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A NOTE ON THE RANK OF THE 2-CLASS GROUP OF THE HILBERT 2-CLASS FIELD OF SOME REAL QUADRATIC NUMBER FIELDS

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Abstract

Let k be a real quadratic number field with 2-class group $C_2(k)$ isomorphic to $Z/2^m Z \times Z/2^n Z,\ m \geq 2,\ n \geq 2,$ such that the discriminant of k is divisible by only positive prime discriminants. Let k^1 be the Hilbert 2-class field of $k,$ and $k_1,\ k_2,\ k_3$ be the three unramified quadratic extensions of $k.$ We prove that if the 2-class number of k is equal to the 2-class number of k_i for $i = 1,\ 2,$ and 3, then either $|C_2(k^1)| = 2$ or rank $(C_2(k^1)) \geq 3.$

1. Introduction

Let k be an algebraic number field, $C_2(k)$ denote the 2-Sylow subgroup of its ideal class group $C(k),$ and k^1 denote the Hilbert 2-class field of k (in the wide sense). Let k^n (for a nonnegative integer n) be

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defined inductively as $k^0 = k$ and $k^{n+1} = (k^n)^1$. Denoting by C the containment symbol, we define $k^0 C k^1 C k^2 C \ldots k^n C \ldots$ to be the 2-class field tower of k. We say that the tower is finite if $k^n = k^{n+1}$ for some n, with length n if n is minimal, and infinite otherwise.

Currently there is no known decision procedure to determine whether or not the 2-class field tower (or the p-class field tower for any prime p) of an algebraic number field k is infinite. It is known though, by purely group theoretical means (cf. [6], [12]) that the 2-class field tower is finite of length at most 3 when the rank, i.e., the minimal number of generators, of $C_2(k^1)$ is less than or equal to 2; and that if we also have $\operatorname{rank}(C_2(k)) = 2$ (resp., 1), then the tower is finite of length at most 2 (resp., 1).

We concentrate here on the case of real quadratic number fields k such that its discriminant d_k is divisible by only positive prime discriminants. In [5], we characterized all real quadratic number fields k for which $C_2(k^1)$ is trivial (hence of rank 0). In [1, 2, 4] and [8], the authors determined in particular those real quadratic number fields k with $\operatorname{rank}(C_2(k)) = 2$ such that d_k is divisible by only positive prime discriminants, for which $C_2(k^1)$ is cyclic and nontrivial (rank 1). Using the notation $(2^m, 2^n)$ to denote $Z/2^m Z \times Z/2^n Z$, $m \geq 1$, $n \geq 1$, it is well known that $C_2(k^1)$ is cyclic when $C_2(k) \approx (2, 2)$ for any number field k, and in [3] we were able to give complete criteria to distinguish between $\operatorname{rank}(C_2(k^1)) = 2$ and $\operatorname{rank}(C_2(k^1)) \geq 3$ for a real quadratic number field k when $C_2(k) \approx (2, 2^n)$, $n \geq 2$, and d_k is divisible by only positive prime discriminants. Furthermore, in [2], we found criteria to determine when $\operatorname{rank}(C_2(k^1)) = 2$ and $\operatorname{rank}(C_2(k^1)) \geq 3$ for $C_2(k) \approx (2^m, 2^n)$, $m \geq 2, n \geq 2$, in particular cases. Denoting $h_2(k)$ (resp., $h(k)$) to represent the 2-class number (resp., class number) of a field k, we determined for these fields k that if $h_2(k_i) > h_2(k)$ for $i = 1, 2, 3$, where the k_i are the three unramified quadratic extensions of k, then rank $(C_2(k^1)) \geq 3$. We also formulated criteria to distinguish between rank $(C_2(k^1)) = 2$ and rank $(C_2(k^1)) \geq 3$ when there exists exactly one unramified quadratic extension k_i such that $h_2(k_i) = h_2(k)$ (cf. Lemma 2).

In this current note, we examine the case in which for all three unramified quadratic extensions, k_1, k_2, and k_3, we have $h_2(k_1) = h_2(k_2) = h_2(k_3) = h_2(k)$. We show in this case that either $|C_2(k^1)| = 2$ or rank $(C_2(k^1)) \geq 3$, and therefore we see that it may possible in this case that k has infinite 2-class field tower for some fields k.

We obtain our main result using properties of metabelian 2-groups of rank 2 and then applying these properties to number fields, as we have done in our previous work (cf. [1, 2, 3, 4]). We start with a brief review of the group theory that we need to utilize.

