SOME LIFTS OF DOUBLE VECTOR BUNDLES RELATED TO A PRODUCT PRESERVING GAUGE BUNDLE FUNCTOR ON VECTOR BUNDLES

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Abstract

We present some lifts (associated to a product preserving bundle functor on vector bundles) of double vector bundles and linear sections on a double vector bundle.

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1. Introduction

Weil functors (product preserving bundle functors on manifolds) were classified by [1], [10] and [5]. These functors were used by many authors (e.g., [2], [7], [8], [9]) to present some lifts of various geometric objects (smooth functions, tensor fields, linear connections on manifolds,...).

Product preserving gauge bundle functor on vector bundles (an example of bundle functors on local categories) were classified in [12]. Similarly to what is done for Weil functors some authors (e.g., [13], [14], [15]) present some lifts of some geometric objects related to product preserving gauge bundle functor on vector bundles.

The most fundamental example of a double vector bundle is the tangent bundle $\mathcal{TE}$ of a vector bundle $(E, M, q)$. The two structures of vector bundles on $\mathcal{TE}$ allow the development of some mathematical tools, namely linear connections, derivative endomorphisms, linear vector fields,...

In this paper, we present some lifts (associated to a product preserving gauge bundle functor on vector bundles) of double vector bundles and linear sections on double vector bundles.

2. Product Preserving Gauge Bundle
Functor on Vector Bundles

2.1. The Weil functor $T^A : \mathcal{Mf} \rightarrow \mathcal{FM}$

We write $\mathcal{Mf}$ for the category of finite dimensional differential manifolds and mappings of class $C^\infty$; moreover, $\mathcal{FM}$ is the category of fibered manifolds and fibered manifolds morphisms.

Let us recall this construction of Weil functors based on [16]. For a Weil algebra $A = \mathbb{R} \cdot 1_A \oplus N$, i.e., a real commutative unital algebra where the ideal of nilpotent elements $N$ is a finite dimensional vector subspace of $A$, and any point $x$ of a differential manifold $M$, let $C^\infty_x(M, \mathbb{R})$
and \( \text{Hom}(C^\infty_x(M, \mathbb{R}), A) \) be the algebra of germs on \( x \) of smooth functions and the set of algebra homomorphisms from \( C^\infty_x(M, \mathbb{R}) \) into \( A \), respectively. If \( \mathcal{Ens} \) denotes the category of sets and mappings, one defines a functor \( T^A : Mf \to \mathcal{Ens} \) by:

\[
T^A M := \bigcup_{x \in M} \text{Hom}(C^\infty_x(M, \mathbb{R}), A) \text{ and } (T^A f)_x(\varphi_x) := \varphi_x \circ f^*_x,
\]

for a manifold \( M \) and \( f \in C^\infty(M, M') \), where \( f^*_x \in \text{Hom}(C^\infty_x(M', \mathbb{R}), C^\infty_x(M, \mathbb{R})) \) is the pull-back algebra homomorphism defined by \( f^*(\text{germ}_{f(x)}(h)) = \text{germ}_x(h \circ f) \).

Now, let \( q_{A,M} : T^A M \to M, (T^A M)_x \ni \varphi \mapsto x \); hence \( (T^A M, M, q_{A,M}) \) is a well-defined fibered manifold. Indeed let \( c = (U, u^i) \), \( 1 \leq i \leq m \) be a chart of \( M \); then the map

\[
\phi_c : (q_{A,M})^{-1}(U) \to U \times N^m
\]

\[
\varphi_x \mapsto (x, \varphi_x(\text{germ}_x(u^i - u^i(x))),
\]

is a local trivialization of \( T^A M \). Given another manifold \( M' \) and a smooth map \( f : M \to M' \), \( T^A f \) is a fibered map. Indeed for charts \( c = (U, u, m), c' = (W, w, m') \) of \( M, M' \) such that \( f(U) \subset W \), \( \phi_{c'} \circ T^A f \circ \phi_c^{-1} \)

