambda x}\overline{H}(u)dudx \leq \int_t^{\infty}\overline{H}(u)du\int_0^{\infty}\int_x^{\infty}e^{\lambda x}\overline{H}(u)dudx,$$

consider

$$\begin{aligned}
\mu_H\int_0^{\infty}\int_{x+t}^{\infty}e^{\lambda x}\overline{H}(u)dudx &= \frac{m}{s}\int_0^{\infty}\int_{x+t}^{\infty}e^{\lambda x}\sum_{k=0}^{\infty}\overline{P}_k\frac{(su)^k}{k!}e^{-su}dudx \\
&= \frac{m}{s}\int_0^{\infty}e^{\lambda x}\sum_{k=0}^{\infty}\overline{P}_k\left[\sum_{r=0}^{k}\frac{(s(x+t))^r}{r!}e^{-s(x+t)}\right]dx \\
&= \frac{m}{s^2}\sum_{k=0}^{\infty}\overline{P}_k\sum_{r=0}^{k}\sum_{j=0}^{r}\binom{r}{j}\frac{(st)^{r-j}}{r!}e^{-st}\int_0^{\infty}(sx)^j e^{-x(s-\lambda)}dx \\
&= \frac{m}{s^3}\sum_{j=0}^{\infty}\sum_{r=j}^{\infty}\sum_{k=r}^{\infty}\overline{P}_k e^{-st}\frac{(st)^{r-j}}{(r-j)!}\left(\frac{s}{s-\lambda}\right)^{j+1} \\
&= \frac{m}{s^3}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=l+j}^{\infty}\overline{P}_k e^{-st}\frac{(st)^l}{l!}\left(\frac{s}{s-\lambda}\right)^{j+1},
\end{aligned}$$

by using the $RNBU_{mgf}$ property,

$$\mu_H \int_0^\infty \int_{x+t}^\infty e^{\lambda x} \overline{H}(u) du dx \le \frac{1}{s^2} \sum_{j=0}^\infty \sum_{r=j}^\infty \overline{P}_r \sum_{l=0}^\infty \sum_{k=l}^\infty \overline{P}_k e^{-st} \frac{(st)^l}{l!} \left(\frac{s}{s-\lambda}\right)^{j+1}$$

$$\le \sum_{k=0}^\infty \overline{P}_k \left[\sum_{l=0}^k \frac{(st)^l}{l!} \frac{e^{-st}}{s}\right] \sum_{r=0}^\infty \overline{P}_r \left[\sum_{j=0}^r \frac{1}{s} \int_0^\infty \frac{(sx)^j}{j!} e^{-x(s-\lambda)} dx\right]$$

$$\le \int_t^\infty \left[\sum_{k=0}^\infty \overline{P}_k \frac{(su)^k}{k!} e^{-su}\right] du \int_0^\infty e^{\lambda x} \left[\int_x^\infty \sum_{r=0}^\infty \overline{P}_r \frac{(su)^r}{r!} e^{-su} du\right] dx$$

$$\le \int_t^\infty \overline{H}(u) du \int_0^\infty \int_x^\infty e^{\lambda x} \overline{H}(u) du dx,$$

which complete the proof.

3. Moments Inequalities

The following result provides moments inequality for the $RNBU_{mgf}$ distributions. In this, as well as subsequent results, all moments are assumed to be exist and finite.

Lemma 3.1. *Let* $\phi(\lambda) = \int_0^\infty e^{\lambda x} dF(x)$. *If* F *belonging to* $RNBU_{mgf}$, *then for all integer* $r \ge 0$,

$$\frac{1}{\lambda^2 (r+1)(r+2)} \mu_{r+2} [\phi(\lambda) - 1] - \frac{1}{\lambda(r+1)(r+2)} \mu \, \mu_{r+2}$$

$$\ge \frac{r!}{\lambda^{r+3}} \mu[\phi(\lambda) - 1] - \frac{r!}{\lambda^{r+2}} \mu^2 - \frac{r!}{\lambda^{r+1}} \mu \sum_{k=0}^r \frac{\lambda^k}{(k+2)!} \mu_{k+2}. \tag{4}$$

