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ON RANDOM $L^0(\mathcal{F})$-CONVEX CONES IN COMPLETE RANDOM NORMED MODULES

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Abstract

Based on the previous results, this paper continues to develop the theory of random convex analysis. First, motivated by the recent work of Ekeland's variational principle on a complete random normed module, we prove that the set of local conical support points of S is dense in the boundary of S under the locally L^0-convex topology, where S is a $\mathcal{T}_c$-closed subset of a random normed module E and S has the countable concatenation property. Then, we prove that it is a nonconvex generalization of the Bishop-Phelps theorem in complete random normed modules. This result is a nontrivial random extension of the corresponding classic result.

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1. Introduction

In [1], we established the Ekeland's variational principle on a complete *RN* module and the Bishop-Phelps theorems in complete *RN* modules under two kinds of topologies. Based on these results, this paper is devoted to prove that the set of local conical support points of S is dense in the boundary of S under the locally L^0-convex topology, where S is a $\mathcal{T}_c$-closed subset of a random normed module E and S has the countable concatenation property. When the probability space $(\Omega, \mathcal{F}, P)$ is trivial, our results reduce to the corresponding classic result [2]. So the extension of our results is nontrivial.

The remainder of this paper is organized as follows: in Section 2, we briefly recall some necessary definitions and facts; in Section 3, we give our main results and proofs.

2. Preliminaries

Throughout this paper, $(\Omega, \mathcal{F}, P)$ denotes a probability space, K the field R of real numbers or C of complex numbers, N the set of positive integers, $\bar{L}^0(\mathcal{F})$ the set of equivalence classes of extended real-valued random variables on Ω and $L^0(\mathcal{F}, K)$ the algebra of equivalence classes of K-valued random variables on Ω under the ordinary scalar multiplication, addition and multiplication operations on equivalence classes, which is denoted by $L^0(\mathcal{F})$ when $K = R$.

It is well known from [3] that $\overline{L}^0(\mathcal{F})$ is a complete lattice under the ordering $\leqslant$: $\xi \leqslant \eta$ iff $\xi^0(\omega) \leqslant \eta^0(\omega)$, for almost all ω in Ω (briefly, a.s.), where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively. Furthermore, every subset A of $\overline{L}^0(\mathcal{F})$ has a supremum and infimum, denoted by $\bigvee A$ and $\bigwedge A$, respectively. It is clear that $L^0(\mathcal{F})$, as a sublattice of $\overline{L}^0(\mathcal{F})$, is also a complete lattice in the sense that every subset with an upper bound has a supremum.

Let $A \in \mathcal{F}$ and ξ and η be in $\overline{L}^0(\mathcal{F})$, we say that $\xi > \eta$ on $A (\xi \geq \eta \text{ on } A)$ if $\xi^0(\omega) > \eta^0(\omega)$ (accordingly, $\xi^0(\omega) \geq \eta^0(\omega)$) for almost all $\omega \in A$, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively. Similarly, one can understand $\xi \neq \eta$ on A and $\xi = \eta$ on A. Specially, $\tilde{I}_A$ stands for the equivalence class of I_A, where $I_A(\omega) = 1$ if $\omega \in A$, and 0 if $w \notin A$.

This paper always employs the following notation: $L^0(\mathcal{F}) = L^0(\mathcal{F}, R)$, $L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) | \xi \geqslant 0\}$, $L^0_{++}(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) | \xi > 0 \text{ on } \Omega\}$.

Let us first recapitulate some known terminology.

Definition 2.1 ([4]). An ordered pair $(S, \|\cdot\|)$ is called *a random normed space* (briefly, an RN space) over K with base $(\Omega, \mathcal{F}, P)$ if S is a linear space over K and $\|\cdot\|$ is a mapping from S to $L^0_+(\mathcal{F})$ such that the following axioms are satisfied:

(RN-1) $\|\alpha x\| = |\alpha| \|x\|$, $\forall \alpha \in K$ and $x \in S$;

(RN-2) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in S$;

(RN-3) $\|x\| = 0$ implies $x = \theta$ (the null vector in S),

where $\|x\|$ is called the random norm of the vector x.