is the map

\[
U \times N^m \to W \times N^m'
\]

\[
(x, n_i) \mapsto (f(x), n'_j),
\]

where \( n'_j = \sum_{a \in [m]} \frac{1}{a!} D_\alpha(w^j \circ f \circ u^{-1})(u(x))n_1^{\alpha_1} \cdots n_m^{\alpha_m}, 1 \leq j \leq m' \), with

\[
D_\alpha F^j = \frac{\partial^{\left|\alpha\right|} F^j}{(\partial x^1)^{\alpha_1} \cdots (\partial x^m)^{\alpha_m}}.
\]
\( T^A : \mathcal{M}f \to \mathcal{F}\mathcal{M} \) is a product preserving bundle functor (see [9]) called the Weil functor associated to \( A \).

Let \( c = (U, u) \) be a chart of \( M \); in all the paper, we’ll use fibered charts \( (q^{-1}_A, u^{i, \alpha}), 1 \leq i \leq m, 0 \leq \alpha \leq \dim N \) of \( T^A M \) associated to the fibered isomorphism \( T^A u \) and defined by \( u^{i, \alpha} = e^*_\alpha \circ T^A(u^i) \), where \( (e^*_\alpha) \) is the dual basis of a fixed basis \( (e_\alpha)_{0 \leq \alpha \leq \dim N} \) of \( A \) such that \( e_0 = 1 \).

### 2.2. Product preserving gauge bundle functor on \( \mathcal{VB} \)

Let \( F : \mathcal{VB} \to \mathcal{F}\mathcal{M} \) be a covariant functor from the category \( \mathcal{VB} \) (of vector bundles and vector bundles homomorphisms) into the category \( \mathcal{F}\mathcal{M} \). Let \( B_{\mathcal{VB}} : \mathcal{VB} \to \mathcal{M}f \) and \( B_{\mathcal{F}\mathcal{M}} : \mathcal{F}\mathcal{M} \to \mathcal{M}f \) be the respective base functors.

**Definition 2.1.** \( F \) is a gauge bundle functor on \( \mathcal{VB} \) when the following conditions are satisfied:

- **(Prolongation)** \( F_{\mathcal{F}\mathcal{M}} \circ F = B_{\mathcal{VB}} \), i.e., \( F \) transforms a vector bundle \( E \to M \) in a fibered manifold \( FE \to M \) and a vector bundle morphism \( E \to G \) over \( M \) in a fibered map \( FE \to FG \) over \( f \).

- **(Localization)** For any vector bundle \( E \to M \) and any inclusion of an open vector subbundle \( i : q^{-1}(U) \to E \), the fibered map \( Fq^{-1}(U) \to p_{E}^{-1}(U) \) over \( id_U \) induced by \( Fi \) is an isomorphism then the map \( Fi \) can be identified to the inclusion \( p_{E}^{-1}(U) \to FE \).
Given two gauge bundle functors $F_1, F_2$ on $\mathcal{VB}$, by a *natural transformation* $\tau : F_1 \to F_2$ we shall mean a system of base preserving fibered maps $\tau_E : F_1 E \to F_2 E$ for every vector bundle $E$ satisfying $F_2 f \circ \tau_E = \tau_G \circ F_1 f$ for every vector bundle morphism $f : E \to G$.

A gauge bundle functor $F$ on $\mathcal{VB}$ is *product preserving* if for any product projections $E_1 \xleftrightarrow{p_{F_1}} E_1 \times E_2 \xrightarrow{p_{F_2}} E_2$ in the category $\mathcal{VB}$, $F E_1 \xleftrightarrow{F p_{F_1}} F(E_1 \times E_2) \xrightarrow{F p_{F_2}} F E_2$ are product projections in the category $\mathcal{FM}$. In other words, the map $(F p_{F_1}, F p_{F_2}) : F(E_1 \times E_2) \to F(E_1) \times F(E_2)$ is a fibered isomorphism over $\text{id}_{M_1 \times M_2}$.