Proof. Since F is $RNBU_{mgf}$, multiplying Equation (1) by t^r and integrating both sides from 0 to ∞, then we have

$$\int_0^\infty t^r \int_0^\infty e^{\lambda x} \overline{W}(x+t) dx dt \le \int_0^\infty t^r \overline{W}(t) dt \int_0^\infty e^{\lambda x} \overline{W}(x) dx,$$

setting,

$$I = \int_0^\infty t^r \int_0^\infty e^{\lambda x}\overline{W}(x+t)dxdt$$

$$= \int_0^\infty t^r e^{-\lambda t}\left[\int_t^\infty e^{\lambda u}\overline{W}(u)du\right]dt$$

$$= \int_0^\infty e^{\lambda u}\overline{W}(u)\left[\int_0^u t^r e^{-t\lambda}dt\right]du,$$

where

$$\int_0^u t^r e^{-\lambda t}dt = \frac{r!}{\lambda^{r+1}}\left[1 - \sum_{k=0}^{r}\frac{(\lambda u)^k}{k!}e^{-\lambda u}\right].$$

Then,

$$I = \frac{r!}{\lambda^{r+1}}\left[\int_0^\infty e^{\lambda u}\overline{W}(u)du - \sum_{k=0}^{r}\frac{\lambda^k}{k!}\int_0^\infty u^k\overline{W}(u)du\right]$$

$$= \frac{1}{\mu}\left[\frac{r!}{\lambda^{r+3}}(\phi(\lambda)-1)\frac{r!}{\lambda^{r+2}}\mu - \frac{r!}{\lambda^{r+1}}\sum_{k=0}^{r}\frac{\lambda^k}{(k+2)!}\mu_{k+2}\right], \tag{5}$$

and

$$II = \int_0^\infty t^r\overline{W}(t)dt\int_0^\infty e^{\lambda x}\overline{W}(x)dx$$

$$= \frac{1}{\mu^2}\left[\int_0^\infty t^r\left(\int_t^\infty \overline{F}(u)du\right)dt\right]\cdot\left[\int_0^\infty e^{\lambda x}\left(\int_x^\infty \overline{F}(v)dv\right)dx\right]$$

$$= \frac{1}{\mu^2(r+1)(r+2)}\mu_{r+2}\left[\frac{1}{\lambda^2}\phi(\lambda) - \frac{1}{\lambda}\mu - \frac{1}{\lambda^2}\right]$$

$$= \frac{1}{\mu^2}\left[\frac{1}{\lambda^2(r+1)(r+2)}\mu_{r+2}(\phi(\lambda)-1) - \frac{1}{\lambda}\mu\,\mu_{r+2}\right]. \tag{6}$$

Hence, from (5) and (6) the result follows.

Note:

When $r = 1$, Equation (4) reduces to

$$\frac{1}{6\lambda^2}\mu_3[\phi(\lambda)-1] \geq \frac{1}{\lambda^4}\mu[\phi(\lambda)-1] - \frac{1}{\lambda^3}\mu^2 - \frac{1}{2\lambda^2}\mu\,\mu_2.$$

4. Testing Exponentiality Against $RNBU_{mgf}$ Class

In this section, a test statistic based on moment inequality is presented for testing $H_0 : F$ is exponential against the alternative; $H_1 : F$ belongs to $RNBU_{mgf}$ class but not exponential, we use $\delta(\lambda, r)$ as a measure of departure from exponentiality.

$$\delta(\lambda, r) = \left[\frac{1}{\lambda^2(r+1)(r+2)}\mu_{r+2} - \frac{r!}{\lambda^{r+3}}\mu\right][\phi(\lambda)-1] + \frac{r!}{\lambda^{r+2}}\mu^2$$
$$- \frac{1}{\lambda(r+1)(r+2)}\mu\,\mu_{r+2} + \frac{r!}{\lambda^{r+1}}\mu\left[\sum_{k=0}^{r}\frac{\lambda^k}{(k+2)!}\mu_{k+2}\right]. \tag{7}$$

At $r = 1$, then $\delta(\lambda, r)$ reduce to

$$\delta(\lambda, 1) = \left[\frac{1}{6\lambda^2}\mu_3 - \frac{1}{\lambda^4}\mu\right][\phi(\lambda)-1] + \frac{1}{\lambda^3}\mu^2 + \frac{1}{2\lambda^2}\mu\,\mu_2. \tag{8}$$

Note that under $H_0 : \delta(\lambda, r) = 0$, while under $H_1 : \delta(\lambda, r) > 0$.