In addition, if S is a left module over the algebra $L^0(\mathcal{F}, K)$ and $\|\cdot\|$ also satisfies the following:

$(RNM\text{-}1)$ $\|\xi x\| = |\xi|\,\|x\|$, $\forall \xi \in L^0(\mathcal{F}, K)$ and $x \in S$,

then such an RN space $(S, \|\cdot\|)$ is called *a random normed module* (briefly, an RN module) over K with base $(\Omega, \mathcal{F}, P)$, such a random norm $\|\cdot\|$ is called an L^0-norm.

The algebra $L^0(\mathcal{F}, K)$ is a special RN module when $\|\cdot\|$ is defined by $\|x\| = |x|$, $\forall x \in L^0(\mathcal{F}, K)$.

Although RN modules are a random generalization of classical normed spaces, its structure can simultaneously induce two kinds of topologies, namely, the (ε, λ)-topology and the locally L^0-convex topology as follows:

Definition 2.2 ([5]). Given an RN module $(E, \|\cdot\|)$ over K with base $(\Omega, \mathcal{F}, P)$. Let ε and λ be any two positive numbers such that $0 < \lambda < 1$, define $N_\theta(\varepsilon, \lambda) = \{x \in E \mid P(\{\omega \in \Omega \mid \|x\|(\omega) < \varepsilon\}) > 1 - \lambda\}$ and let $\mathcal{N}_\theta = \{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1\}$. Then $\mathcal{N}_\theta$ becomes a local base at θ of some Hausdorff linear topology, called the (ε, λ)*-topology* for $(E, \|\cdot\|)$.

Then for every RN module, we always denote by $\mathcal{T}_{\varepsilon,\lambda}$ the (ε, λ)-topology. The introduction of the (ε, λ)-topology owes to Schweizer and Sklar [6]. In fact, the (ε, λ)-topology is frequently used for the research of probabilistic normed spaces [7-10]. It is clear that a net $\{x_\alpha, \alpha \in \wedge\}$ in E converges in the (ε, λ)-topology to $x \in E$ if and only if $\{\|x_\alpha - x\|, \alpha \in \wedge\}$ converges in probability P to 0.

The locally L^0-convex topology was first introduced by Filipović et al. [11].

Definition 2.3 ([11])**.** Given an RN module $(E, \|\cdot\|)$ over K with base $(\Omega, \mathcal{F}, P)$, then $\mathcal{U}_\theta = \{B(\varepsilon) | \varepsilon \in L^0_{++}(\mathcal{F})\}$ is a local base at $\theta \in E$ of some Hausdorff locally L^0-convex topology, called the *locally* L^0*-convex topology* induced by $\|\cdot\|$, where $B(\varepsilon) = \{y \in E \,|\, \|y\| \leqslant \varepsilon\}$.

From now on, for each RN module, we always denote by $\mathcal{T}_c$ the locally L^0-convex topology induced by $\|\cdot\|$.

To introduce the main results of this paper, let's recall a very important notion.

Definition 2.4 ([12])**.** Let E be an $L^0(\mathcal{F}, K)$-module and G be a subset of E. G is said to have the countable concatenation property if for each sequence $\{g_n : n \in N\}$ in G and each countable partition $\{A_n, n \in N\}$ of Ω to $\mathcal{F}$ there always exists $g \in G$ such that $\tilde{I}_{An} g = \tilde{I}_{An} g_n$ for each $n \in N$. If E has the countable concatenation property, $H_{cc}(G)$ denotes the countable concatenation hull of G, namely, the smallest set containing G and having the countable concatenation property.

Now let us recall the notion of a random conjugate space, which is crucial in random functional analysis.