2. Properties of Some Metabelian 2-Groups of Rank 2 and Preliminary Results

Let $G = Gal(k^2/k)$ and G' denote the commutator subgroup of G. By class field theory it is well known that $G/G' \approx Gal(k^1/k) \approx C_2(k)$, and $G' \approx Gal(k^2/k^1) \approx C_2(k^1)$. We recall that a group G is said to be metabelian if its commutator group G' is abelian. From our earlier work we begin with the following lemma (cf. Proposition 1 in [4]).

Lemma 1. *Let G be a finite metabelian 2-group such that $G/G' \approx (2^m, 2^n)$, $m \geq 1$, $n \geq 1$, with two generators a and b such that $a\wedge(2^m) \equiv b\wedge(2^n) \equiv 1$ mod G' (where the symbol ^ denotes exponentiation). Let*

$H_1 =< a, b^2, G' >, H_2 =< ab, b^2, G' >,$ *and* $H_3 =< a^2, b, G' >$ *be the three maximal subgroups of G, and* $J_0 =< a^2, b^2, G' >= H_1 \cap H_2 \cap H_3$, *which is a normal subgroup of G of index* 4. *Then*

(i) $|G'| = 1$ *if and only if* $(G' : H_j') = 1$ *for some (equivalently all)* $j \in \{1, 2, 3\}$;

(ii) $|G'| = 2$ *if and only if* $(G' : J_0{}') = 2$.

Lemma 1 has the following immediate field theoretical corollary, where $h_2(k)$ (resp., $h(k)$) denotes the 2-class number (resp., class number) of an algebraic number field k (cf. Theorem 1 in [4]).

Corollary 1. *Let* k *be a number field such that* $C_2(k) \approx (2^m, 2^n)$, $m \geq 1, n \geq 1$. *Then*

(i) $h_2(k^1) = 1$ *if and only if* $h_2(k_j) = (1/2)h_2(k)$ *for any (or equivalently all) of the three unramified quadratic extensions* k_j *of* k, $j = 1, 2, 3$;

(ii) $h_2(k^1) = 2$ *if and only* $h_2(k_0) = (1/2)h_2(k)$, *where* $K_0 = k_1k_2k_3$.

In our earlier work (cf. Theorem 1 in [2]), we made further use of group theory to obtain more information about fields k as above when $C_2(k^1)$ is cyclic with $h_2(k^1) \geq 4$, and we will also make use of this group theory to obtain our current results. We let G be a metabelian 2-group such that $G/G' \approx (2^m, 2^n)$, $m \geq 2$, $n \geq 2$, where $G =< a, b >: a\wedge(2^m) \equiv b\wedge(2^n) \equiv 1 \mod G'$. In our previous work (cf. [2]; see also [7]), we calculated the commutator subgroups of the three maximal subgroups, H_1, H_2, H_3, and seven normal subgroups of index four, $J_0, J_{11}, J_{12}, J_{21}, J_{22}, J_{31}, J_{32}$, in G. We described these subgroups as follows: $H_1 = < a, b^2, G' >, H_2 =< ab, b^2, G' >, H_3 =< a^2, b, G' >, J_0 =< a^2, b^2, G' >,$

$J_{11} =< a, b^4, G' >$, $J_{12} =< ab^2, b^4, G' >$, $J_{21} =< ab, b^4, G' >$, $J_{22} =< a^3b, b^4, G' >$, $J_{31} =< a^4, b, G' >$, $J_{32} =< a^4, a^2b, G' >$. In order to calculate all the above commutator subgroups, we made use of the lower central series $\{G_k\}$ of G, defined inductively as $G_1 = G$, $G_2 = G' = [G, G_1]$, $G_{k+1} = [G, G_k]$, for all $k \geq 1$, where $[A, B] =< \{[a, b] = a^{-1}b^{-1}ab\} \mid a \in A, b \in B >$. It is well known that the lower central series terminates in finitely many steps at the identity subgroup I for all finite p-groups (i.e., groups of p-power order for any prime p are nilpotent) (cf. [10]). Furthermore, we defined $[x_1, \ldots, x_k]$ inductively on k as $[[x_1, \ldots, x_{k-1}], x_k]$ for all $x_j \in G$ and $k > 2$.

