Journal of Mathematical Sciences: Advances and Applications
Volume 53, 2018, Pages 57-73
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100121996

SOME LIFTS OF DOUBLE VECTOR BUNDLES RELATED TO A PRODUCT PRESERVING GAUGE BUNDLE FUNCTOR ON VECTOR BUNDLES

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Abstract

We present some lifts (associated to a product preserving bundle functor on vector bundles) of double vector bundles and linear sections on a double vector bundle.

2010 Mathematics Subject Classification: 58A32.
Keywords and phrases: double vector bundle, linear section, Weil bundle, gauge bundle functor, lift.
Received August 29, 2018; Revised September 22, 2018

1. Introduction

Weil functors (product preserving bundle functors on manifolds) were classified by [1], [10] and [5]. These functors were used by many authors (e.g., [2], [7], [8], [9]) to present some lifts of various geometric objects (smooth functions, tensor fields, linear connections on manifolds,...).

Product preserving gauge bundle functor on vector bundles (an example of bundle functors on local categories) were classified in [12]. Similarly to what is done for Weil functors some authors (e.g., [13], [14], [15]) present some lifts of some geometric objects related to product preserving gauge bundle functor on vector bundles.

The most fundamental example of a double vector bundle is the tangent bundle TE of a vector bundle (E, M, q). The two structures of vector bundles on TE allow the development of some mathematical tools, namely linear connections, derivative endomorphisms, linear vector fields,....

In this paper, we present some lifts (associated to a product preserving gauge bundle functor on vector bundles) of double vector bundles and linear sections on double vector bundles.

2. Product Preserving Gauge Bundle Functor on Vector Bundles

2.1. The Weil functor $T^A : \mathcal{M}f \to \mathcal{FM}$

We write $\mathcal{M}f$ for the category of finite dimensional differential manifolds and mappings of class C^∞; moreover, $\mathcal{FM}$ is the category of fibered manifolds and fibered manifolds morphisms.

Let us recall this construction of Weil functors based on [16]. For a Weil algebra $A = \mathbb{R} \cdot 1_A \oplus N$, i.e., a real commutative unital algebra where the ideal of nilpotent elements N is a finite dimensional vector subspace of A, and any point x of a differential manifold M, let $C_x^\infty(M, \mathbb{R})$

and $Hom(C_x^\infty(M, \mathbb{R}), A)$ be the algebra of germs on x of smooth functions and the set of algebra homomorphisms from $C_x^\infty(M, \mathbb{R})$ into A, respectively. If $\mathcal{E}ns$ denotes the category of sets and mappings, one defines a functor $T^A : \mathcal{M}f \to \mathcal{E}ns$ by:

$$T^A M := \bigcup_{x \in M} Hom(C_x^\infty(M, \mathbb{R}), A) \text{ and } \left(T^A f\right)_x(\varphi_x) := \varphi_x \circ f_x^*,$$

for a manifold M and $f \in C^\infty(M, M')$, where $f_x^* \in Hom\Big(C_{f(x)}^\infty (M', \mathbb{R}), C_x^\infty(M, \mathbb{R})\Big)$ is the pull-back algebra homomorphism defined by $f^*\left(germ_{f(x)}(h)\right) = germ_x(h \circ f)$.

Now, let $q_{A,M} : T^A M \to M, \left(T^A M\right)_x \ni \varphi \mapsto x;$ hence $\left(T^A M, M, q_{A,M}\right)$ is a well-defined fibered manifold. Indeed let $c = \left(U, u^i\right), 1 \le i \le m$ be a chart of M; then the map

$$\phi_c : (q_{A,M})^{-1}(U) \to U \times N^m$$

$$\varphi_x \mapsto (x, \varphi_x(germ_x(u^i - u^i(x))),$$

is a local trivialization of $T^A M$. Given another manifold M' and a smooth map $f : M \to M'$, $T^A f$ is a fibered map. Indeed for charts $c = (U, u, m), c' = (W, w, m')$ of M, M' such that $f(U) \subset W$, $\phi_{c'} \circ T^A f \circ \phi_c^{-1}$ is the map

$$U \times N^m \to W \times N^{m'}$$

$$(x, n_i) \mapsto (f(x), n'_j),$$

where $n'_j = \sum_{\alpha \in \mathbb{N}^m \setminus \{0\}} \frac{1}{\alpha!} D_\alpha(w^j \circ f \circ u^{-1})(u(x)) n_1^{\alpha_1} \cdots n_m^{\alpha_m}$, $1 \le j \le m'$, with

$$D_\alpha F^j = \frac{\partial^{|\alpha|} F^j}{(\partial x^1)^{\alpha_1} \cdots (\partial x^m)^{\alpha_m}}.$$

$T^A : \mathcal{M}f \to \mathcal{FM}$ is a product preserving bundle functor (see [9]) called the *Weil functor* associated to A.