Definition 2.5 ([12])**.** Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. Then $E^*_{\varepsilon,\lambda} = \{f : E \to L^0(\mathcal{F}, K) | f$ is a continuous module homomorphism from $(E, \mathcal{T}_{\varepsilon,\lambda})$ to $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})\}$ and $E^*_c = \{f : E \to L^0(\mathcal{F}, K) | f$ is a continuous module homomorphism from $(E, \mathcal{T}_c)$ to $(L^0(\mathcal{F}, K), \mathcal{T}_c)\}$, are called the random conjugate spaces of $(E, \|\cdot\|)$ under $\mathcal{T}_{\varepsilon,\lambda}$ and $\mathcal{T}_c$, respectively.

Under $\mathcal{T}_{\varepsilon,\lambda}$ and $\mathcal{T}_c$, an *RN* module $(E, \|\cdot\|)$ over K with base $(\Omega, \mathcal{F}, P)$ has the same random conjugate space, namely, $E^*_{\varepsilon,\lambda} = E^*_c$. Thus they can be denoted by the same notation E^* [12]. Further, define $\|\cdot\|^* : E^* \to L^0_+(\mathcal{F})$ by $\|f\|^* = \wedge\{\xi \in L^0_+(\mathcal{F}) : |f(x)| \leqslant \xi \cdot \|x\|, \forall x \in E\}$, then $(E^*, \|\cdot\|^*)$ is also an *RN* module over K with base $(\Omega, \mathcal{F}, P)$ and $\|f\|^* = \vee\{|f(x)| : x \in E$ and $\|x\| \leqslant 1\}$ for any $f \in E^*$. Besides, it is well known that E^* is $\mathcal{T}_{\varepsilon,\lambda}$-complete, so E^* must have the countable concatenation property [12].

Let E be a left module over the algebra $L^0(\mathcal{F}, K)$, a nonempty subset M of E is called $L^0(\mathcal{F})$-convex if $\xi x + \eta y \in M$ for any x and $y \in M$ and ξ and $\eta \in L^0_+(\mathcal{F})$ such that $\xi + \eta = 1$. In addition, it is called an $L^0(\mathcal{F})$-convex cone if $\xi x + \eta y \in M$ for any x and $y \in M$ and ξ and $\eta \in L^0_+(\mathcal{F})$, further M is called pointed if $M \bigcap (-M) = \theta$.

3. On Random $L^0(\mathcal{F})$-Convex Cones in Complete Random Normed Modules

In this section, we establish a nonconvex generalization of the Bishop-Phelps theorems in complete *RN* modules, namely, Theorem 3.9 below. To introduce it, we need a series of preparations.

Definition 3.1. Let E be an *RN* module over R with base $(\Omega, \mathcal{F}, P)$, $f \in E^*$ and $k \in L^0_{++}(\mathcal{F})$. Define

$$C(f, k) = \{y \in E : k\|y\| \leq f(y)\}.$$

It is easy to see that $C(f, k)$ is a pointed, closed and $L^0(\mathcal{F})$-convex random cone under each of $\mathcal{T}_{\varepsilon,\lambda}$ and $\mathcal{T}_c$.

Definition 3.2 ([1]). Let X be a Hausdorff space and $f : X \to \overline{L}^0(\mathcal{F})$, then f is bounded from below (resp., bounded from above) if there exists $\xi \in L^0(\mathcal{F})$ such that $f(x) \geq \xi$ (accordingly, $f(x) \leq \xi$) for any $x \in X$.

Lemma 3.3 below is the Ekeland's variational principle on a complete random normed module, which was established by us in [1].

Lemma 3.3 ([1]). *Let $(E, \|\cdot\|)$ be a $\mathcal{T}_c$-complete RN module over R with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property, $k \in L^0_{++}(\mathcal{F})$, $G \subset E$ a $\mathcal{T}_c$-closed subset with the countable concatenation property. Further, if $f \in E^*$ is bounded from above on G, and $z \in G$, then there exists $x_0 \in G$ such that*

(1) $x_0 \in C(f, k) + z$;

(2) $G \bigcap (C(f, k) + x_0) = \{x_0\}$.

Before the proof of Lemma 3.7, we first give the hyperplane separation theorems in *RN* modules under the locally L^0-convex topology, namely, Proposition 3.5 below. The proof of Lemma 3.7 is based on Proposition 3.5.

