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# ON A CHOICE MODEL CONCERNING ALL COHERENT PREVISIONS OF TWO OR MORE THAN TWO RANDOM GAINS

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## Abstract

We show a choice model based on application the principles of the economic theory of preferences of consumers to the two-dimensional convex set of all coherent previsions of two or more than two random gains. Such a model is well-founded because we establish an essential analogy between the properties of consumer preferences about consumption bundles and the ones of coherent previsions of random quantities. We deal with a unified approach to an integrated and simplified formulation of decision-making theory in its two components, probability and utility. The path followed in order to study coherence properties of prevision is guided by the economic criteria of the

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decision-making theory which is however presented in a simplified form. Indeed, the fundamental hypothesis of additivity of prevision tells us that the decision-maker is not risk-averse but he is risk-neutral, so the certain gain equivalent to a random quantity viewed as a random gain coincides with a particular and coherent prevision of this random gain. We are the first in the world to do this kind of work and for this reason we believe that it is not inappropriate that our references limit themselves to those pioneering works which will always be very meaningful from heuristic point of view too. We prove that it is occasionally necessary to look further back than one could usually think in order to imagine works developing interesting and original ideas.

## 1. Introduction

It is methodologically fundamental to distinguish the logic of certainty from the logic of probable, that is to say, possibility from probability: what is objectively certain or impossible or possible is different from what is subjectively probable. It makes sense to express one's subjective and non-predetermined opinion only in respect of what is possible or uncertain at a given instant. We always mean uncertainty as a simple ignorance: it ceases only when we receive certain information. What is logical is exact but it says nothing, so we have to consider the importance of what is extralogical: probability is exactly an extralogical notion because it is an additional notion with respect to the logic of certainty. It is also a psychological notion because its value does not transcend the psychological value that it has with regard to each individual. Moreover, probability is not independent of such a psychological value, so we deal with a living, elastic and psychological logic coinciding with the logic of probable. We believe that probability does not exist outside of us. Its nature is absolutely unitary and it does not have an absolute and objective value which is independent of our thought, sensations and assessments. Therefore, any random quantity can be dealt with by the logic of certainty as well as by the logic of probable. We recognize two different and extreme aspects concerning the logic of certainty. At first we distinguish a more or less extensive class of alternatives which appear possible to us in the current state of our information. Afterwards we definitively observe which is the true alternative to be verified among the ones logically possible. The probability comes into play after constituting the range of possibility and before knowing which is the true alternative to be verified: the logic of probable will fill in this range by considering a probabilistic mass distributed upon it in a coherent way. An individual correctly makes a prevision of a random quantity when he leaves the objective domain of the logically possible in order to distribute his subjective sensations of probability among all the possible alternatives and in the way which will appear most appropriate to him ([14], [15], [16]). After assigning a subjective probability $p_i$ to each possible value $x_i$, $i = 1, \ldots, n$, of $X$ belonging to the set $I(X) = \{x_1, \ldots, x_n\}$, with $x_1 < \ldots < x_n$, we obtain a coherent prevision of $X$ if and only if we have $0 \leq p_i \leq 1$ as well as $\sum_{i=1}^{n} p_i = 1$ ([2], [3], [11]). When $p_i$ varies while $x_i$ is constant we obtain all coherent previsions of $X$ which always recognize a continuous set. This set is a convex set. Thus, the logic of probable deals with continuous sets while the logic of certainty deals with discrete sets. Into our choice model, we consider the two aspects of the logic of certainty into a linear space coinciding with the two-dimensional real space $\mathbb{R}^2$ over the field $\mathbb{R}$ of real numbers where we deal with two random quantities. We have two orthogonal axes to each other: a same Cartesian coordinate system is chosen on every axis. The real space $\mathbb{R}^2$ has an Euclidean structure and it is evidently our space $\mathbb{S}$ of alternatives. The set $\mathcal{Q}$ related to two random quantities which are jointly considered is a discrete subset of the two-dimensional real space $\mathbb{R}^2$. Every possible value for the two random quantities under consideration belonging to the set $\mathcal{Q}$ definitively becomes 0 or 1 when we make an empirical observation referring to it. We observe that the set $\mathcal{Q}$ of two random quantities jointly considered into a linear space becomes a Boolean algebra whose two idempotent numbers are into a discrete subset of the two-dimensional real space $\mathbb{R}^2$ over the field $\mathbb{R}$ of real numbers.

Conversely, all coherent previsions of two random quantities which are jointly considered turn out to be represented by a two-dimensional convex set. This set is a continuous set. All points of it are formally admissible in terms of coherence but the decision-maker must always choose one point of them. Into our choice model, we wonder if there is a point of the two-dimensional convex set under consideration which must be chosen by the decision-maker because it is the best point among all admissible points.