4.1. Empirical test statistic for $RNBU_{mgf}$

To estimate $\delta(\lambda, r)$, let $X_1, X_2, \ldots, X_n$ be a random sample from F. Let $\overline{F}_n(x)$ denote the empirical distribution of the survival function $\overline{F}(x)$, where

$$\overline{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i > x), \qquad dF_n(x) = \frac{1}{n}.$$

And let $\hat{\delta}(\lambda, r)$ be the empirical estimate of $\delta(\lambda, r)$, where

$$\hat{\delta}(\lambda, r) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \Bigg[\left(\frac{1}{\lambda^2 (r+1)(r+2)} X_i^{r+2} - \frac{r!}{\lambda^{r+3}} X_i \right) (e^{\lambda X_j} - 1) + \frac{r!}{\lambda^{r+2}} X_i X_j$$

$$- \frac{1}{\lambda (r+1)(r+2)} X_i X_j^{r+2} + \frac{r!}{\lambda^{r+1}} X_i \left(\sum_{k=0}^{r} \frac{\lambda^k}{(k+2)!} X_j^{k+2} \right) \Bigg]. \tag{9}$$

To make the test statistic $\hat{\delta}(\lambda, r)$ scale invariant, we set $\hat{\xi}(\lambda, r) = \frac{\hat{\delta}(\lambda, r)}{\bar{X}^{r+3}}$, then

$$\hat{\xi}(\lambda, r) = \frac{1}{n^2 \bar{X}^{r+3}} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi(X_i, X_j), \tag{10}$$

where

$$\phi(X_i, X_j) = \left(\frac{1}{\lambda^2 (r+1)(r+2)} X_i^{r+2} - \frac{r!}{\lambda^{r+3}} X_i \right) (e^{\lambda X_j} - 1) + \frac{r!}{\lambda^{r+2}} X_i X_j$$

$$- \frac{1}{\lambda (r+1)(r+2)} X_i X_j^{r+2} + \frac{r!}{\lambda^{r+1}} X_i \left(\sum_{k=0}^{r} \frac{\lambda^k}{(k+2)!} X_j^{k+2} \right). \tag{11}$$

The following theorem summarizes the asymptotic normality of $\hat{\xi}(\lambda, r)$.

Theorem 4.1. *As* $n \to \infty$, $\sqrt{n}\left(\hat{\xi}(\lambda, r) - \delta(\lambda, r)\right)$ *is asymptotically normal with mean* 0 *and variance* σ^2 *given in Equation* (13). *Under* H_0, *the variance* σ^2 *reduces to Equation* (14).

Proof.

Let

$$\eta_1(X_1) = E[\phi(X_1,\ X_2)|X_1]$$

$$= \frac{1}{\lambda(1-\lambda)(r+1)(r+2)} X_1^{r+2}$$

$$+ \left[\frac{r!}{\lambda^{r+1}}\left(\sum_{k=0}^{r}\lambda^k\right) - \frac{r!}{\lambda^{r+1}(1-\lambda)} - \frac{r!}{\lambda}\right] X_1,$$

and

$$\eta_2(X_2) = E[\phi(X_1,\ X_2)|X_2]$$

$$= \left(\frac{r!}{\lambda^2} - \frac{r!}{\lambda^{r+3}}\right)(e^{\lambda X_2} - 1) - \frac{1}{\lambda(r+1)(r+2)} X_2^{r+2}$$

$$+ \frac{r!}{\lambda^{r+2}} X_2 + \frac{r!}{\lambda^{r+1}}\left(\sum_{k=0}^{r}\frac{\lambda^k}{(k+2)!} X_2^{k+2}\right).$$