Proposition 3.4 ([12]). *Let E be a left module over the algebra $L^0(\mathcal{F}, K)$ and M and G be any two nonempty subsets of E such that $\tilde{I}_A M + \tilde{I}_{A^c} M \subset M$ and $\tilde{I}_A G + \tilde{I}_{A^c} G \subset G$. If $H_{cc}(M) \cap H_{cc}(G) = \emptyset$, then there exists an $\mathcal{F}$-measurable subset $H(M, G)$ unique a.s. such that the following are satisfied*:

(1) $P(H(M, G)) > 0$;

(2) $\tilde{I}_A M \cap \tilde{I}_A G = \emptyset$ *for all* $A \in \mathcal{F}$, $A \subset H(M, G)$ *with* $P(A) > 0$;

(3) $\widetilde{I}_A M \cap \widetilde{I}_A G \neq \emptyset$ *for all* $A \in \mathcal{F}$, $A \subset \Omega \setminus H(M, G)$ *with* $P(A) > 0$.

Let E, M, and G be the same as in Proposition 3.4 such that $H_{cc}(M) \cap H_{cc}(G) = \emptyset$, then $H(M, G)$ is called the hereditarily disjoint stratification of H and M, and $P(H(M, G))$ is called the hereditarily disjoint probability of H and G.

Proposition 3.5 ([13])**.** *Let* $(E, \|\cdot\|)$ *be an RN module over K with base* $(\Omega, \mathcal{F}, P)$ *and G and M be two nonempty* $L^0(\mathcal{F})$*-convex subsets of E such that the* $\mathcal{T}_c$*-interior* G^o *of G is not empty and* $H_{cc}(G^o) \bigcap H_{cc}(M) = \emptyset$. *Then there exists* $f \in E^*$ *such that*

$$(\mathrm{Re}\, f)(x) \leq (\mathrm{Re}\, f)(y) \text{ for all } x \in G \text{ and } y \in M,$$

and

$$(\mathrm{Re}\, f)(x) < (\mathrm{Re}\, f)(y) \text{ on } H(G^o, M) \text{ for all } x \in G^o \text{ and } y \in M.$$

Proposition 3.6 ([13])**.** *Let* $(E, \|\cdot\|)$ *be an RN module over K with base* $(\Omega, \mathcal{F}, P)$*. If a subset G of E has the countable concatenation property, then so does the* $\mathcal{T}_c$*-interior* G^o *of G.*

Now, we can give Lemma 3.7 and its proof as follows.

Lemma 3.7. *Let* $(E, \|\cdot\|)$ *be a* $\mathcal{T}_c$*-complete RN module over R with base* $(\Omega, \mathcal{F}, P)$ *such that E has the countable concatenation property,* $G \subset E$ *an a.s. bounded (namely,* $\bigvee\{\|p\| : p \in G\} \in L^0_+(\mathcal{F})$*),* $\mathcal{T}_c$*-closed* $L^0(\mathcal{F})$*-convex nonempty subset of E with* $0 \notin G$ *and G has the countable concatenation property,* $C := L^0_+(\mathcal{F}) \cdot G$*, and* $F \subset E$ *a* $\mathcal{T}_c$*-closed subset of E and F has the countable concatenation property. If* $z \in F$ *and* $F \bigcap (C + z)$ *is a.s. bounded, then there exists* $z_0 \in F \bigcap (C + z)$ *such that*

$$F \bigcap (C + z_0) = \{z_0\}.$$

Proof. First, we prove that there exist $g \in E^*$ and $k \in L^0_{++}(\mathcal{F})$ such that $C := L^0_+(\mathcal{F}) \cdot G \subset C(g, k)$ as follows.