Let $c = (U, u)$ be a chart of M; in all the paper, we'll use fibered charts $\left(q_{A,M}^{-1}(U), u^{i,\alpha}\right)$, $1 \le i \le m$, $0 \le \alpha \le \dim N$ of $T^A M$ associated to the fibered isomorphism $T^A u$ and defined by $u^{i,\alpha} = e_\alpha^* \circ T^A(u^i)$, where (e_α^*) is the dual basis of a fixed basis $(e_\alpha)_{0 \le \alpha \le \dim N}$ of A such that $e_0 = 1$.

2.2. Product preserving gauge bundle functor on $\mathcal{VB}$

Let $F : \mathcal{VB} \to \mathcal{FM}$ be a covariant functor from the category $\mathcal{VB}$ (of vector bundles and vector bundles homomorphisms) into the category $\mathcal{FM}$. Let $B_{\mathcal{VB}} : \mathcal{VB} \to \mathcal{M}f$ and $B_{\mathcal{FM}} : \mathcal{FM} \to \mathcal{M}f$ be the respective base functors.

Definition 2.1. F is a gauge bundle functor on $\mathcal{VB}$ when the following conditions are satisfied:

- **(Prolongation)** $\mathcal{B}_{\mathcal{FM}} \circ F = B_{\mathcal{VB}}$, i.e., F transforms a vector bundle $E \xrightarrow{q} M$ in a fibered manifold $FE \xrightarrow{p_E} M$ and a vector bundle morphism $E \xrightarrow{f} G$ over $M \xrightarrow{\bar{f}} N$ in a fibered map $FE \xrightarrow{Ff} FG$ over $\bar{f}$.

- **(Localization)** For any vector bundle $E \xrightarrow{q} M$ and any inclusion of an open vector subbundle $i : q^{-1}(U) \hookrightarrow E$, the fibered map $Fq^{-1}(U) \to p_E^{-1}(U)$ over id_U induced by Fi is an isomorphism then the map Fi can be identified to the inclusion $p_E^{-1}(U) \hookrightarrow FE$.

Given two gauge bundle functors F_1, F_2 on $\mathcal{VB}$, by *a natural transformation* $\tau : F_1 \to F_2$ we shall mean a system of base preserving fibered maps $\tau_E : F_1 E \to F_2 E$ for every vector $_M$bundle E satisfying $F_2 f \circ \tau_E = \tau_G \circ F_1 f$ for every vector bundle morphism $f : E \to G$.

A gauge bundle functor F on $\mathcal{VB}$ is *product preserving* if for any product projections $E_1 \xleftarrow{pr_1} E_1 \times E_2 \xrightarrow{pr_2} E_2$ in the category $\mathcal{VB}$, $FE_1 \xleftarrow{Fpr_1} F(E_1 \times E_2) \xrightarrow{Fpr_2} FE_2$ are product projections in the category $\mathcal{FM}$. In other words, the map $(Fpr_1, Fpr_2) : F(E_1 \times E_2) \to F(E_1) \times F(E_2)$ is a fibered isomorphism over $id_{M_1 \times M_2}$.

Example 2.1. Let $A = \mathbb{R} \cdot 1_A \oplus N$ be a Weil algebra.

(a) Each Weil functor T^A induces a product preserving gauge bundle functor $T^A : \mathcal{VB} \to \mathcal{FM}$ in a natural way.