**Example 2.1.** Let $A = \mathbb{R} \cdot 1_A \oplus N$ be a Weil algebra.

(a) Each Weil functor $T^A$ induces a product preserving gauge bundle functor $T^A : \mathcal{VB} \to \mathcal{FM}$ in a natural way.

(b) The gauge bundle functor $T^{A, V} : \mathcal{VB} \to \mathcal{FM}$: Let $V$ be an $A$-module such that $\dim_{\mathbb{R}}(V) < \infty$. For a vector bundle $(E, M, q)$ and $x \in M$, let

$$T^A_{x, V} E = \{(\varphi_x, \psi_x) / \varphi_x \in \text{Hom}(C^\infty_x(M, \mathbb{R}), A) \text{ and } \psi_x \in \text{Hom}_{\varphi_x}(C^\infty_x f, l(E), V)\},$$

where $\text{Hom}(C^\infty_x(M, \mathbb{R}), A)$ is the set of algebra homomorphisms $\varphi_x$ from the algebra $C^\infty_x(M, \mathbb{R}) = \{\text{germ}_x(g) / g \in C^\infty(M, \mathbb{R})\}$ into $A$ and $\text{Hom}_{\varphi_x}(C^\infty_x f, l(E), V)$ is the set of module homomorphisms $\psi_x$ over $\varphi_x$ from the $C^\infty_x(M, \mathbb{R})$-module $C^\infty_x f, l(E, \mathbb{R}) = \{\text{germ}_x(h) / h : E \to \mathbb{R} \text{ is fiberwise linear}\}$ into $V$. Let $T^{A, V} E = \bigcup_{x \in M} T^A_{x, V} E$ and $p_{E}^{A, V} : T^{A, V} E \to M$, $T^A_{x, V} E \ni (\varphi, \psi) \mapsto x$. $(T^{A, V} E, M, p^{A, V}_E)$ is a well-defined fibered
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manifold. Indeed let \( c = (q^{-1}(U), x^i = u^i \circ q, y^j) \), with \( 1 \leq i \leq m, 1 \leq j \leq n \)
be a fibered chart of \( E \); then the map

\[
\phi_e : (p^A_{E,V})^{-1}(U) \rightarrow U \times N \times V
\]

\[
(\varphi_x, \psi_x) \mapsto (x, \varphi_x(\text{germ}_x(u^i - u^i(x))), \psi_x(\text{germ}_x(y^j))),
\]
is a local trivialization for a bundle structure on \( T^{A,V,E} \). Given another vector bundle \( (G, M', q') \) and a vector bundle homomorphism \( f : E \rightarrow G \) over \( \tilde{f} : M \rightarrow M' \), let

\[
T^{A,V}_f : T^{A,V}_E \rightarrow T^{A,V}_G
\]

\[
(\varphi_x, \psi_x) \mapsto (\varphi_x \circ \tilde{f}_x^*, \psi_x \circ f_x^*),
\]

where \( \tilde{f}_x^* : C^\infty_{\tilde{f}(x)}(N) \rightarrow C^\infty_x(M) \) and \( f_x^* : C^\infty_{\tilde{f}(x)}(G) \rightarrow C^\infty_x(f,1)(E) \) are given by the pull-back with respect to \( \tilde{f} \) and \( f \). Then \( T^{A,V}_f \) is a fibered map over \( \tilde{f} \). \( T^{A,V} : \mathcal{VB} \rightarrow \mathcal{FM} \) is a product preserving gauge bundle functor (see [12]).

**Remark 2.1.** Let \( F : \mathcal{VB} \rightarrow \mathcal{FM} \) be a product preserving gauge bundle functor.

(a) \( F \) associates the pair \( (A^F, V^F) \), where \( A^F = F(id_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}) \) is a Weil algebra and \( V^F = F(\mathbb{R} \rightarrow pt) \) is a \( A^F \)-module such that \( \dim_{\mathbb{R}}(V^F) < \infty \).