## 2. Events as Idempotent Numbers

An event is a mental separation between subjective sensations. It is actually a statement which you do not know yet to be true or false. The statements of which you can say if they are true or false on the basis of an empirical observation which is well-determined and always possible, theoretically at least, have an objective meaning ([7], [8], [9]). Such statements are said propositions if one is thinking more in terms of the expressions in which they are formulated or events if one is thinking more in terms of the situations and circumstances to which their being true or false corresponds. Therefore, proposition and event are the same thing. An event is certain or impossible for an individual when he already knows the outcome before knowing if it is true or false: the proposition "in a throw of a die having six faces, with each of them showing a different number from 1 to 6, the face that is uppermost when it comes to rest is 1 or 2 or 3 or 4 or 5 or 6" is an event but it is not a random event as well as "in a throw of a die having six faces, with each of them showing a different number from 1 to 6, the face that is uppermost when it comes to rest is 41". The intermediate case of uncertainty is clearly the only that allows evaluations of probability. Every evaluation expresses a subjective degree of belief in the occurrence of a single event. An evaluation of probability is always attributed by a given individual with a given set of information at a given instant ([10], [12], [13]). The logic of certainty facilitates us to fix our attention on sensations so, through the convention true = 1 and false = 0, it is the necessary tool of every reasoning in those cases where it is only relevant the occurrence or not of an event: if $A$ and $B$ are events, the negation of $A$ is $\overline{A} = 1 - A$ and such an event is true if $A$ is false, while if $A$ is true it is false; the negation of $B$ is similarly $\overline{B} = 1 - B$. The logical product of $A$ and $B$ is $A \wedge B = AB$ and such an event is true if $A$ is true and $B$ is true, otherwise it is false; the logical sum of $A$ and $B$ is $(A \vee B) = \left(\overline{\overline{A} \wedge \overline{B}}\right) = 1 - (1 - A)(1 - B)$ from which it follows that such an event is true if at least one of events is true, where we have $A \vee B = A + B$ when $A$ and $B$ are incompatible events because it is impossible for them both to occur. Thought is essential to elaboration of the logic: our mind is necessarily finite so the logic of certainty is also finite. Its role consists in deducing the truth or falsity of certain consequences from the truth or falsity of certain premises. Therefore, when one considers, for example, a logical product of infinite events or one wonders if infinite events of a set are all true, one can never verify if such statements are true or false. These statements are conceptually meaningless because they do not coincide with any mental separation between subjective sensations. Hence, it is not a logical restriction to consider finite partitions of incompatible and exhaustive events identified by the idempotent numbers 0 and 1.

## 3. Random Quantities as Finite Partitions of Events

For any individual who does not certainly know the true value of a quantity $X$, which is random in a non-redundant usage for him, there are two or more than two possible values for $X$ that will always be real numbers in this context. The set of these values is denoted by $I(X)$: in any case only one is the true value of each random quantity and the meaning that you have to give to random is the one of unknown by the individual of whom you consider his state of uncertainty. Thus, random does not mean undetermined but it means established in an unequivocal fashion, so a supposed bet based upon it would be decided at the appropriate time. The possible values for $X$ are objective because, although it is personal their determination, they depend on objective circumstances which consist in the imperfect state of information of an individual, that is to say, in his degree of ignorance. Indeed, when a given individual outlines the domain of uncertainty he does not use his subjective opinions on what he does not know because the values of $X$ depend only on what he objectively knows or not. In this context, each possible value for $X$ belonging to the set $I(X) = \{x_1, \ldots, x_n\}$, with $x_1 < \ldots < x_n$, will always be a non-negative real number. A prevision of $X$ is given by $\mathbf{P}(X) = x_1 p_1 + \ldots + x_n p_n$, with $0 \leq p_i \leq 1$, $i = 1, \ldots, n$, and $\sum_{i=1}^{n} p_i = 1$: it is rendered as a function of the probabilities $p_i$ of the possible values for $X$. The prevision of $X$ is usually called the mathematical expectation of $X$ or its mean value. The above result, which is also interpretable as center of mass or barycentre of a discrete system of $n$ point masses placed on a line on which a Cartesian coordinate system has been chosen, can be extended to the general case when we introduce the notion of a probability density function as well as the one of a cumulative distribution function. These notions are more complicated than the concept of prevision and they require the use of more advanced mathematical tools than is actually necessary. On the other hand, the use of Riemann-Stieltjes integral allows to calculate the barycentre in case of a continuous system when a probability density function is not defined and this fact strengthens the essentiality of the concept of mean value unlike the one of exact distribution of mass which can never be determined in practice. The same symbol $\mathbf{P}$ evidently denotes both prevision of a random quantity and probability of an event because we identify any event $E$ with a random quantity called the indicator of $E$ which takes values 1 or 0 according as $E$ is true or false. Probability calculus has a very special character according to subjectivistic conception of probability because common sense plays the most essential role and it is analytically expressed as objective conditions of coherence.