Considering

$$\eta(X) = \eta_1(X) + \eta_2(X). \tag{12}$$

Using Equation (12), then the variance of $\hat{\xi}(\lambda,\ r)$ is given by

$$\sigma^2 = \mathrm{Var}\left[\left(\frac{r!}{\lambda^2} - \frac{r!}{\lambda^{r+3}}\right)(e^{\lambda X} - 1) + \left(\frac{r!}{\lambda^{r+2}} - \frac{r!}{\lambda^{r+1}(1-\lambda)} - \frac{r!}{\lambda}\right) X \right.$$

$$\left. + \frac{1}{(1-\lambda)(r+1)(r+2)} X^{r+2} + \frac{r!}{\lambda^{r+1}} \sum_{k=0}^{r}\left(\frac{\lambda^k}{(k+2)!} X^{k+2} + \lambda^k X\right)\right]. \tag{13}$$

At $r = 1,$ and under H_0 the variance reduces to

$$\sigma_0^2(\lambda,\ 1) = \mathrm{Var}\left[\frac{1}{6\lambda(1-\lambda)} X^3 + \frac{1}{2\lambda^2} X^2 + \frac{1-\lambda-\lambda^2}{\lambda^3(1-\lambda)} X + \frac{1-\lambda^2}{\lambda^4}(1 - e^{\lambda X})\right]$$

$$= \frac{2(7-5\lambda)}{(1-\lambda)^4(1-2\lambda)}. \tag{14}$$

5. The Pitman Asymptotic Efficiency

To asses the quality of this procedure, we evaluate the "Pitman's Asymptotic Efficiency" which is defined as

$$PAE(\delta(\theta)) = \frac{1}{\sigma_0} \left| \frac{d}{d\theta} \delta(\theta) \right|_{\theta \to \theta_0},$$

for some commonly used distributions in reliability,

(i) Linear failure rate family: $\overline{F}_1(x) = \exp(-x - \theta x^2 / 2)$, $x \geq 0$, $\theta \geq 0$;

(ii) Makeham family: $\overline{F}_2(x) = \exp(-x - \theta(x + e^{-x} - 1))$, $x \geq 0$, $\theta \geq 0$;

(iii) Weibull family: $\overline{F}_3(x) = \exp(-x^{\theta})$, $x \geq 0$, $\theta \geq 1$.

Note that the exponential distribution is attained at $\theta_0 = 0$ in (i), (ii), and at $\theta_0 = 1$ in (iii).

Since

$$\delta_\theta(\lambda, r) = \left[\frac{1}{\lambda^2 (r+1)(r+2)} \mu_{(r+2)\theta} - \frac{r!}{\lambda^{r+3}} \mu_\theta\right] [\phi_\theta(\lambda) - 1] + \frac{r!}{\lambda^{r+2}} \mu_\theta^2$$

$$- \frac{1}{\lambda(r+1)(r+2)} \mu_\theta \, \mu_{(r+2)\theta} + \frac{r!}{\lambda^{r+1}} \mu_\theta \left[\sum_{k=0}^{r} \frac{\lambda^k}{(k+2)!} \mu_{(k+2)\theta}\right].$$

The $PAE(\delta_\theta(\lambda, r))$ can be written as

$$PAE(\delta_\theta(\lambda, r)) = \frac{1}{\sigma_0} \left| \left[\frac{1}{\lambda^2 (r+1)(r+2)} \mu_{(r+2)\theta} - \frac{r!}{\lambda^{r+3}} \mu_\theta\right] \phi_\theta(\lambda) \right.$$

$$+ \left[\frac{1}{\lambda^2 (r+1)(r+2)} \mu_{(r+2)\theta} - \frac{r!}{\lambda^{r+3}} \mu_\theta\right] [\phi_\theta(\lambda) - 1]$$

$$+ \frac{2r!}{\lambda^{r+2}} \mu_\theta \mu_\theta - \frac{1}{\lambda(r+1)(r+2)} [\mu_\theta \mu_{(r+2)\theta} + \mu_\theta \mu_{(r+2)\theta}]$$

$$+ \frac{r!}{\lambda^{r+1}} \sum_{k=0}^{r} \frac{\lambda^k}{(k+2)!} [\mu_\theta \mu_{(k+2)\theta} + \mu_\theta \mu_{(k+2)\theta}] \Bigg|. \tag{15}$$