(b) There is a natural isomorphism \( \Theta : F \rightarrow T^{A^F,V^F} \) defined on the vector bundle \( (E, M, q) \) as follows: For \( \xi \in F_xE, \ \Theta_E(\xi) = (\varphi_x, \psi_x) \) with \( \varphi_x(\text{germ}_x f) = F(f \circ q)(\xi) \) and \( \psi_x(\text{germ}_x h) = F(h)(\xi) \). In particular, the product preserving gauge bundle functor \( T^A : \mathcal{VB} \rightarrow \mathcal{FM} \) is equivalent to \( T^{A,A} \).
(c) For two vector bundles \( E_1, E_2 \) with the same base \( M \) and \( p_i : E_1 \oplus_M E_2 \to E_i, i = 1, 2 \) the projections, the map \( (F p_1, F p_2) : F(E_1 \oplus_M E_2) \to F E_1 \times F E_2 \) induces a vector bundle isomorphism \( F(E_1 \oplus_M E_2) \to F E_1 \oplus_{FM} F E_2 \) in a natural way.

(d) For a vector bundle \((E, M, q), (FE, FM, Fq)\) is also a vector bundle where \( FM = T^{AFE} M \); if \( f : E \to G \) is a morphism of vector bundles over \( \bar{f} : M \to M' \), then \( F \bar{f} : FE \to FG \) is also a morphism of vector bundles over \( F \bar{f} : FM \to FM' \). The addition, the scalar multiplication and the zero section of \( FE \to FM \to FM \) are respectively, given by

\[
\begin{align*}
FE \oplus_{FM} FE & \to FE & \mathbb{R} \times FE & \to FE \\
(u, \bar{u}) & \mapsto F(ad^E)(u, \bar{u})' & (\lambda, \bar{u}) & \mapsto F(m^E_\lambda)(\bar{u}),
\end{align*}
\]

and \( F0_E : FM \to FE \),

with \( ad^E \), \( m^E \), and \( 0_E \) the addition, scalar multiplication, and zero section of \( E \), respectively.

3. Lifts of Double Vector Bundles

3.1. Double vector bundles

Definition 3.1. A double vector bundle structure is a system \((D, A, B, M)\) of four vector bundles structures

\[
\begin{array}{ccc}
D & \xrightarrow{q_D^D} & B \\
q_A^D & \downarrow & \downarrow q_B \\
A & \xrightarrow{q_A} & M
\end{array}
\]

where \( D \) is a vector bundle on bases \( A \) and \( B \), which are themselves vector bundles on \( M \), such that each of the four structure maps of each
vector bundle structure on $D$ (namely, the bundle projection, addition, scalar multiplication and the zero section) is vector bundle morphism with respect to other structure.

**Remark 3.1.** Let $(D, A, B, M)$ be a double vector bundle. The last part of the definition means that:

(1) $q^D_B$ and $q^D_A$ are morphisms of vector bundles over $q_A$ and $q_B$, respectively; in particular $q_A \circ q^D_A = q_B \circ q^D_B : D \to M$.

(2) The zero sections $0^D_A : A \to D$, $0^D_B : B \to D$ are morphisms of vector bundles over the zero sections $0^A_M : M \to A$, $0^B_M : M \to B$, respectively.

(3) Since the sum $D \oplus_B D$ is a subbundle of the vector bundle

$$\left( q^D_A \times q^D_A \right)^{-1} (A \oplus_M A) \mapsto A \oplus_M A$$

by (1) and (2), the addition $ad^D_B : D \oplus_B D \to D$ is a morphism of vector bundles over the addition $ad^A_M : A \oplus_M A \to A$. Similarly, the addition $ad^D_A : D \oplus_A D \to D$ is a morphism of vector bundles over the addition $ad^B_M : B \oplus_M B \to M$.