Indeed, regarding an evaluation of probability, known over any finite set of possible events and interpretable as the opinion of a given individual, we can only judge if it is coherent or not. The calculus of probability can be based on only one restriction according to which it would be incoherent not to think that the probability of the logical sum of two incompatible events has to increase when the probabilities of these two events increase; putting it differently, with $A$ and $B$ which are two incompatible events, since we have to consider $A \vee B = A + B$, after evaluating both $A$ and $B$ in a coherent fashion, the same individual who evaluates the event-sum $A \vee B$ in such a way as to obtain $\mathbf{P}(A \vee B) \neq \mathbf{P}(A) + \mathbf{P}(B)$ is not coherent ([1], [4], [5], [6]). We have coherently both $0 \leq \mathbf{P}(A) \leq 1$ and $0 \leq \mathbf{P}(B) \leq 1$.

## 4. Coherence Properties of P

We deal with a unified approach to an integrated and simplified formulation of decision-making theory in its two components, probability and utility. A random quantity is always a random gain in this context. In general, given $X$ which is a random gain whose possible values are monetary values, we call $\mathbf{P}(X)$ the certain gain which is considered equivalent to $X$ according to a fixed individual, so $X$ is preferred, or not, to a certain gain $x$ according as $x$ is less than $\mathbf{P}(X)$ or $x$ is greater than $\mathbf{P}(X)$ into a subjective scale of preference represented by a cardinal utility function. It is convenient to consider two distinct and orthogonal axes into a Cartesian coordinate plane having an Euclidean structure in order to examine coherence properties of $\mathbf{P}$ under conditions of uncertainty. If $X_1$ and $Y_1$ are two random gains whose possible values belonging to the set $I(X_1)$ and to the set $I(Y_1)$ are represented on the horizontal axis, we have

$$\mathbf{P}(X_1 + Y_1) = \mathbf{P}(X_1) + \mathbf{P}(Y_1)$$

because **P** is an additive function. We have

$$\inf I(X_1 + Y_1) \leq \mathbf{P}(X_1 + Y_1) \leq \sup I(X_1 + Y_1),$$

with $X_1 + Y_1 = Z_1$, because **P** is convex. If $X_2$ and $Y_2$ are two random gains whose possible values belonging to the set $I(X_2)$ and to the set $I(Y_2)$ are represented on the vertical axis, we have similarly

$$\mathbf{P}(X_2 + Y_2) = \mathbf{P}(X_2) + \mathbf{P}(Y_2),$$

and

$$\inf I(X_2 + Y_2) \leq \mathbf{P}(X_2 + Y_2) \leq \sup I(X_2 + Y_2),$$

with $X_2 + Y_2 = Z_2$. Such properties are necessary and sufficient conditions for coherence, that is to say, for avoiding undesirable decisions which lead to a certain loss. They can be taken as the foundation for the entire theory of probability because when we consider a particular random quantity which is an event because its possible values are only two we obtain that **P** is a probability of it. More generally, by considering a random quantity $X$ whose possible values are not monetary values, $\mathbf{P}(X)$ is called prevision of $X$. The term prevision is valid in all cases and we will use it. From additivity and convexity we notice that **P** is also linear because we have, for every real number $a$,

$$\mathbf{P}(aZ_1) = a\mathbf{P}(Z_1)$$

as well as

$$\mathbf{P}(aZ_2) = a\mathbf{P}(Z_2).$$

More generally, we have

$$\mathbf{P}(aZ_1' + bZ_1'' + cZ_1''' + \ldots) = a\mathbf{P}(Z_1') + b\mathbf{P}(Z_1'') + c\mathbf{P}(Z_1''') + \ldots,$$

for any finite number of random gains $Z_1', Z_1'', Z_1''' \ldots$ or

$$\mathbf{P}(aZ_2' + bZ_2'' + cZ_2''' + \ldots) = a\mathbf{P}(Z_2') + b\mathbf{P}(Z_2'') + c\mathbf{P}(Z_2''') + \ldots,$$

for any finite number of summands $Z_2', Z_2'', Z_2''', \ldots,$ with $a, b, c, \ldots$ any real numbers. In particular, additivity of **P** allows to extend a line segment on the horizontal axis when such a line segment is recognized by convexity of **P** as well as a line segment on the vertical axis when such a line segment is recognized by convexity of **P**. Indeed, $\mathbf{P}(X_1)$ and $\mathbf{P}(Y_1)$ have the same masses or probabilities of $\mathbf{P}(X_1 + Y_1),$ while the absolute value of each element of the set $I(X_1 + Y_1)$ is not lower than the one of each element of $I(X_1)$. The same goes when we consider $\mathbf{P}(X_2), \mathbf{P}(Y_2), \mathbf{P}(X_2 + Y_2), I(X_2 + Y_2)$ and $I(X_2)$. This is the reason why we will later consider $Z_1$ on the horizontal axis and $Z_2$ on the vertical one into our choice model.