We utilized commutator and lower central series relations to calculate the above commutator subgroups. In particular, we utilized the notation $y_{1,1} = y_{11} = [a, b]$, and for $r, s \in N$ (the natural numbers), $y_{r,s} = y_{rs} = [a, b, x_1, \ldots, x_k]$ with $x_j = a$ or b, $k = r + s - 2$, and where r (resp., s) is the number of occurrences of a (resp., b) in the commutator (cf. [2]). From the commutator identities $[xy, z] = [x, z]\ [x, z, y][y, z]$ and $[x, yz] = [x, z][x, y][x, y, z]$, we know that y_{rs} is independent of the order of the $x_1, \ldots, x_k$; $G_n =< \{y_{rs} : r, s \in N, r + s = n\}, G_{n+1} >$; $G_k = 1$ for sufficiently large k (see above); and thus we have $G_2 =< \{y_{rs} : r, s \in N, r + s \leq k\} >$ (cf. [2]). We also know that the group exponent $\exp(G_3/G_4)$ divides $\exp(G_2/G_3)$, and more generally, $\exp(G_n/G_{n+1})$ divides $\exp(G_{n-1}/G_n)$ for any $n > 2$ (cf. [11], p. 266).

We utilized the above group theory to obtain a listing of all the above commutator subgroups, which enabled us to prove that if k is a real quadratic number field with discriminant $d_k = d_1d_2d_3$, where the d_j, $j = 1, 2, 3$, are positive prime discriminants, $C_2(k) \approx (2^m, 2^n)$, $m \geq 2$, $n \geq 2$, and $h_2(k_j) = h_2(k)$ for all three unramified quadratic extensions k_j of k, (which implies that $C_2(k^1)$ is not trivial by Corollary 1) then $C_2(k^1)$ is not cyclic with $h_2(k) \geq 4$ (cf. Theorem 5 in [2]). We also proved that if k is as above but with (*w*log) $h_2(k_i) = h_2(k_j) = h_2(k)$, $h_2(k_1) > h_2(k)$, then $C_2(k^1)$ is not cyclic (cf. Theorem 6 in [2]). Furthermore, we established a corollary to our group theoretical results (cf. Theorems 1, 2, 3, and Corollary 2 in [2]) to obtain the following result for the other possible two cases for the 2-class numbers of the three unramified quadratic extensions of k, assuming that $C_2(k^1)$ is nontrivial by means of Corollary 1.

Lemma 2. *Let k be a number field such that $C_2(k) \approx (2^m, 2^n)$, $m \geq 1$, $n \geq 1$, and assume that $h_2(k_i) = h_2(k)$ for exactly one of the three unramified quadratic extensions k_i of k. Then rank $(C_2(\mathrm{k}^1)) = 2$ if $h_2(k_{ij}) = h_2(k)$ and rank $(C_2(k^1)) \geq 3$ if $h_2(k_{ij}) > h_2(k)$, for either (both) unramified quadratic cyclic extensions K_{ij} of k containing k_i. Furthermore, if $h_2(k_i) > h_2(k)$ for all three k_i, then rank $(C_2(k^1)) \geq 3$.*

3. Distinguishing Between Rank $(C_2(k^1)) = 2$ and Rank $(C_2(k^1)) \geq 3$

As we see from the above discussion and Lemma 2, for k a real quadratic number field with $C_2(k) \approx (2^m, 2^n)$, $m \geq 2$, $n \geq 2$, where the discriminant d_k is divisible by only positive prime discriminants, the open questions in regard to the rank of $C_2(k^1)$ are for the cases when $h_2(k_i) = h_2(k)$ for all three unramified quadratic extensions k_i of k, and when $h_2(k_i) = h_2(k)$ for exactly two of the k_i. We now establish our main result, which is that either $|C_2(k^1)| = 2$ or rank $(C_2(k^1)) \geq 3$ when $h_2(k_i) = h_2(k)$ for all three unramified quadratic extensions k_i of k as above. From Lemma 3 of [2] and the proof of Theorem 2 in [2], we know that for k as above with $h_2(k_i) = h_2(k)$ for $i = 1, 2, 3,$ we have $(G_2 : G_3) = 2,$ and that if rank $(G_2) = 2$ then $G_2/G_4 \approx (2, 2).$ To establish our main result, we begin with the following lemma, whose proof is due to Chip Snyder, where for G' nontrivial, $c = |\{j \in \{1, 2, 3\} : (G' : H_j') = 2\}|$, and where the symbol $d(G)$ denotes the rank of a group G.