In this case, we obtain

$$PAE(\delta_\theta(\lambda, 1), \bar{F}_1(x)) = \frac{1}{\sigma_0} \left| \frac{2}{(1-\lambda)^2} \right|,$$

$$PAE(\delta_\theta(\lambda, 1), \bar{F}_2(x)) = \frac{1}{\sigma_0} \left| \frac{3}{8(2 - 3\lambda + \lambda^2)} \right|,$$

$$PAE(\delta_\theta(\lambda, 1), \bar{F}_3(x)) = \frac{1}{\sigma_0} \left| - \frac{\lambda(2+3\lambda) + 2(1+\lambda)\log[1-\lambda]}{3\lambda^2(1-\lambda)} \right|.$$

Table 1. Pitman asymptotic efficiencies for various values of λ

Distribution	$\lambda = 0.01$	$\lambda = 0.001$	$\lambda = 1.5$	$\lambda = 2.5$	$\lambda = 3.5$
LFR	0.53105	0.53418	2.82843	1.20605	1.06904
Makeham	0.04954	0.05005	0.53033	0.67840	0.33408
Weibull	0.22060	0.22251	–	–	–

Table 1 gives the efficiencies of our proposed test $\delta_\theta(\lambda, s)$ for various values of λ.

6. Monte Carlo Null Distribution Critical Points

In practice, simulated percentiles for small samples are commonly used by applied statisticians and reliability analyst. We have simulated the upper percentile values for 90%, 95%, 98%, and 99%. Table 2 presents these percentile values of the statistics $\hat{\xi}(\lambda, r)$ and the calculations are based on 10000 simulated samples of sizes $n = 5(5)50$, 29, 43 at $r = 1$ and $\lambda = 0.01$.

Table 2. The upper percentile of $\hat{\xi}(\lambda, r)$

n	90%	95%	98%	99%
5	0.42647	0.63558	0.98456	1.33650
10	0.38006	0.54105	0.83123	1.06132
15	0.32198	0.41284	0.56607	0.73496
20	0.28651	0.35809	0.46665	0.57325
25	0.26249	0.31605	0.40156	0.47571
29	0.24886	0.29879	0.37626	0.43004
30	0.24795	0.29802	0.37135	0.42359
35	0.23369	0.27692	0.33919	0.38258
40	0.22097	0.25932	0.30989	0.34963
43	0.22123	0.25467	0.30169	0.34131
45	0.21904	0.25270	0.29475	0.32901
50	0.21216	0.24384	0.28816	0.31959

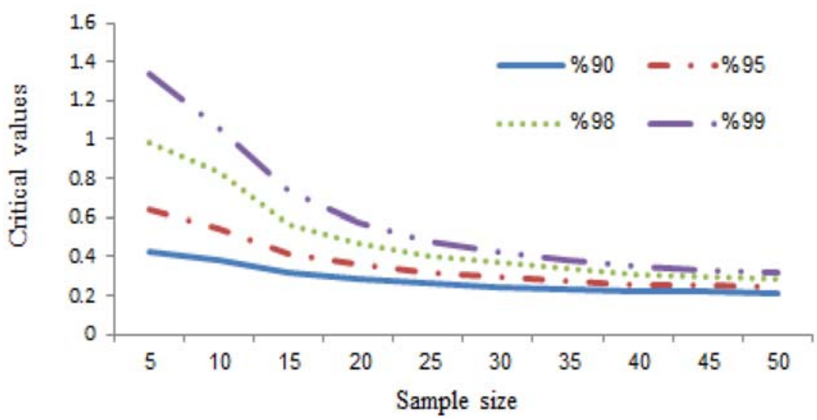

Figure 1. Relation between critical values, sample size and confidence levels at $r = 1$, $\lambda = 0.01$.