## 5. Space of Alternatives and set of Coherent Previsions

When we consider one random quantity *X*, every possible value of it, for a given individual at a certain instant, is an element of the set $I(X) = \{x_1, \ldots, x_n\},$ with $x_1 < \ldots < x_n,$ coinciding with the set $\mathcal{Q}$. More in general, each possible value $x_i, i = 1, \ldots, n,$ is a real number in the space $\mathbb{S}$ of alternatives coinciding with a line on which an origin, a unit of length and an orientation are chosen. The set $\mathcal{Q}$ is a subset of $\mathbb{S}$. If *X* belongs to a half-line, $X \leq x,$ or to an interval, $x' \leq X \leq x'',$ or to any arbitrary set, $X \in \mathcal{J},$ the space $\mathbb{S}$ always coincides with such a line. We consider *en masse* all events concerning *X*. Every point of a line is assumed to correspond to a real number and every real number to a point: the real line is a vector space of dimension 1 over the field $\mathbb{R}$ of real numbers, that is to say, over itself. After assigning a subjective probability $p_i$ to each possible value $x_i, i = 1, \ldots, n,$ of *X* it turns out to be $\mathbf{P}(X) = x_1 p_1 + \ldots + x_n p_n,$ where we have $0 \leq p_i \leq 1$ and $\sum_{i=1}^{n} p_i = 1,$ with $\mathbf{P}(X)$ which is a prevision of *X*. All coherent previsions of *X* are obtained when $p_i$ varies while $x_i$ is constant and they coincide with a part of a line that is bounded by two distinct end points, $x_1 = \inf I(X)$ and $x_n = \sup I(X)$, because we have coherently in $\inf I(X) \le \mathbf{P}(X) \le \sup I(X)$. We recognize a one-dimensional convex set. Into our choice model we jointly consider two random quantities, $X_1$ and $X_2$, viewed as random gains into a Cartesian coordinate plane. We suppose that the possible values of $X_1$ and $X_2$ are on two different and orthogonal lines on which an origin, a unit of length and an orientation are chosen whose intersection is given by the point $(0, 0)$. It is absolutely the same thing if every possible value of a random quantity is viewed as a particular ordered pair of real numbers or as a single real number. All marginal and coherent previsions of $X_1$ and $X_2$ recognize two different segments on these two lines because we have respectively in $\inf I(X_1) \le \mathbf{P}(X_1) \le \sup I(X_1)$ and in $\inf I(X_2) \le \mathbf{P}(X_2) \le \sup I(X_2)$. The set $\mathcal{Q}$ of the possible points for the random point $(X_1, X_2)$ consists of pairs of possible values for $X_1$ and $X_2$. These quantities are said to be logically independent because if $X_1$ and $X_2$ have respectively *r* possible values and *s* possible values, then all the *rs* pairs are possible for $(X_1, X_2)$. Into our choice model, we suppose that all possible values for $X_1$ and $X_2$ are non-negative. The set $\mathcal{P}$ of all coherent previsions $\mathbf{P}$ is a subset into a Cartesian coordinate plane: the possible pairs $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ are the Cartesian coordinates of a possible point of this subset. We always project $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ onto the two orthogonal axes whose intersection is given by the point $(0, 0)$ because we are interested in marginal and coherent previsions into our choice model. Moreover, when we project $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ onto the two axes under consideration it is always the same thing if every marginal and coherent prevision is viewed as a particular ordered pair of real numbers or as a single real number. Every point of a Cartesian coordinate plane is assumed to correspond to an ordered pair of real numbers and vice versa: $\mathbb{R}^2$ is a vector space of dimension 2 over the field $\mathbb{R}$ of real numbers and it is called the two-dimensional real space. The set $\mathcal{P}$ of all coherent previsions $\mathbf{P}$ is geometrically a two-dimensional convex set: it is the closed convex hull of the set $\mathcal{Q}$ of the possible values of $X_1$ and $X_2$. The set $\mathcal{P}$ is analytically a two-dimensional convex set because the linear inequality between the random quantities $X_1$ and $X_2$ is given by

$$c_1 X_1 + c_2 X_2 \leq c.$$

It must be satisfied by the corresponding previsions $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$, so we have

$$c_1 \mathbf{P}(X_1) + c_2 \mathbf{P}(X_2) \leq c.$$

The expression given by

$$c_1 \mathbf{P}(X_1) + c_2 \mathbf{P}(X_2) = c \tag{1}$$

is the equation of a line whose slope is $-\frac{c_1}{c_2}$, horizontal intercept is given by $\frac{c}{c_1}$ while vertical intercept is given by $\frac{c}{c_2}$. Such a line is a hyperplane in the vector space $\mathbb{R}^2$ and a point $\mathbf{P}$ of $\mathcal{P}$ is a coherent prevision because the line given by (1) does not separate it from the set $\mathcal{Q}$ of the possible points for $X_1$ and $X_2$. We suppose that this line passes through the point $(\sup I(X_1), \sup I(X_2))$ of a Cartesian coordinate plane.