Since $0 \notin G$, then there exists $\varepsilon \in L^0_{++}(\mathcal{F})$ such that $N_\varepsilon(0) \cap G = \emptyset$ and hence $N^o_\varepsilon(0) \cap G = \emptyset$. It is easy to check that $N_\varepsilon(0)$ has the countable concatenation property. Thus the $\mathcal{T}_c$-interior $N^o_\varepsilon(0)$ has the countable concatenation property by Proposition 3.5. It follows that $H_{cc}(N^o_\varepsilon(0)) \bigcap H_{cc}(G) = \emptyset$. By Proposition 3.5, there exists $g \in E^*$ such that

$$g(x) \leq g(y) \text{ for all } x \in N_\varepsilon(0) \text{ and } y \in G,$$

and hence we have $\gamma := \vee g(N_\varepsilon(0)) \leq \wedge g(G)$.

Since G is a.s. bounded, there exists $M \in L^0_{++}(\mathcal{F})$ such that $\|y\| \leq M, \forall y \in G$. Thus, we have $\frac{\gamma}{M} \cdot \|y\| \leq \gamma \leq g(y), \forall y \in G$. Hence taking $k = \frac{\gamma}{M}$, we have $G \subset C(G, k)$ and hence $C \subset C(g, k)$.

Since $g \in E^*$ and $F \bigcap (C + z)$ is a.s. bounded, it implies that g is bounded from above on $F \bigcap (C + z)$. It is easy to see that $F \bigcap (C + z)$ is $\mathcal{T}_c$-closed and has the countable concatenation property. Applying Lemma 3.3 to $F \bigcap (C + z)$, we have that there exists $z_0 \in F \bigcap (C + z)$ such that

$$\{z_0\} = F \bigcap (C + z) \bigcap (C(g, k) + z_0) \supset F \bigcap (C + z_0) \supset \{z_0\},$$

and hence we have $\{z_0\} = F \bigcap (C + z_0)$.

Thus it is clear that z_0 is just desired.

Definition 3.8. Let E be an RN module over R with base $(\Omega, \mathcal{F}, P)$, $G \subset E$ a subset.

(1) $f \in E^* \setminus \{0\}$ such that f is bounded from above on G. If $x \in G$ is such that $f(x) = \bigvee f(G)$, then x is called a support point of f and f is called an a.s. bounded random linear functional supporting G at x;

(2) $x \in G$ is called a conical support point of G, if there exists a closed L^0-convex random cone C with vertex 0 such that $G \cap (C + x) = \{x\}$;

(3) $x \in G$ is called a local conical support point of G, if there exist $\varepsilon \in L^0_{++}(\mathcal{F})$ and a closed L^0-convex random cone C with nonempty interior such that $G \cap (C + x) \cap N_\varepsilon(x) = \{x\}$.

We now state the main result of this section, namely, Theorem 3.9 below. It shows that the set of local conical support points of S is $\mathcal{T}_c$-dense in the $\mathcal{T}_c$-boundary of S (briefly, $\partial_c(S)$).

Theorem 3.9. *Let $(E, \|\cdot\|)$ be a $\mathcal{T}_c$-complete RN module over R with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property, $G \subset E$ be a $\mathcal{T}_c$-closed subset of E and G has the countable concatenation property, $\varepsilon \in L^0_{++}(\mathcal{F})$, and $z_0 \in \partial_c(G)$. Then there exist $\delta \in L^0_{++}(\mathcal{F})$, C a $\mathcal{T}_c$-closed $L^0(\mathcal{F})$-convex cone, and $x_0 \in S$ such that*

$$\|x_0 - z_0\| < \varepsilon \text{ on } \Omega \text{ and } G \cap (C + x_0) \cap N_\delta(x_0) = \{x_0\}.$$

Proof. Let y_0 be in $E \setminus G$ such that $\|y_0 - z_0\| \le \frac{\varepsilon}{4}$. And taking $g_0 \in G$ such that $\|y_0 - g_0\| \le \frac{5r}{4}$, where $r := \bigwedge\{\|y_0 - z\| : z \in G\}$. We can, without loss of generality, suppose $g_0 = 0$. Since $r \le \|y_0 - z_0\| \le \frac{\varepsilon}{4}$, one can have $\|y_0 - g_0\| = \|y_0\| \le \frac{5r}{4} \le \frac{5}{4} \cdot \frac{\varepsilon}{4} \le \frac{\varepsilon}{3}$. By $r \le \|y_0\|$, we have $0 \,\overline{\in}\, B := N_{\frac{r}{4}}(y_0)$.