(4) The scalar multiplication of $D \to B$, $m^D_B : \mathbb{R} \times D \to D$ is a morphism of vector bundles over the scalar multiplication $m^A_M : \mathbb{R} \times A \to A$. Similarly, of $D \to A$, $m^D_A : \mathbb{R} \times D \to D$ is a morphism of vector bundles over the scalar multiplication $m^B_M : \mathbb{R} \times B \to B$.

**Definition 3.2.** Let $(D, A, B, M)$ be a double vector bundle. The bundle $D \to B$ is called the horizontal bundle structure on $D$ and $D \to A$ is called the vertical bundle structure on $D$. 


Example 3.1. (a) For a differential manifold $M$, let us consider trivial vector bundles $A = M \times \mathbb{R}^n$, $B = M \times \mathbb{R}^p$, $D = M \times \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$ over $A$ and $B$; then $(D, A, B, M)$ is a double vector bundle.

(b) The tangent bundle of a vector bundle $(E, M, p)$

\[
\begin{array}{ccc}
TE & \xrightarrow{T(p)} & TM \\
\pi_E & \downarrow & \downarrow \\
E & \xrightarrow{p} & M
\end{array}
\]

is a double vector bundle.

**Definition 3.3.** A morphism of double vector bundles $(\varphi, \varphi_A, \varphi_B, f) : (D, A, B, M) \to (D', A', B', M')$ consists of morphisms of vectors bundles

\[
\begin{array}{cccccccc}
D & \xrightarrow{\varphi} & D' & D & \xrightarrow{\varphi} & D' & A & \xrightarrow{\varphi_A} & A' & B & \xrightarrow{\varphi_B} & B' \\
\downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\varphi_A} & A' & B & \xrightarrow{\varphi_B} & B' & M & \xrightarrow{f} & M' & M & \xrightarrow{f} & M'
\end{array}
\]

If $M = M'$ and $f = id_M$, $\varphi$ is called a morphism over $M$; if further $A = A'$ and $\varphi_A = id_A$, $\varphi$ is said over $A$. If $A = A'$, $B = B'$ and both $\varphi_A, \varphi_B$ are identities, we say that $\varphi$ preserves the side bundles.

3.2. **Double vector bundles as homogeneous structures**

Let us recall the following notions according to [3].

A *smooth action* of the multiplicative monoid $(\mathbb{R}_+, \cdot)$ on a smooth manifold $F$, is a smooth map $h : \mathbb{R}_+ \times F \to F$, $(t, x) \mapsto h(t, x) = h_t(x)$ such that $h(1, x) = x$ and $h_t \circ h_s = h_{ts}$. 
Given a smooth action \( h : \mathbb{R}_+ \times F \to F \), let \( M = h_0(F) \) the set of fixed points of the projection \( h_0 \) and the smooth map \( \mathcal{V} : F \to TF, x \mapsto \frac{d}{dt} \big|_{t=0} h(t, x) \in T_{h_0(x)}F. \)

It is clear that \( M \subset \mathcal{V}^{-1}(0) \), the inverse image of the set of zeros of \( TF \); if \( M = \mathcal{V}^{-1}(0) \) \( h \) is called a *homogeneous structure* on \( F \). In this case, there is a structure of vector bundle \( E \to M \) on \( E = \mathcal{V}(F) \) and \( \mathcal{V} : F \to E \) is a diffeomorphism. The vector bundle structure \( F \xrightarrow{h_0} M \) carried by this diffeomorphism is the unique vector bundle structure on \( F \) whose homotheties coincide with \( h \). Conversely homotheties of a vector bundle \( F \) associate a homogeneous structure \( h : \mathbb{R}_+ \times F \to F, (t, x) \mapsto t \cdot x. \) This implies that vector bundles correspond with homogeneous structures.