## 6. By Constructing our Choice Model

At first we jointly consider two monetary quantities $X_1$ and $X_2$ having the meaning of non-negative and random gains whose sets of possible values are respectively, $I(X_1) = \{x_{11}, \ldots, x_{1n}\}$ and $I(X_2) = \{x_{21}, \ldots, x_{2n}\}$. Then, the inequality $c_1 \mathbf{P}(X_1) + c_2 \mathbf{P}(X_2) \leq c$ follows from the linear inequality $c_1 X_1 + c_2 X_2 \leq c$: the point $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ of a Cartesian coordinate plane is a point of the two-dimensional closed convex hull of the set $\mathcal{Q}$ of the possible points for the random point $(X_1, X_2)$. There evidently exists a dichotomy between $I(X_1) = \{x_{11}, \ldots, x_{1n}\}$ and $\mathbf{P}(X_1)$, between $I(X_2) = \{x_{21}, \ldots, x_{2n}\}$ and $\mathbf{P}(X_2)$ as well as between $\mathcal{Q}$ and the set of points of a Cartesian coordinate plane whose coordinates are given by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$. Any point of $\mathcal{Q}$ is expressed by $(x_{1i}, x_{2j})$, where we have $i, j = 1, \ldots, n$. Therefore, the sets $I(X_1)$, $I(X_2)$ and $\mathcal{Q}$ contain a finite number of possible values unlike the sets $\mathbf{P}(X_1)$, $\mathbf{P}(X_2)$ and $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ containing an infinite number of values. Afterwards, we jointly consider two random gains $Z_1$ and $Z_2$ into a same Cartesian coordinate plane because we must take into account additivity property of $\mathbf{P}$ referring to marginal and coherent previsions. Thus, the two-dimensional convex set of all coherent previsions of $Z_1$ and $Z_2$ consists of an infinite number of points. Each point of this set is an ordered pair $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$ of real numbers that we always project onto the two orthogonal axes under consideration. This set is bounded by a line segment on the horizontal axis whose lower end-point is the number 0 on the horizontal line and whose higher end-point is the highest possible value of $Z_1$ on the same line, by a line segment on the vertical axis whose lower end-point is the number 0 on the vertical line and whose higher end-point is the highest possible value of $Z_2$ on the same line, and by a line with a negative slope whose equation is given by $c_1 \mathbf{P}(Z_1) + c_2 \mathbf{P}(Z_2) = c$: it is a hyperplane in the vector space $\mathbb{R}^2$ having a Euclidean structure. After projecting $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$ onto the two axes under consideration it is evident that we take into account a continuous set of real numbers on the horizontal axis as well as on the vertical one because these sets respectively contain all marginal and coherent previsions of $Z_1$ and $Z_2$. Only one point $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$ of the ones of the two-dimensional convex set under consideration is the one chosen by the decision-maker under conditions of uncertainty: what can we say about this point? By analogy with an economic model of consumer behaviour, can we say that the decision-maker chooses the best things he can afford? Which is the best decision-maker choice about $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$ and consequently about marginal and coherent prevision of $Z_1$ and $Z_2$? Random gains as random quantities can obviously be an infinite number but we consider only the case of two random gains because it is more general than you might think at first, since we can often interpret one of random gains as representing everything else the decision-maker might want to evaluate. Moreover, when we consider only the case of two random gains, we could graphically represent decision-maker choice. In this way, we could evidently represent decision-maker choices involving many random gains by using two-dimensional diagrams. We call prevision bundles the objects of decision-maker choice. Therefore, every point $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$ of a Cartesian coordinate plane can be imagined as a consumption bundle, with $\mathbf{P}(Z_1)$ and $\mathbf{P}(Z_2)$ that tell us how much the decision-maker is choosing to foresee of $\mathbf{P}(Z_1)$ and how much the decision-maker is choosing to foresee of $\mathbf{P}(Z_2)$. The prices of $\mathbf{P}(Z_1)$ and $\mathbf{P}(Z_2)$ are respectively, $c_1$ and $c_2$, while the amount of money the decision-maker has to spend is $c$. The expression $c_1\mathbf{P}(Z_1) + c_2\mathbf{P}(Z_2) \leq c$ represents the budget constraint of the decision-maker because the amount of money spent on $\mathbf{P}(Z_1)$ and on $\mathbf{P}(Z_2)$ must be no more than the total amount the decision-maker has to spend. The decision-maker's affordable prevision bundles are those that do not cost any more than $c$. This set of affordable prevision bundles at prices $(c_1, c_2)$ and income $c$ is the budget set of the decision-maker. The budget set is a two-dimensional convex set and it is an extension of $\mathcal{P}$ referring to $(\mathbf{P}(X_1), \mathbf{P}(X_2))$. The expression $c_1\mathbf{P}(Z_1) + c_2\mathbf{P}(Z_2) = c$ represents the budget line which is the set of prevision bundles that cost exactly $c$. The budget constraint will take the form $c_1\mathbf{P}(Z_1) + \mathbf{P}(Z_2) \leq c$ if $\mathbf{P}(Z_2)$ represents everything else the decision-maker might want to foresee other than $\mathbf{P}(Z_1)$. We say that random gain $Z_2$ represents a composite random quantity and its price will automatically be $c_2 = 1$ as well as the one of $\mathbf{P}(Z_2)$. We suppose that the decision-maker can rank various prevision possibilities. They are all formally admissible in terms of coherence: the subjective way in which he ranks the prevision bundles describes his preferences into our choice model. Economics tells us that well-behaved preferences are monotonic, because more is better, and convex, because averages are weakly preferred to extremes. We are obviously talking about goods, not bads. Indifference curves are characterized by a negative slope and they are used to represent different kinds of preferences in a graphical way. Our choice model has indifference curves which are parallel lines referring to the first quadrant of a two-dimensional Cartesian coordinate system: we can intuitively think of indifference curves representing perfect substitutes, so the weighted average of two indifferent and extreme prevision bundles is not preferred to the two extreme prevision bundles but it is as good as the two extreme prevision bundles. Moreover, we can use an ordinal utility function because it is simply a way to represent a preference ordering which is related to all coherent previsions of $(Z_1, Z_2)$ that we always project onto the two axes under consideration. Every prevision bundle is getting a utility level and those prevision bundles on higher indifference curves are getting larger utility levels but the numerical magnitudes of utility levels have no intrinsic meaning at this step of our choice model. Thus, we established an essential analogy between properties related to coherent previsions of two random gains that we always project onto the two axes under consideration and the ones related to well-behaved preferences. It is good because additivity and convexity of $\mathbf{P}$ referring to marginal and coherent previsions correspond to monotonicity and convexity of well-behaved preferences. After projecting $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$ onto the two orthogonal axes of a