Let $C := L^0_+(\mathcal{F}) \cdot B$, $\delta_0 = \frac{\|y_0\|}{2}$ and $G_1 = N_{\delta_0}(0) \bigcap G$. It is clear that G_1 is $\mathcal{T}_c$-closed and has the countable concatenation property. Applying G_1 to Lemma 3.8, then there exists $x_0 \in G_1 \bigcap C$ such that $\{x_0\} = G_1 \cap (C + x_0)$.

Since $x_0 \in C$, we can suppose $x_0 = \alpha \cdot u$, where $\alpha \in L^0_+(\mathcal{F})$, $u \in B$. Since $x_0 \in N_{\delta_0}(0)$, it is easy to check that $\tilde{I}_A \cdot N_{\delta_0}(0) \bigcap \tilde{I}_A \cdot S = \emptyset$, where $S := \{ky | k \in L^0_{++}(\mathcal{F})$ and $k \ge 1,\ y \in B\}$. Thus we have $\alpha < 1$ on Ω. Hence, it follows that

$$\|x_0 - y_0\| = \|\alpha u - y_0\| = \|\alpha(u - y_0) - (1-\alpha)y_0\| \le \alpha \cdot \frac{r}{4} + (1-\alpha)\|y_0\|$$

$$\le \frac{r}{4} + (1-\alpha) \cdot \frac{5r}{4} \le (5 - 4\alpha) \cdot \frac{r}{4} \le \frac{5r}{4}. \ (*)$$

By $r \le \frac{\varepsilon}{4}$, one can have $\|x_0 - y_0\| \le \frac{5r}{4} \le \frac{5\varepsilon}{16}$, which implies

$$\|x_0 - z_0\| \le \|x_0 - y_0\| + \|y_0 - z_0\| \le \frac{5\varepsilon}{16} + \frac{\varepsilon}{4} < \varepsilon \text{ on } \Omega.$$

Since $x_0 \in G$, we have $\|x_0 - y_0\| \geq r.$. Thus from $(*)$, it is easy to have $r \leq \|x_0 - y_0\| \leq (5-4\alpha)\cdot\frac{r}{4}$, which implies $\alpha \leq \frac{1}{4}$, and hence $\|u\| \leq \|u - y_0\| + \|y_0\| \leq \frac{r}{4} + \frac{5r}{4} = \frac{3r}{2}$. Thus we have

$$\|x_0\| = \alpha\|u\| \leq \frac{1}{4}\cdot\frac{3r}{2} < \frac{r}{2} \leq \frac{\|y_0\|}{2} = \delta_0 \text{ on } \Omega.$$

Let $\delta = \delta_0 - \|x_0\| > 0$ on Ω. If $\|\eta - x_0\| < \delta$ on Ω, then it is easy to have $\|\eta\| \leq \|\eta - x_0\| + \|x_0\| < \delta + \|x_0\| = \delta_0$ on Ω, which implies $N_\delta(x_0) \subset N_{\delta_0}(0)$. Thus it follows that

$$x_0 \in G\bigcap(C + x_0)\bigcap N_\delta(x_0) \subset G\bigcap(C + x_0)\bigcap N_{\delta_0}(0) \subset (C + x_0)\bigcap G_1 = \{x_0\}.$$

Hence one can have

$$G \cap (C + x_0) \cap N_\delta(x_0) = \{x_0\}.$$

Thus C, δ, and x_0 are desired.

From Theorem 3.9, one can obtain Corollary 3.10 below, which is the Bishop-Phelps theorem in complete *RN* module. It was established by us in [1].

Corollary 3.10 ([1]). *Let $(E, \|\cdot\|)$ be a $\mathcal{T}_c$-complete RN module over R with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property and G be a $\mathcal{T}_c$-closed $L^0(\mathcal{F})$-convex subset of E such that G has the countable concatenation property. Then the set of support points of G is $\mathcal{T}_c$-dense in $\partial_c G$.*

Proof. Suppose K is a $\mathcal{T}_c$-closed $L^0(\mathcal{F})$-convex random cone with vertex 0 and nonempty interior in E.