The *Euler vector field* of a vector bundle \( F \to M \) is the smooth vector field \( \Delta_F \) on \( F \) given by \( \Delta_F(x) = \frac{d}{dt} \big|_{t=1} t \cdot x. \) The global flow \( F^{\Delta_E} : \mathbb{R} \times F \to F \) of \( \Delta_F \) is given by \( F^{\Delta_E}(t, x) = e^t \cdot x = h(e^t, x). \)

Two homogeneous structures \( h_1, h_2 : \mathbb{R}_+ \times F \to F \) are called *commuting homogeneous structures* if \( h_1^1 \circ h_2^2 = h_2^2 \circ h_1^1 \) for all \( (s, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \) i.e., \( [\Delta^1, \Delta^2] = 0 \), since \( h_1, h_2 \) come from global flows of \( \Delta^1, \Delta^2 \). If one denotes \( E^i = h_0^i(F), i = 1, 2 \) the corresponding bases and \( M = E^1 \cap E^2 \), the system
is a double vector bundle structure. Conversely, the homotheties of a double vector bundle

\[
\begin{array}{ccc}
D & \xrightarrow{q^D_B} & B \\
Q^D_A & \downarrow & \downarrow q_B \\
A & \xrightarrow{q^A} & M \\
\end{array}
\]

commute. Indeed, the scalar multiplication of \( D \to B, m^D_B : \mathbb{R} \times D \to D \) is a morphism of vector bundles over the scalar multiplication \( m^A_M : \mathbb{R} \times A \to A \), hence for \( (s, a) \in \mathbb{R} \times A \), the induced map \( \{s\} \times D_a \to D_{s \cdot a} \), \( (s, d) \mapsto s \cdot d \) is linear; in particular for all \( t \in \mathbb{R}, s \cdot \left[ t \cdot A^d \right] = t \cdot \left[ s \cdot A^d \right] \).

This shows that a double vector bundle can be equivalently defined as a smooth manifold equipped with two vector bundle structures whose Euler vector fields \( \Lambda^1, \Lambda^2 \) commute.

3.3. Natural transformations \( Q(a) : F \to F \)

Let \( F : \mathcal{VB} \to \mathcal{FM} \) be a product preserving gauge bundle functor.

For a vector bundle \((E, M, q)\), the scalar multiplication \( m^E_M : \mathbb{R} \times E \to E \), is a vector bundle morphism over the projection \( pr_2 : \mathbb{R} \times M \to M \), hence for any \( a \in A^F \), there is a natural transformation \( Q(a) : F \to F \) given by

\[
Q(a)_E = Fm^E_M(a, \cdot) : FE \to FE. \tag{3.1}
\]

\( Q(a) \) is entirely determined by the algebra homomorphism \( Q(a)_{\mathbb{R} \to \mathbb{R}} = id_{A^F} \) and the module homomorphism \( Q(a)_{\mathbb{R} \to \mathbb{R}} : V^F \to V^F, v \mapsto a \cdot v \), over \( id_{A^F} \) (Theorem 3.5 [12]). The following result is clear.
Proposition 3.1. We have

1. \( Q(1_A) = id_{\tau A} \),
2. \( Q(a + b) = Q(a) + Q(b) \),
3. \( Q(\lambda \cdot a) = \lambda \cdot Q(a) \),
4. \( Q(ab) = Q(a) \circ Q(b) \),

for all \( \lambda \in \mathbb{R} \) and \( (a, b) \in A^2 \).

3.4. Lifts of double vector bundles and linear sections

Let \( F : VB \to FM \) be a product preserving gauge bundle functor and \( (D, A, B, M) \) a double vector bundle.

Theorem 3.2. The system \((FD, FA, FB, FM)\) is also a double vector bundle.

Proof. (1) \( Fq_B^D \) and \( Fq_A^D \) are morphisms of vector bundles over \( Fq_A \) and \( Fq_B \), respectively by Remark 3.1(d).

(2) The zero sections \( F0_A^D : FA \to FD, F0_B^D : FB \to FD \) are morphisms of vector bundles over the zero sections \( F0_M^A : FM \to FA, F0_M^B : FM \to FB \) by the same remark.

