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THE COMPLETE CHARACTERIZATIONS OF WEAKLY *Po* AND T_0-IDENTIFICATION *P* PROPERTIES REVISITED WITH A CORRECTION

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Abstract

In recent papers T_0-identification spaces were used to define weakly *Po* spaces and properties, and T_0-identification *P* properties. Initially, the search for properties that are weakly *Po* or T_0-identification *P* was by trial and error, motivating a 2017 paper in which weakly *Po* spaces and properties were characterized and T_0-identification *P* properties were thought to be characterized. Within this paper, a counterexample is given to the previously thought characterization of T_0-identification *P* properties and necessary corrections are made.

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1. Introduction and Preliminaries

T_0-identification spaces were introduced in the 1936 paper [15].

Definition 1.1. Let (X, T) be a space, R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the natural map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

Within the 1936 paper [15], T_0-identification spaces were used to jointly characterize pseudometrizable and metrizable: A space is pseudometrizable iff its T_0-identification space is metrizable.

Similarly, in the 1975 paper [14], the R_1 separation axiom and T_0-identification spaces were used to further characterize the T_2 property.

Definition 1.2. A space (X, T) is R_1 iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$ [1].

Since for any topological property P and any space with property P, its T_0-identification space exists, then there are no restrictions on spaces for which its T_0-identification space exists. Thus attention shifted from properties of spaces (X, T) for which its T_0-identification space $(X_0, Q(X, T))$ exists to the properties of the T_0-identification spaces $(X_0, Q(X, T))$, motivating the definition and work below.

Definition 1.3. A topological property P is a T_0-identification space property iff there exists a space (X, T), whose T_0-identification space has property P.

In the 1936 paper [15], it was shown that T_0-identification spaces satisfy the T_0 separation axiom. Thus, for a topological property to be a T_0-identification space property, (P and T_0), denoted by Po, would have to exist. Within the 2007 paper [2], it was shown that a space is T_0 iff the natural map N from the space onto its T_0-identification space is a homeomorphism. Thus, for each topological property P for which Po exists, Po is a T_0-identification space property. Hence $\{P \mid P$ is a topological property and a T_0-identification space property$\} = \{P \mid P$ is a topological property and Po exists$\}$.

Within the 1977 paper [3], several topological properties, including R_1, were shown to be simultaneously shared by a space and its T_0-identification space. Thus R_1 is a T_0-identification space property that is not $(R_1)o = T_2$ [1], raising questions about other topological properties that are T_0-identification space properties P for which $P \neq Po$.

In the 2015 paper [4], the use of T_0-identifications space to characterize each of metrizable and T_2, as given above, motivated the introduction and investigation of weakly Po spaces and properties.

Definition 1.4. Let P be a topological property for which Po exists. Then a space (X, T) is weakly Po iff its T_0-identification space $(X_0, Q(X, T))$ has property P. A topological property Qo for which weakly Qo exists is called a weakly Po property.

Since the T_0-identification space of each space is T_0, then for a topological property Q for which weakly Qo exists, a space (X, T) is weakly Qo iff $(X_0, Q(X, T))$ has property Qo, and, within $(X_0, Q(X, T))$, Q and Qo are equivalent.

By the results above, R_1 = weakly $(R_1)o$ = weakly T_2, which will be used later. Hence R_1 is weakly *Po*, and R_2 is a weakly *Po* property. Also, in the 2015 paper [4], it was shown that for a topological property Q for which weakly Qo exists, weakly Qo is simultaneously shared by both a space and its T_0-identification space, which when combined with the results above, led to the introduction and investigation of T_0-identification P properties.

Definition 1.5. A topological property S is a T_0-identification P property iff S is simultaneously shared by both a space and its T_0-identification space [5].

Then, by definition, for a topological property Q, Q is weakly *Po* iff Q is a T_0-identification P property and weakly *Po* and T_0-identification P are equivalent properties.

In the 2015 paper [4], the search for topological properties that fail to be weakly *Po* properties led to the need and use of T_0 and "not-T_0" revealing T_0 and "not-T_0" as useful topological properties, motivating the addition of the long-neglected topological property "not-P" into the study of topology, where P is a topological property for which "not-P" exists. Thus far, the addition and use of "not-P" in the study of topology has led to the discovery of the never before imagined least of all topological properties $L = (T_0$ or "not-T_0"$)$ [6] and that there is no strongest topological property [7]. As is expected, the existence of the never before imagined topological property L revealed needed changes in classical topology, including both product [8] and subspace properties [9] leading to new, meaningful, never before imagined properties and examples for each of those two properties, expanding and changing the study of topology forever.