Cartesian coordinate plane when we say that more is better we mean that a line segment is increasingly large on the horizontal axis and a line segment is increasingly large on the vertical one.

## 7. Revealed Coherent Previsions of Random Quantities

Among all points formally admissible in terms of coherence we suppose that the point chosen by the decision-maker is $(r_1, r_2)$, with $r_1 = \mathbf{P}(Z_1)$ and $r_2 = \mathbf{P}(Z_2)$. We always project this point onto the two axes under consideration. Economics tells us that the prevision bundle $(r_1, r_2)$ must represent an optimal choice for the decision-maker. Given the budget $(c_1, c_2, c)$, the decision-maker can choose, if he wants, the prevision bundle $(s_1, s_2)$, with $s_1 = \mathbf{P}(Z_1)$ and $s_2 = \mathbf{P}(Z_2)$ that we project onto the two axes under consideration, where we have $s_1 \neq r_1$ and $s_2 \neq r_2$, and he can even have leftover money. When we say that the decision-maker can choose the prevision bundle $(s_1, s_2)$ at prices $(c_1, c_2)$ and income $c$, we mean that $(s_1, s_2)$ satisfies the budget constraint

$$c_1 s_1 + c_2 s_2 \leq c.$$

Given this budget, the prevision bundle $(r_1, r_2)$ is actually chosen, that is to say, it must satisfy the budget constraint with equality, so we have

$$c_1 r_1 + c_2 r_2 = c.$$

Putting these two expressions together, the fact that the decision-maker can choose the prevision bundle $(s_1, s_2)$ when it satisfies the budget constraint $(c_1, c_2, c)$, it means that

$$c_1 r_1 + c_2 r_2 \geq c_1 s_1 + c_2 s_2.$$

In other words, we establish:

**The principle of revealed coherent prevision.** Let $(r_1, r_2)$ be, with $r_1 = \mathbf{P}(Z_1)$ and $r_2 = \mathbf{P}(Z_2)$ that we project onto the two orthogonal axes of a Cartesian coordinate plane, the chosen prevision bundle at prices $(c_1, c_2)$ and let $(s_1, s_2)$ be, with $s_1 = \mathbf{P}(Z_1)$ and $s_2 = \mathbf{P}(Z_2)$ that we project onto the two orthogonal axes under consideration, where we have $s_1 \neq r_1$ and $s_2 \neq r_2$, another prevision bundle such that $c_1 r_1 + c_2 r_2 \geq c_1 s_1 + c_2 s_2$: then, if the decision-maker is choosing the most preferred prevision bundle he can afford, we must have that the $r$-bundle is strictly preferred to the $s$-bundle.

If the inequality $c_1 r_1 + c_2 r_2 \geq c_1 s_1 + c_2 s_2$ is satisfied and $(s_1, s_2)$ is actually a different prevision bundle from $(r_1, r_2)$, we say that $(r_1, r_2)$ is directly revealed preferred to $(s_1, s_2)$ in the sense that $(r_1, r_2)$ is chosen instead of $(s_1, s_2)$. All the prevision bundles that could have been chosen but were not, because they have been rejected in favor of $(r_1, r_2)$, are revealed worse than the chosen prevision bundle $(r_1, r_2)$. The choices that decision-makers make are preferred to the choices that they could have made so, if the point $(r_1, r_2)$ of a Cartesian coordinate plane is directly revealed preferred to $(s_1, s_2)$, then $(r_1, r_2)$ is actually preferred to the point $(s_1, s_2)$. Therefore, we have the following:

**Weak axiom of revealed coherent prevision.** If the $r$-bundle is directly revealed preferred to the $s$-bundle and the two prevision bundles are different, then it cannot happen that the $s$-bundle is directly revealed preferred to the $r$-bundle.