(b) The gauge bundle functor $T^{A,V} : \mathcal{VB} \to \mathcal{FM}$: Let V be a A-module such that $\dim_{\mathbb{R}}(V) < \infty$. For a vector bundle (E, M, q) and $x \in M$, let

$$T_x^{A,V} E = \{(\varphi_x, \psi_x) / \varphi_x \in Hom(C_x^\infty(M, \mathbb{R}), A) \text{ and } \psi_x \in Hom_{\varphi_x}(C_x^{\infty, f, l}(E), V)\},$$

where $Hom(C_x^\infty(M, \mathbb{R}), A)$ is the set of algebra homomorphisms φ_x from the algebra $C_x^\infty(M, \mathbb{R}) = \{germ_x(g) / g \in C^\infty(M, \mathbb{R})\}$ into A and $Hom_{\varphi_x}(C_x^{\infty, f, l}(E), V)$ is the set of module homomorphisms ψ_x over φ_x from the $C_x^\infty(M, \mathbb{R})$-module $C_x^{\infty, f, l}(E, \mathbb{R}) = \{germ_x(h) / h : E \to \mathbb{R}$ is fiberwise linear$\}$ into V. Let $T^{A,V} E = \bigcup_{x \in M} T_x^{A,V} E$ and $p_E^{A,V} : T^{A,V} E \to M$, $T_x^{A,V} E \ni (\varphi, \psi) \mapsto x$. $\left(T^{A,V} E, M, p_E^{A,V}\right)$ is a well-defined fibered manifold. Indeed let $c = (q^{-1}(U), x^i = u^i \circ q, y^j), 1 \leq i \leq m, 1 \leq j \leq n$ be a fibered chart of E; then the map

$$\phi_c : (p_E^{A,V})^{-1}(U) \to U \times N^m \times V^n$$

$$(\varphi_x, \psi_x) \mapsto (x, \varphi_x(germ_x(u^i - u^i(x))), \psi_x(germ_x(y^j))),$$

is a local trivialization for a bundle structure on $T^{A,V}E$. Given another vector bundle (G, M', q') and a vector bundle homomorphism $f : E \to G$ over $\bar{f} : M \to M'$, let

$$T^{A,V}f : T^{A,V}E \to T^{A,V}G$$

$$(\varphi_x, \psi_x) \mapsto (\varphi_x \circ \bar{f}_x^*, \psi_x \circ f_x^*),$$

where $\bar{f}_x^* : C_{\bar{f}(x)}^{\infty}(N) \to C_x^{\infty}(M)$ and $f_x^* : C_{\bar{f}(x)}^{\infty, f, l}(G) \to C_x^{\infty, f, l}(E)$ are given by the pull-back with respect to $\bar{f}$ and f. Then $T^{A,V}f$ is a fibered map over $\bar{f}$. $T^{A,V} : \mathcal{VB} \to \mathcal{FM}$ is a product preserving gauge bundle functor (see [12]).

Remark 2.1. Let $F : \mathcal{VB} \to \mathcal{FM}$ be a product preserving gauge bundle functor.

(a) F associates the pair $\left(A^F, V^F\right)$, where $A^F = F(id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R})$ is a Weil algebra and $V^F = F(\mathbb{R} \to pt)$ is a A^F-module such that $\dim_{\mathbb{R}}\left(V^F\right) < \infty$.

(b) There is a natural isomorphism $\Theta : F \to T^{A^F, V^F}$ defined on the vector bundle (E, M, q) as follows : For $\xi \in F_x E$, $\Theta_E(\xi) = (\varphi_x, \psi_x)$ with $\varphi_x(germ_x f) = F(f \circ q)(\xi)$ and $\psi_x(germ_x(h)) = F(h)(\xi)$. In particular, the product preserving gauge bundle functor $T^A : \mathcal{VB} \to \mathcal{FM}$ is equivalent to $T^{A,A}$.

(c) For two vector bundles E_1, E_2 with the same base M and $p_i : E_1 \oplus_M E_2 \to E_i$, $i = 1, 2$ the projections, the map $(Fp_1, Fp_2) : F(E_1 \oplus_M E_2) \to FE_1 \times FE_2$ induces a vector bundle isomorphism $F(E_1 \oplus_M E_2) \to FE_1 \oplus_{FM} FE_2$ in a natural way.

(d) For a vector bundle (E, M, q), (FE, FM, Fq) is also a vector bundle where $FM = T^{A^F}M$; if $f : E \to G$ is a morphism of vector bundles over $\bar{f} : M \to M'$, then $Ff : FE \to FG$ is also a morphism of vector bundles over $F\bar{f} : FM \to FM'$. The addition, the scalar multiplication and the zero section of $FE \to FM \to FM$ are respectively, given by

$$FE \oplus_{FM} FE \to FE \qquad\qquad \mathbb{R} \times FE \to FE$$

$$(\tilde{u}, \tilde{v}) \mapsto F\left(ad^E\right)(\tilde{u}, \tilde{v}), \qquad (\lambda, \tilde{u}) \mapsto F\left(m_\lambda^E\right)(\tilde{u}),$$

and $F0_E : FM \to FE$,

with ad^E, m^E, and 0_E the addition, scalar multiplication, and zero section of E, respectively.