(3) The additions \( Fad_B^D : FD \oplus FB \to FD, Fad_A^D : FD \oplus FA \to FD \) are morphisms of vector bundles over the addition \( ad_M^A : A \oplus_M A \to A, ad_M^B : B \oplus_M B \to M \), respectively.

(4) The scalar multiplications of \( FD \to FB, FD \to FA \)

\[
\begin{align*}
m_{FB}^D : \mathbb{R} \times FD & \to FD \\
m_{FA}^D : \mathbb{R} \times FD & \to FD \\
(\lambda, \vec{d}) & \mapsto F(m_{FB}^D)(\vec{d}), \\
(\lambda, \vec{d}) & \mapsto F(m_{FA}^D)(\vec{d})
\end{align*}
\]
are morphisms of vector bundles over the scalar multiplications

\[ m_{FM}^{FA} : \mathbb{R} \times FA \to FA \quad m_{FM}^{FB} : \mathbb{R} \times FB \to FB \]

\[(\lambda, \tilde{\alpha}) \mapsto F(m_{LM}^{A})(\tilde{\alpha}), \quad (\lambda, \tilde{\alpha}) \mapsto F(m_{LM}^{B})(\tilde{\alpha}),\]

respectively, since by for each \( \lambda \in \mathbb{R} \), the partial maps \( m_{LM}^{D} \) and \( m_{LM}^{B} \) are morphisms of vector bundles over \( m_{LM}^{A} \) and \( m_{LM}^{B} \), respectively.

**Remark 3.2.** Let us denote \( h_{s}^{A}, h_{t}^{B}(s, t) \in \mathbb{R}^{*} \times \mathbb{R}^{*} \) the homotheties of \( FD \to FA, FD \to FB \), respectively. These are commuting vector bundle morphisms, hence \( Fh_{s}^{A}, Fh_{t}^{B}(s, t) \in \mathbb{R}^{*} \times \mathbb{R}^{*} \) commute and since they are exactly homotheties of \( FD \to FA, FD \to FB \),

\[
\begin{array}{ccc}
FD & \xrightarrow{Fh_{0}^{B}} & FB \\
\downarrow & & \downarrow \\
FA & \xrightarrow{Fh_{0}^{B}} & FM
\end{array}
\]

is a double vector bundle.

Let \((D, A, B, M)\) be a double vector bundle. We fix one of structures of vector bundles on \( D, q_{B}^{D} : D \to B \) for instance.

**Definition 3.4.** A smooth section \( \sigma \in \Gamma(q_{B}^{D}) \) is said linear if it is a morphism of vector bundles

\[
\begin{array}{ccc}
B \xrightarrow{\sigma} D \\
\downarrow & & \downarrow \\
M \xrightarrow{\bar{\sigma}} A
\end{array}
\]

i.e., \( h_{t}^{A} \circ \sigma = \sigma \circ h_{t} \), for all \( t \in \mathbb{R}^{+} \), where \( h, h^{A} \) are homotheties of \( B \to M, D \to A \), respectively (Theorem 2.4 [3]).
**Example 3.2.** Linear sections of the double vector bundle $(TE, TM, E, M)$ associated to the vector bundle structure $\pi_E : TE \to E$ are called linear vector fields on $E$. Linear vector fields were studied in [11] and classified in [13]. Some properties of lifts of such vector fields were studied in [14] and [15].

**Definition 3.5.** For a smooth linear section $\sigma : B \to D$ over $\overline{\sigma} : M \to A$ its $a$-lift ($a \in A^F$) related to $F$ is given by $\sigma^{(a)} = Q(a)_D \circ F\sigma : FB \to FD$, where $Q(a) : F \to F$ is the natural transformation (3.1).