The above axiom tells us that when the $r$-bundle is chosen at prices $(c_1, c_2)$ and the different $s$-bundle is chosen at prices $(d_1, d_2)$, then if we have

$$c_1 r_1 + c_2 r_2 \geq c_1 s_1 + c_2 s_2,$$

it must not be the case that

$$d_1 s_1 + d_2 s_2 \geq d_1 r_1 + d_2 r_2.$$

In other words, we mean that if the $s$-bundle is affordable when the $r$-bundle is chosen, then when the $s$-bundle is chosen, the $r$-bundle must not be affordable. Now we suppose that the prevision bundle $(s_1, s_2)$ is chosen at prices $(d_1, d_2)$ and that it is itself revealed preferred to another prevision bundle $(t_1, t_2)$, where we have $t_1 = \mathbf{P}(Z_1)$ and $t_2 = \mathbf{P}(Z_2)$ that we project onto the two orthogonal axes under consideration, with $t_1 \neq s_1$ and $t_2 \neq s_2$. Then, we have

$$d_1 s_1 + d_2 s_2 \geq d_1 t_1 + d_2 t_2.$$

Therefore, we know that the $r$-bundle is strictly preferred to the $s$-bundle and that the $s$-bundle is strictly preferred to the $t$-bundle, so we can conclude that the $r$-bundle is indirectly revealed preferred to the $t$-bundle. We evidently use the transitivity assumption about consumer preference, so we can say that the decision-maker definitely wants the $r$-bundle rather than the $t$-bundle. We have consequently the following:

**Strong axiom of revealed coherent prevision.** If the $r$-bundle is directly revealed preferred to the $s$-bundle or the $r$-bundle is indirectly revealed preferred to the $s$-bundle, with the $r$-bundle and the $s$-bundle which are not the same, then the $s$-bundle cannot be directly or indirectly revealed preferred to the $r$-bundle.

The chain of direct comparisons can obviously be of any finite length. Moreover, when we change prices and income we observe that the budget line changes its negative slope. Anyway, our choice model tells us that such a line must pass through the point $(\sup I(X_1), \sup I(X_2))$ of a Cartesian coordinate plane. The optimal choice of the decision-maker is evidently that prevision bundle in the decision-maker's budget set which lies on the highest indifference curve. By considering $\mathbf{P}(Z_1)$ and $\mathbf{P}(Z_2)$, the highest indifference curve for the decision-maker coincides with the line $c_1\mathbf{P}(Z_1)+c_2\mathbf{P}(Z_2)=c$ whose slope is negative. This means that when we increase $\mathbf{P}(Z_1)$ we must decrease $\mathbf{P}(Z_2)$ and vice versa in order to move on this line. Therefore, we have

$$\frac{\Delta\mathbf{P}(Z_2)}{\Delta\mathbf{P}(Z_1)}=-\frac{c_1}{c_2}. \tag{2}$$

Thus, the optimal choice of the decision-maker is any point of the line given by $c_1\mathbf{P}(Z_1)+c_2\mathbf{P}(Z_2)=c$: he can freely move along it according to the equality (2). Now we must identify it in order to complete our choice model.

## 8. Cardinal Utility Function into our Choice Model

Coherence properties of $\mathbf{P}$ referring to marginal previsions operationally follow from economic criteria of decision-making theory. Such a theory is however presented in a simplified form. We always project every point $(\mathbf{P}(Z_1),\mathbf{P}(Z_2))$ onto the two axes of a Cartesian coordinate plane. Given $Z_1$ and $Z_2$ which are two random gains, $\mathbf{P}(Z_1)$ and $\mathbf{P}(Z_2)$, respectively represent the subjective and fair prices of $Z_1$ and $Z_2$. Each of them is the price that one is willing to pay in order to purchase the right to take part in a bet characterized by random conditions that we respectively denote by $Z_1$ and $Z_2$. According to the state of information and subjective judgments of a certain individual, fair prices of $Z_1$ and $Z_2$ coincide with the certain gains equivalent to $Z_1$ and to $Z_2$ and this happens when such an individual is not either risk-averse or risk-lover but he is risk-neutral. More generally, any price always measures a preference which must constantly manifest itself in one way or another: given $\mathbf{P}(Z_1')$ and $\mathbf{P}(Z_1'')$, with $\mathbf{P}(Z_1')\neq\mathbf{P}(Z_1'')$, we prefer $Z_1'$ to $Z_1''$ if $\mathbf{P}(Z_1')$ is higher than $\mathbf{P}(Z_1'')$ or we prefer $Z_1''$ to $Z_1'$ if $\mathbf{P}(Z_1'')$ is higher than $\mathbf{P}(Z_1')$. The same evidently goes for $\mathbf{P}(Z_2')$ and $\mathbf{P}(Z_2'')$.