3. Lifts of Double Vector Bundles

3.1. Double vector bundles

Definition 3.1. A double vector bundle structure is a system (D, A, B, M) of four vector bundles structures

$$\begin{array}{ccccc} & D & \xrightarrow{q_B^D} & B & \\ q_A^D & \downarrow & & \downarrow & q_B, \\ & A & \xrightarrow{q_A} & M & \end{array}$$

where D is a vector bundle on bases A and B, which are themselves vector bundles on M, such that each of the four structure maps of each vector bundle structure on D (namely, the bundle projection, addition, scalar multiplication and the zero section) is vector bundle morphism with respect to other structure.

Remark 3.1. Let (D, A, B, M) be a double vector bundle. The last part of the definition means that:

(1) q_B^D and q_A^D are morphisms of vector bundles over q_A and q_B, respectively; in particular $q_A \circ q_A^D = q_B \circ q_B^D : D \to M$.

(2) The zero sections $0_A^D : A \to D$, $0_B^D : B \to D$ are morphisms of vector bundles over the zero sections $0_M^A : M \to A$, $0_M^B : M \to B$, respectively.

(3) Since the sum $D \oplus_B D$ is a subbundle of the vector bundle

$$\left(q_A^D \times q_A^D\right)^{-1}(A \oplus_M A) \mapsto A \oplus_M A$$

by (1) and (2), the addition $ad_B^D : D \oplus_B D \to D$ is a morphism of vector bundles over the addition $ad_M^A : A \oplus_M A \to A$. Similarly, the addition $ad_A^D : D \oplus_A D \to D$ is a morphism of vector bundles over the addition $ad_M^B : B \oplus_M B \to M$.

(4) The scalar multiplication of $D \to B$, $m_B^D : \mathbb{R} \times D \to D$ is a morphism of vector bundles over the scalar multiplication $m_M^A : \mathbb{R} \times A \to A$. Similarly, of $D \to A$, $m_A^D : \mathbb{R} \times D \to D$ is a morphism of vector bundles over the scalar multiplication $m_M^B : \mathbb{R} \times B \to B$.

Definition 3.2. Let (D, A, B, M) be a double vector bundle. The bundle $D \xrightarrow{q_B^D} B$ is called the horizontal bundle structure on D and $D \xrightarrow{q_A^D} A$ is called the vertical bundle structure on D.

Example 3.1. (a) For a differential manifold M, let us consider trivial vector bundles $A = M \times \mathbb{R}^n$, $B = M \times \mathbb{R}^p$, $D = M \times \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$ over A and B; then (D, A, B, M) is a double vector bundle.

(b) The tangent bundle of a vector bundle (E, M, p)

$$\begin{array}{ccccc} & TE & \xrightarrow{T(p)} & TM & \\ \pi_E & \downarrow & & \downarrow & \pi_M \\ & E & \xrightarrow{p} & M & \end{array}$$

is a double vector bundle.

Definition 3.3. A morphism of double vector bundles

$$(\varphi, \varphi_A, \varphi_B, f) : (D, A, B, M) \to (D', A', B', M')$$

consists of morphisms of vectors bundles

$$\begin{array}{ccc} D & \xrightarrow{\varphi} & D' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\varphi_A} & A' \end{array}, \quad \begin{array}{ccc} D & \xrightarrow{\varphi} & D' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\varphi_B} & B' \end{array}, \quad \begin{array}{ccc} A & \xrightarrow{\varphi_A} & A' \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M' \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{\varphi_B} & B' \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M' \end{array}$$

If $M = M'$ and $f = id_M$, φ is called a morphism over M; if further $A = A'$ and $\varphi_A = id_A$, φ is said over A. If $A = A'$, $B = B'$ and both φ_A, φ_B are identities, we say that φ preserves the side bundles.

3.2. Double vector bundles as homogeneous structures

Let us recall the following notions according to [3].

A *smooth action* of the multiplicative monoid $(\mathbb{R}_+, \cdot)$ on a smooth manifold F, is a smooth map $h : \mathbb{R}_+ \times F \to F$, $(t, x) \mapsto h(t, x) = h_t(x)$ such that $h(1, x) = x$ and $h_t \circ h_s = h_{ts}$.