\[ \sigma^{(a)} \in \Gamma(Fq^D_B) \] is linear over $\overline{\sigma}^{(a)} \in \Gamma(Fq_A)$ since

\[
Fq^D_A \circ \sigma^{(a)} = Fq^D_A \circ Q(a)_D \circ F\sigma \\
= Q(a)_A \circ Fq^D_A \circ F\sigma \\
= Q(a)_A \circ F(\overline{\sigma} \circ q_B) \\
= \overline{\sigma}^{(a)} \circ Fq_B,
\]

and

\[
Fh^A_t \circ \sigma^{(a)} = Fh^A_t \circ Q(a)_D \circ F\sigma \\
= Q(a)_A \circ Fh^A_t \circ F\sigma \\
= Q(a)_A \circ F(h^A_t \circ \sigma) \\
= Q(a)_A \circ F(\sigma \circ h_t) = \sigma^{(a)} \circ Fh_t,
\]

for all $t \in \mathbb{R}$. 
Remark 3.3. Given a fibered chart \( \varphi : (q_B^D)^{-1}(\Omega) \to \mathbb{R}^n \times \mathbb{R}^q \) of \( D \to B \) and the local frame \( \{ e_j, e'_k, 1 \leq j \leq n \text{ and } 1 \leq k \leq q \} \) on \( \Omega \) defined by

\[
\begin{align*}
  e_j(b) &= \varphi^{-1}(w(b), e_j, 0), \\
  e'_k(b) &= \varphi^{-1}(w(b), 0, e'_k),
\end{align*}
\]

where \((e_j), (e'_k)\) are basis of \( \mathbb{R}^n, \mathbb{R}^q \), respectively, let us denote \( \{ \varepsilon_{j\beta}, \varepsilon'_{k\beta}, 1 \leq j \leq n, 1 \leq k \leq q \} \) and the local frame associated the fibered chart \( F\varphi \). We have

\[
e_{j\beta} = e_j^{(e)} \quad \text{and} \quad e'_{k\beta} = e_k^{(e')}.
\] (3.2)

Indeed, let \( c_j : \Omega \to \mathbb{R}^n, b \mapsto e_j \) and \( c'_0 : \Omega \to \mathbb{R}^q, b \mapsto 0; Fc_j \) and \( Fc'_0 \) are constant maps \( \tilde{b} \mapsto e_{j0} \), and \( \tilde{b} \mapsto 0 \) : 

\[
e_j^{(e_{\alpha})}(\tilde{b}) = Q(e_{\alpha})_{p^{-1}(U)} \circ Fc_j(\tilde{b})
\]

\[
= Q(e_{\alpha})_{p^{-1}(U)} \circ F(\varphi^{-1} \circ (\omega \times c_j \times c'_0))(\tilde{b})
\]

\[
= \varphi^{-1} \circ Q(e_{\alpha})_{p(U \times \mathbb{R}^n)}(FW(\tilde{b}), e_{j0}, 0)
\]

\[
= \varphi^{-1}(FW(\tilde{b}), e_{\alpha}, e_{j0}, 0) = \varphi^{-1}(FW(\tilde{b}), e_{j0}, 0) = \varepsilon_{j\alpha}(\tilde{x}),
\]

for \( \tilde{b} \in FW \).

Corollary 3.3. Sections \( \sigma^{(a)}, \sigma \in \Gamma(q_B^D) \) and \( a \in A^F \), generate the module \( \Gamma(FD) \) of smooth sections of the vector bundle \( (FD, FB, F(q_B^D)) \).
References

DOI: https://doi.org/10.1016/0022-4049(86)90076-9

DOI: https://doi.org/10.1017/S0027763000004931

DOI: https://doi.org/10.1016/j.geomphys.2009.06.009

DOI: https://doi.org/10.1016/j.geomphys.2011.09.004


DOI: https://doi.org/10.1007/BF00133034

DOI: https://doi.org/10.1016/0926-2245(92)90006-9

DOI: https://doi.org/10.1007/978-3-662-02950-3


DOI: https://doi.org/10.4064/cm90-2-7


DOI: https://doi.org/10.5817/AM2014-3-161


DOI: https://doi.org/10.5817/AM2016-3-131