Anyway, because of risk aversion, it is not true that if one is willing to purchase a random quantity $A$ at the price $\mathbf{P}(A)$ and a random quantity $B$ at the price $\mathbf{P}(B)$, one must be willing to purchase both of them together at the price $\mathbf{P}(A) + \mathbf{P}(B)$. The reason is that the purchase of one of them can affect the desirability of the other. The same obviously goes when we consider $Z_1'$ and $Z_1''$ or $Z_2'$ and $Z_2''$ instead of $A$ and $B$. Conversely, when we accept the simplifying hypothesis of additivity, we are willing to purchase $A$ and $B$ at the price $\mathbf{P}(A) + \mathbf{P}(B)$ as well as $Z_1'$ and $Z_1''$ at the price $\mathbf{P}(Z_1') + \mathbf{P}(Z_1'')$ or $Z_2'$ and $Z_2''$ at the price $\mathbf{P}(Z_2') + \mathbf{P}(Z_2'')$. It follows from such a simplification that if a certain individual is indifferent to the exchange of $A$ for $\mathbf{P}(A)$ and of $B$ for $\mathbf{P}(B)$, he is also indifferent to the exchange of $A + B$ for $\mathbf{P}(A) + \mathbf{P}(B)$; nevertheless, $P$ always expresses a subjective judgment, so the value for which such an individual is indifferent to the exchange of $A + B$ is, by definition, $\mathbf{P}(A + B)$. Therefore, according to additivity property of $\mathbf{P}$ we have $\mathbf{P}(A + B) = \mathbf{P}(A) + \mathbf{P}(B)$. Now we must extend this property to $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$, so we introduce a cardinal utility function into our choice model in order to get the conclusion of all reasoning that we made. Cardinal utility function of the decision-maker is a line with a positive slope. It is a strictly increasing linear function. In particular, the point $(0, 0)$ of a Cartesian coordinate plane is a point of such a line. When we consider all the certain gains equivalent to the random gain $Z_1$, we have $y = mx$, with $m \in \mathbb{R}^+$, while all the certain gains equivalent to the random gain $Z_2$ are recognized by means of $x = \frac{1}{m}y$. The utility function $y = mx$ is the inverse function of $x = \frac{1}{m}y$ and $x = \frac{1}{m}y$ is the inverse of $y = mx$. These two functions have the same two-dimensional diagram. The two variables $x$ and $y$ of the function $y = mx$ represent the argument and the value of it: they are respectively fair prices of $Z_1$ and $Z_2$ into our choice model. Conversely, the two variables $y$ and $x$ of the function $x = \frac{1}{m} y$ represent the argument and the value of it: they are respectively fair prices of $Z_2$ and $Z_1$ into our choice model. The optimal choice for the decision-maker is that point of a Cartesian coordinate plane where the utility function under consideration intersects a line with a negative slope which bounds the two-dimensional convex set of all coherent previsions of $(Z_1, Z_2)$ that we always project onto the two orthogonal axes under consideration. Thus, in this context, the certain gain equivalent to $Z_1$ optimally chosen by the decision-maker is a point on the horizontal axis and it coincides with the corresponding expected utility of $Z_1$, while the certain gain equivalent to $Z_2$ optimally chosen by the decision-maker is a point on the vertical axis and it coincides with the corresponding expected utility of $Z_2$. All the points of $y = mx$ coincide with the expected utility of $Z_1$ as well as all the points of $x = \frac{1}{m} y$ coincide with the expected utility of $Z_2$ into our choice model.

## 9. Conclusion

By considering two random gains, the two-dimensional convex set of all their coherent previsions has an infinite number of points constituting a subset into a Cartesian coordinate plane having an Euclidean structure. We always project all coherent previsions of two random gains onto the two orthogonal axes of a Cartesian coordinate plane because we are interested in their marginal and coherent previsions. Therefore, we divide every coherent prevision referring to two random gains into two marginal and coherent previsions referring to only one random gain of the pair of random gains under consideration. By virtue of an essential analogy that we established between properties related to coherent previsions of two random gains and the ones related to well-behaved preferences, we suppose that the points of the two-dimensional convex set can be ranked as to their desirability, so the decision-maker can establish whether one of prevision bundles is strictly better than the other or he can decide he is indifferent between two prevision bundles. Our choice model tells us which is the best choice when the decision-maker is risk-neutral. Therefore, his optimal choice is that point of a Cartesian coordinate plane where his cardinal utility function intersects a line with a negative slope which bounds the two-dimensional convex set of all coherent previsions of two random gains. This line is a hyperplane in the two-dimensional real space. Cardinal utility function of the decision-maker is a line whose slope is positive, with the point $(0, 0)$ of a Cartesian coordinate plane which is one of points of it. Such a function is a strictly increasing linear function and all points of it that we project onto the two orthogonal axes of a Cartesian coordinate plane are fair prices of the two random gains under consideration: fair prices of $Z_1$ are on the horizontal axis while fair prices of $Z_2$ are on the vertical one.

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