Given a smooth action $h : \mathbb{R}_+ \times F \to F$, let $M = h_0(F)$ the set of fixed points of the projection h_0 and the smooth map

$$\mathcal{V} : F \to TF,\ x \mapsto \frac{d}{dt}|_{t=0} h(t, x) \in T_{h_0(x)}F.$$

It is clear that $M \subset \mathcal{V}^{-1}(0)$, the inverse image of the set of zeros of *TF*; if $M = \mathcal{V}^{-1}(0)$ h is called a *homogeneous structure* on *F*. In this case, there is a structure of vector bundle $E \to M$ on $E = \mathcal{V}(F)$ and $\mathcal{V} : F \to E$ is a diffeomorphism. The vector bundle structure $F \overset{h_0}{\to} M$ carried by this diffeomorphism is the unique vector bundle structure on F whose homotheties coincide with *h*. Conversely homotheties of a vector bundle *F* associate a homogeneous structure $h : \mathbb{R}_+ \times F \to F,\ (t, x) \mapsto\ t \cdot x.$ This implies that vector bundles correspond with homogeneous structures.

The *Euler vector field* of a vector bundle $F \to M$ is the smooth vector field Δ_F on F given by $\Delta_F(x) = \frac{d}{dt}|_{t=1} t \cdot x.$ The global flow $Fl^{\Delta_E} : \mathbb{R} \times F \to F$ of Δ_F is given by $Fl^{\Delta_E}(t, x) = e^t \cdot x = h(e^t, x)$.

Two homogeneous structures $h_1, h_2 : \mathbb{R}_+ \times F \to F$ are called *commuting homogeneous structures* if $h_t^1 \circ h_s^2 = h_s^2 \circ h_t^1$ for all $(s, t) \in \mathbb{R}_+ \times \mathbb{R}_+$ i.e., $[\Delta^1, \Delta^2] = 0$, since h_1, h_2 come from global flows of Δ^1, Δ^2. If one denotes $E^i = h_0^i(F)$, $i = 1, 2$ the corresponding bases and $M = E^1 \cap E^2$, the system

$$\begin{array}{cccc} & F & \xrightarrow{h_0^2} & E^2 & \\ h_0^1 & \downarrow & & \downarrow & \overline{h_0^1} \\ & E^1 & \xrightarrow{\overline{h_0^2}} & M & \end{array}$$

is a double vector bundle structure. Conversely, the homotheties of a double vector bundle

$$\begin{array}{ccccc} & D & \xrightarrow{q_B^D} & B & \\ q_A^D & \downarrow & & \downarrow & q_B \\ & A & \xrightarrow{q_A} & M & \end{array}$$

commute. Indeed, the scalar multiplication of $D \to B$, $m_B^D : \mathbb{R} \times D \to D$ is a morphism of vector bundles over the scalar multiplication $m_M^A : \mathbb{R} \times A \to A$, hence for $(s, a) \in \mathbb{R} \times A$, the induced map $\{s\} \times D_a \to D_{s \cdot a}$, $(s, d) \mapsto s \underset{B}{\cdot} d$ is linear; in particular for all $t \in \mathbb{R}$, $s \underset{B}{\cdot} \left(t \underset{A}{\cdot} d\right) = t \underset{A}{\cdot} \left(s \underset{B}{\cdot} d\right)$. This shows that a double vector bundle can be equivalently defined as a smooth manifold equipped with two vector bundle structures whose Euler vector fields Δ^1, Δ^2 commute.

3.3. Natural transformations $Q(a) : F \to F$

Let $F : \mathcal{VB} \to \mathcal{FM}$ be a product preserving gauge bundle functor.

For a vector bundle (E, M, q), the scalar multiplication $m_M^E : \mathbb{R} \times E \to E$, is a vector bundle morphism over the projection $pr_2 : \mathbb{R} \times M \to M$, hence for any $a \in A^F$, there is a natural transformation $Q(a) : F \to F$ given by

$$Q(a)_E = Fm_M^E(a, \cdot) : FE \to FE. \tag{3.1}$$

$Q(a)$ is entirely determined by the algebra homomorphism $Q(a)_{\mathbb{R} \xrightarrow{id_{\mathbb{R}}} \mathbb{R}} = id_{A^F}$ and the module homomorphism $Q(a)_{\mathbb{R} \to pt} : V^F \to V^F$, $v \mapsto a \cdot v$, over id_{A^F} (Theorem 3.5 [12]). The following result is clear.