We now prove that any point of S which is a conical support point with respect to K is in fact a support point of G. We need to prove that $K + x$ and G satisfy the hypotheses of Proposition 3.5 as follows:

(1) It is easy to see that $K + x$ and G are both $L^0(\mathcal{F})$-convex.

(2) Since K is an $L^0(\mathcal{F})$-convex cone, it is easy to check that $K + x$ has the countable concatenation property. Thus $(K + x)^o$ has the countable concatenation property by Proposition 3.6.

(3) We can now prove $\tilde{I}_A \cdot (K + x)^o \bigcap \tilde{I}_A \cdot G = \emptyset$ for any $A \in \mathcal{F}$ with $P(A) > 0$ as follows.

First, from $(K + x) \bigcap G = \{x\}$, one can have $(K + x)^o \bigcap G = \emptyset$ since $x \in \partial_c(K + x)$.

Second, from $(K + x) \bigcap G = \{x\}$, we can deduce $\tilde{I}_A \cdot (K + x) \bigcap \tilde{I}_A \cdot G = \tilde{I}_A \cdot \{x\}$ for any $A \in \mathcal{F}$ with $P(A) > 0$. Otherwise, there exists some $B \in \mathcal{F}$ with $P(B) > 0$ and $\hat{y} \in E$ such that $\tilde{I}_B \cdot \hat{y} \in \tilde{I}_B \cdot (K + x) \bigcap \tilde{I}_B \cdot G$ and $\tilde{I}_B \cdot \hat{y} \neq \tilde{I}_B \cdot x$. Let us take $z = \tilde{I}_B \cdot \hat{y} + \tilde{I}_{B^c} \cdot x$, then it is easy to see that $\tilde{I}_{B^c} \cdot x \in \tilde{I}_{B^c} \cdot ((K + x) \bigcap G) \subset \tilde{I}_{B^c} \cdot (K + x) \bigcap \tilde{I}_{B^c} \cdot G$. Thus we can have $z \in (K + x) \bigcap G = \{x\}$, which implies $\tilde{I}_B \cdot \hat{y} = \tilde{I}_B \cdot x$, a contradiction.

Third, we consider the problem in the relative topology. Since $\tilde{I}_A \cdot (K + x)^o$ is the relative $\mathcal{T}_c$-interior of $\tilde{I}_A \cdot (K + x)$ in $\tilde{I}_A \cdot E$ and $\tilde{I}_A \cdot x$ is a relative $\mathcal{T}_c$-boundary point of $\tilde{I}_A \cdot (K + x)$ in $\tilde{I}_A \cdot E$, we can have $\tilde{I}_A \cdot (K + x)^o \bigcap \tilde{I}_A \cdot G = \emptyset$, for any $A \in \mathcal{F}$ with $P(A) > 0$.

Then by Proposition 3.5, there exists $f \in E^*$ such that

$$f(p) \leq f(q) \text{ for all } p \in G \text{ and } q \in K + x,$$

which implies $f(x) \leq \bigvee f(G) \leq \bigwedge f(K + x) \leq f(x)$, and hence $f(x) = \bigvee f(G)$.

Further, by $\{x\} = G \bigcap (K + x) \bigcap N_\delta(x) \subset G \bigcap (K + x) = \{x\}$, it shows that a local conical support point of G is a conical support point.

Thus we can get the conclusion from Theorem 3.9.

Remark 3.11. We prove that the set of local conical support points of S is dense in the boundary of S under the locally L^0-convex topology, where S is a $\mathcal{T}_c$-closed subset of a random normed module E and S has the countable concatenation property; Then we prove that this result is a nonconvex generalization of the Bishop-Phelps theorem in a complete random normed module. A $\mathcal{T}_{\varepsilon,\lambda}$-complete $L^0(\mathcal{F})$-convex subset S must have the countable concatenation property, but we wonder whether Theorem 3.9 is true or not under the (ε, λ)-topology?.

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