In view of Table 2 and Figure 1, it is noticed that the critical values are increasing as the confidence level increasing and decreasing as the sample size increasing.

6.1. The power estimates

In this subsection, we present the power estimates of the test statistic $\hat{\xi}(\lambda, r)$ at the significance levels $\alpha = 0.05$ and $\alpha = 0.01$, respectively. These powers are estimated for LFR, Weibull, and Gamma distribution. The estimates are based on 10000 simulated samples for sizes $n = 10$, 20, and 30 with parameter $\theta = 2, 3,$ and 4, at $r = 1$, $\lambda = 0.01$.

Table 3. Power estimates using $\alpha = 0.05$

Distribution	θ	$n = 10$	$n = 20$	$n = 30$
Weibull	2	0.8161	0.9993	1.0000
	3	0.7126	1.0000	1.0000
	4	0.6434	1.0000	1.0000
Gamma	2	0.9999	0.9838	0.9804
	3	1.0000	0.9945	0.9947
	4	1.0000	0.9981	0.9977
LFR	2	0.3147	0.8894	1.0000
	3	0.1211	0.7073	1.0000
	4	0.0439	0.4688	1.0000

Table 4. Power estimates using $\alpha = 0.01$

Distribution	θ	$n = 10$	$n = 20$	$n = 30$
Weibull	2	0.9948	1.0000	1.0000
	3	0.9993	1.0000	1.0000
	4	0.9999	1.0000	1.0000
Gamma	2	1.0000	0.9918	0.9939
	3	1.0000	0.9967	0.999
	4	1.0000	0.9987	0.9994
LFR	2	0.7964	1.0000	1.0000
	3	0.6068	1.0000	1.0000
	4	0.4225	1.0000	1.0000

From Tables 3 and 4, it is noted that the power of the test increases by increases the value of the parameter θ and sample size n.

7. Applications

In this section, we apply our test to some real data-sets in the case of non censored data at 95% confidence level.

Data-set # 1.

Consider the data set is from Kotz and Johnson [16] and represents the survival times (in years) after diagnosis of 43 patients with a certain kind of leukemia. In this case, we get at $r = 1, \lambda = 0.01, \hat{\xi}(\lambda, r) = 0.324448,$ and these value exceeds the tabulated critical value in Table 3. It is evident that at the significant level 0.05 this data set has $RNBU_{mgf}$ property.

Data-set # 2.

Consider the data set in Abouammoh et al. [3]. These data represent set of 40 patients suffering from blood cancer from one of ministry of health hospitals in Saudi Arabia. In this case, we get at $r = 1,$ $\lambda = 0.01,$ $\hat{\xi}(\lambda, r) = 0.202384,$ and these value less than the tabulated critical value in Table 3. This means that the set of data have exponential property.

Data-set # 3.

The following data in Keating et al. [13] set on the time, in operating days, between successive failures of air conditioning equipment in an aircraft. We can see that the value of test statistic at $r = 1,$ $\lambda = 0.01,$ $\hat{\xi}(\lambda, r) = 0.357586,$ and these values greeter than the tabulated critical value in Table 3. This means that the set of data have $RNBU_{mgf}$ property.

Data-set # 4.

Consider the data set given in Grubbs [12]. This data set gives the times between arrivals of 25 customers at a facility. It is easily to show that at $r = 1$, $\lambda = 0.01$, $\hat{\xi}(\lambda, r) = 0.184959$, which is less than the critical value of Table 3. Then we accept H_0 which states that the data set have exponential property.

Data-set # 5.

Consider the well-known Darwin data Fisher [11] that represent the differences in heights between cross- and self-fertilized plants of the same pair grown together in one pot. It is easily to show that at $r = 1$, $\lambda = 0.01$, $\hat{\xi}(\lambda, r) = 0.692288$, which is greater than the critical value of Table 3. Then we accept H_1 which states that the data set have $RNBU_{mgf}$ property.

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