Proposition 3.1. *We have*

(1) $Q(1_A) = id_{T^A}$,

(2) $Q(a + b) = Q(a) + Q(b)$,

(3) $Q(\lambda \cdot a) = \lambda \cdot Q(a)$,

(4) $Q(ab) = Q(a) \circ Q(b)$,

for all $\lambda \in \mathbb{R}$ *and* $(a, b) \in A^2$.

3.4. Lifts of double vector bundles and linear sections

Let $F : \mathcal{VB} \to \mathcal{FM}$ be a product preserving gauge bundle functor and (D, A, B, M) a double vector bundle.

Theorem 3.2. *The system* (FD, FA, FB, FM) *is also a double vector bundle.*

Proof. (1) Fq_B^D and Fq_A^D are morphisms of vector bundles over Fq_A and Fq_B, respectively by Remark 3.1(d).

(2) The zero sections $F0_A^D : FA \to FD, F0_B^D : FB \to FD$ are morphisms of vector bundles over the zero sections $F0_M^A : FM \to FA$, $F0_M^B : FM \to FB$ by the same remark.

(3) The additions $Fad_B^D : FD \oplus_{FB} FD \to FD, Fad_A^D : FD \oplus_{FA} FD \to FD$ are morphisms of vector bundles over the addition $ad_M^A : A \oplus_M A \to A$, $ad_M^B : B \oplus_M B \to M$, respectively.

(4) The scalar multiplications of $FD \to FB$, $FD \to FA$

$$m_{FB}^{FD} : \mathbb{R} \times FD \to FD \qquad\qquad m_{FA}^{FD} : \mathbb{R} \times FD \to FD$$

$$(\lambda, \tilde{d}) \mapsto F\left(m_{B\lambda}^D\right)(\tilde{d}), \qquad\qquad (\lambda, \tilde{d}) \mapsto F\left(m_{A\lambda}^D\right)(\tilde{d})$$

are morphisms of vector bundles over the scalar multiplications

$$m_{FM}^{FA} : \mathbb{R} \times FA \to FA \qquad\qquad m_{FM}^{FB} : \mathbb{R} \times FB \to FB$$

$$(\lambda, \tilde{a}) \mapsto F\big(m_{M\lambda}^{A}\big)(\tilde{a}), \qquad\qquad (\lambda, \tilde{a}) \mapsto F\big(m_{M\lambda}^{B}\big)(\tilde{a}),$$

respectively, since by for each $\lambda \in \mathbb{R}$, the partial maps $m_{B\lambda}^{D}$ and $m_{A\lambda}^{D}$ are morphisms of vector bundles over $m_{M\lambda}^{A}$ and $m_{M\lambda}^{B}$, respectively. □

Remark 3.2. Let us denote $h_s^A, h_t^B (s, t) \in \mathbb{R}^* \times \mathbb{R}^*$ the homotheties of $FD \to FA$, $FD \to FB$, respectively. These are commuting vector bundle morphisms, hence $Fh_s^A, Fh_t^B (s, t) \in \mathbb{R}^* \times \mathbb{R}^*$ commute and since they are exactly homotheties of $FD \to FA$, $FD \to FB$,

$$\begin{array}{ccccc} & FD & \xrightarrow{Fh_0^B} & FB & \\ Fh_0^A & \downarrow & & \downarrow & F\overline{h_0^A} \\ & FA & \xrightarrow{F\overline{h_0^B}} & FM & \end{array}$$

is a double vector bundle.

Let (D, A, B, M) be a double vector bundle. We fix one of structures of vector bundles on D, $q_B^D : D \to B$ for instance.

Definition 3.4. A smooth section $\sigma \in \Gamma(q_B^D)$ is said linear if it is a morphism of vector bundles

$$\begin{array}{ccccc} & B & \xrightarrow{\sigma} & D & \\ q_B & \downarrow & & \downarrow & q_A^D, \\ & M & \xrightarrow{\overline{\sigma}} & A & \end{array}$$

i.e., $h_t^A \circ \sigma = \sigma \circ h_t$, for all $t \in \mathbb{R}_+$, where h, h^A are homotheties of $B \to M$, $D \to A$, respectively (Theorem 2.4 [3]).