Initially, the search for properties that are weakly *Po* or equivalently T_0-identification P was by trial and error. As established above, for a topological property Q for which Qo exists, a topological property W was sought such that for a space with property W its T_0-identification space has property Qo, which, in turn, implies the initial space has property W. Since the trial and error search process was tedious, time consuming, uncertain, and never ending, there was a need to completely characterize each of weakly *Po* spaces and properties. Within the 2017 paper [10], when it was not realized that weakly *Po* and T_0-identification P are equivalent topological properties, weakly *Po* was characterized and T_0-identification P was thought to be characterized. Below a counterexample is given for the once believed characterization of T_0-identification P and necessary changes are made.

2. Preliminaries and a Counterexample

Within the 2017 paper [10], for a topological property Q for which Qo exists, a property *QNO* was defined.

Definition 2.1. Let Q be a topological property such that Qo exists. A space (X, T) has property *QNO* iff (X, T) is "not-T_0" and $(X_0, Q(X, T))$ has property Qo.

In that paper [10], it was shown that for a topological property for which Qo exists, *QNO* exists and is a topological property, and a space has property (Qo or *QNO*) iff its T_0-identification space has property (Qo or *QNO*). Thus for a topological property Q for which Qo exists, (Qo or *QNO*) is a T_0-identification P property and (Qo or *QNO*) = weakly (Qo or *QNO*)o. Since *QNO* is "not-T_0", then (Qo and *QNO*)o = Qo. Thus $\{Uo \mid U$ is a topological property for which Uo exists$\} \subseteq \{Uo \mid U$ is a topological property and Uo is a weakly *Po* property$\}$ = $\{Uo \mid U$ is a topological property and T_0-identification $P\}$ and since $\{Uo \mid U$ is a topological property and Uo is a weakly Po property} $\subseteq$ {Uo | U is a topological property for which Uo exists}, then the three sets are equal and the weakly Po properties are completely characterized replacing the uncertainty of selecting Qo in the trial and error search process by certainty.

Also, the following statement was thought to be true.

Statement 2.1. For a topological property Q for which both Qo and $(Q$ and "not-T_0"$)$ exist, Q is a T_0-identification P property, QNO = $(Q$ and "not-T_0"$)$, and Q = weakly Qo = (Qo or $(Q$ and "not-T_0"$)$).

The following example shows Statement 2.1 not to be true.

Counterexample. Let $W = R_1$. Then $Wo = (R_1$ and $T_0) = T_2$ [1] exists. Since R_1 is a T_0-identification P property, then $WNO = (R_1$ and "not-T_0"$)$. Let $Q = (T_0$ or $(R_1$ and "not-T_0"$)$. Then $Qo = T_0$ and $(Q$ and "not-T_0"$) = (R_1$ and "not-T_0"$)$ exist and by Statement 2.1, Q is a T_0-identification P property. Hence Q is weakly Po and Q = weakly Qo = weakly $(T_0$ or $(R_1$ and "not-T_0"$))o = T_0$, but, since L = weakly Lo = weakly T_0 [11], then $L = (T_0$ or $(R_1$ and "not-T_0"$))$, which is a contradiction. Thus Statement 2.1 is not true.

3. A Correction

Theorem 3.1. *Let Q be a topological property. Then the following are equivalent*: (a) *Q is a T_0-identification P property*, (b) *Q is weakly Po*, (c) *both Qo and $(Q$ and "not-T_0"$)$ exists, and $(Q$ and "not-T_0"$) = QNO$, and* (d) $Q = (Qo$ *and* $QNO)$.

Proof. Clearly, by the results above, (a) and (b) are equivalent.

(b) implies (c): Let (X, T) be a space with property Q. Then (X, T) has property Q = weakly Qo, Qo exists, and $(X_0, Q(X, T))$ has property Qo, which implies (X, T) has property (Qo or QNO), where Qo and QNO are distinct topological properties. Thus Q is a T_0-identification P property and $Q = (Qo$ or $(Q$ and "not-T_0")) [12], which implies (Q and "not-T_0") = QNO.

(c) implies (d): Since both Qo and (Q and "not-T_0") exist, then $Q = (Qo$ or $(Q$ and "not-T_0")) = (Qo or QNO), which is a T_0-identification P property.

Thus, the uncertainty in the trial and error search process for selecting the starting place Qo is resolved leaving only the uncertainty of determining weakly Qo. Within the paper [13], each of the T_0-identification space and weakly Po processes were internalized greatly simplifying the search for weakly Qo.

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