Example 3.2. Linear sections of the double vector bundle (TE, TM, E, M) associated to the vector bundle structure $\pi_E : TE \to E$ are called linear vector fields on E. Linear vector fields were studied in [11] and classified in [13]. Some properties of lifts of such vector fields were studied in [14] and [15].

Definition 3.5. For a smooth linear section $\sigma : B \to D$ over $\overline{\sigma} : M \to A$ its a-lift $(a \in A^F)$ related to F is given by $\sigma^{(a)} = Q(a)_D \circ F\sigma : FB \to FD$, where $Q(a) : F \to F$ is the natural transformation (3.1).

$\sigma^{(a)} \in \Gamma(Fq_B^D)$ is linear over $\overline{\sigma}^{(a)} \in \Gamma(Fq_A)$ since

$$\begin{aligned} Fq_A^D \circ \sigma^{(a)} &= Fq_A^D \circ Q(a)_D \circ F\sigma \\ &= Q(a)_A \circ Fq_A^D \circ F\sigma \\ &= Q(a)_A \circ F(\overline{\sigma} \circ q_B) \\ &= \overline{\sigma}^{(a)} \circ Fq_B, \end{aligned}$$

and

$$\begin{aligned} Fh_t^A \circ \sigma^{(a)} &= Fh_t^A \circ Q(a)_D \circ F\sigma \\ &= Q(a)_A \circ Fh_t^A \circ F\sigma \\ &= Q(a)_A \circ F\left(h_t^A \circ \sigma\right) \\ &= Q(a)_A \circ F(\sigma \circ h_t) = \sigma^{(a)} \circ Fh_t, \end{aligned}$$

for all $t \in \mathbb{R}$.

Remark 3.3. Given a fibered chart $\varphi : \left(q_B^D\right)^{-1}(\Omega) \to w(\Omega) \times \mathbb{R}^n \times \mathbb{R}^q$ of $D \to B$ and the local frame $\{\varepsilon_j, \varepsilon'_k, 1 \le j \le n$ and $1 \le k \le q\}$ on Ω defined by

$$\begin{cases} \varepsilon_j(b) = \varphi^{-1}(w(b), e_j, 0), \\ \varepsilon'_k(b) = \varphi^{-1}(w(b), 0, e'_k), \end{cases}$$

where $(e_j), (e'_k)$ are basis of $\mathbb{R}^n, \mathbb{R}^q$, respectively, let us denote $\{\varepsilon_{j\beta}$, $\varepsilon'_{k\beta}$, $1 \le j \le n, 1 \le k \le q$ and the local frame associated the fibered chart $F\varphi$. We have

$$\varepsilon_{j\beta} = \varepsilon_j^{(e_\beta)} \text{ and } \varepsilon'^*_{k\beta} = \varepsilon'^{(e_\beta)}_k. \tag{3.2}$$

Indeed, let $c_j : \Omega \to \mathbb{R}^n, b \mapsto e_j$ and $c'_0 : \Omega \to \mathbb{R}^q, b \mapsto 0$; Fc_j and Fc'_0 are constant maps $\widetilde{b} \mapsto e_{j0}$, and $\widetilde{b} \mapsto 0_{(V^F)^q}$:

$$\begin{aligned} \varepsilon_j^{(e_\alpha)}\left(\widetilde{b}\right) &= Q(e_\alpha)_{p^{-1}(U)} \circ F\varepsilon_j\left(\widetilde{b}\right) \\ &= Q(e_\alpha)_{p^{-1}(U)} \circ F(\varphi^{-1} \circ (\omega \times c_j \times c'_0))\left(\widetilde{b}\right) \\ &= F\varphi^{-1} \circ Q(e_\alpha)_{u(U) \times \mathbb{R}^n}\left(Fw\left(\widetilde{b}\right), e_{j0}, 0\right) \\ &= F\varphi^{-1}\left(Fw\left(\widetilde{b}\right), e_\alpha, e_{j0}, 0\right) = F\varphi^{-1}\left(Fw\left(\widetilde{b}\right), e_{j\alpha}, 0\right) = \varepsilon_{j\alpha}(\widetilde{x}), \end{aligned}$$

for $\widetilde{b} \in FW$.

Corollary 3.3. *Sections* $\sigma^{(a)}$, $\sigma \in \Gamma\left(q_B^D\right)$ *and* $a \in A^F$, *generate the module* $\Gamma(FD)$ *of smooth sections of the vector bundle* $\left(FD, FB, F\left(q_B^D\right)\right)$.

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