Lemma 3. *Let G be a finite metabelian 2-group of rank 2, and assume that $d(G_2) = 2$ and $c = 3$. Then $J_0' = G_4$.*

Proof. Since $(G_2 : G_3) = 2$ and $G_2/G_4 \approx (2, 2),$ *w*log $G_3 =< y_{12}, G_4 >.$ By the presentation of J_0 we have $J_0' =< {y_{11}}^4 {y_{12}}^2 {y_{21}}^2 y_{22}, \{{y_{r+1,s}}^2 y_{r+2,s}\}, \{{y_{r,s+1}}^2 y_{r,s+2}\} > CG_4$ (cf. [2]). If $G_4 = 1,$ then our result is trivially true, so we assume that $G_4 \neq 1.$ We claim that in this case $J_0' G_{m+1} = G_m$ for $m \geq 4.$ To prove our claim, notice that since $G_3 =< y_{12}, G_4 >,$ then $y_{21} \equiv 1 \mod G_4$ or $y_{21} \equiv y_{12} \mod G_4.$ Using commutator relations (cf. [2], [7]) it follows that $G_4 =< y_{13}, G_5 > CJ_n' G_5\, CG_4,$ and therefore $J_0' G_5 = G_4.$ Proceeding by induction, we obtain that $J_0' G_6 = G_5,$ and we continue in this way to establish our claim that $J_0' G_{m+1} = G_m$ for $m \geq 4.$ Since G is nilpotent, we conclude that $G_4 = J_0' G_5 = J_0' G_6 = \ldots = J_0',$ which proves our result.

We now make use of Lemma 3 to prove our main result.

Theorem 1. *Let k be a real quadratic number field with discriminant $d_k = d_1 d_2 d_3,$ where the $d_j,\ j = 1, 2, 3,$ are positive prime discriminants, $C_2(k) \approx (2^m, 2^n),\ m \geq 2,\ n \geq 2,$ and $h_2(k_j) = h_2(k)$ for all three unramified quadratic extensions k_j of k. Then either $|C_2(k^1)| = 2$ or $d(C_2(k^1)) \geq 3$. Furthermore, $d(C_2(k^1)) \geq 3$ if and only if $h_2(K_0) \geq h_2(k),$ where $K_0 = k_1 k_2 k_3$.*

Proof. From Theorem 5 of [2] we see that $h_2(K_0) \neq h_2(k)$ (notice that K_0 is the fixed field of J_0), and this implies by Corollary 1 of [2] that either $|C_2(k^1)| \leq 2$ or $C_2(k^1$ is not cyclic. Since $h_2(k_j) = h_2(k)$ for $i = 1, 2, 3,$ we know from Corollary 1 above that $C_2(k^1) \neq 1$. Furthermore, $h_2(K_0) \neq h_2(k)$ implies that $(G_2 : J_0') \neq 4$, and therefore from Lemma 3 we know that $d(G_2) = d(C_2(\mathrm{k}^1)) \neq 2$. This implies that either $|C_2(k^1)| = 2$ or $d(C_2(k^1)) \geq 3$, and from Corollary 1 we obtain that $d(C_2(k^1)) \geq 3$ if and only if $h_2(K_0) \geq h_2(k)$, which establishes our theorem.

In the case where $h_2(k_i) = h_2(k)$ for exactly two k_i, we have not been able to distinguish between $d(C_2(k^1)) = 2$ and $d(C_2(k^1)) \geq 3$ in general, but we have obtained a preliminary result in this direction (however, see Remark 1 below), which we formulate initially with the following group theoretical lemma, where the case $h_2(k_i) = h_2(k)$ for exactly two k_i is equivalent to $c = 2$ (see above), and G^{ab} denotes the abelianzation G/G' for any group G.

Lemma 4. *Let G be a finite metabelian 2-group of rank 2, and assume that $d(G_2) = 2$ and $c = 2$. Then $(G_2 : G_3) \geq 4$ and $|J_0{}^{ab}| \geq |G^{ab}|$, $|J_{uv}{}^{ab}| \geq |G^{ab}|$ for all $u = 1, 2, 3$ and $v = 1, 2$.*

Proof. We see that $(G_2 : G_3) \geq 4$ from Theorem 2 of [2]. Let $J = J_0$ or J_{uv} for $u = 1, 2, 3$ and $v = 1, 2$. Since $(G_2 : G_3) \geq 4$, from the formulations of J_0' and J_{uv}' given in [2], we have $(G_2 : J') \geq (G_2 : G_2{}^4$ $G_3) = 4$ and thus $|J^{ab}| \geq |G^{ab}|$.

Lemma 4 has the following field theoretical corollary.

Corollary 2. *Let* k *be a real quadratic number field with discriminant* $d_k = d_1 d_2 d_3$, *where the* d_j, $j = 1, 2, 3$, *are positive prime discriminants,* $C_2(k) \approx (2^m, 2^n)$, $m \geq 2$, $n \geq 2$, *and assume that* $h_2(k_i) = h_2(k)$ *for exactly two* k_i. *Furthermore, assume that* $h_2(K) = h_2(k)/2$ *for some unramified quadratic extension K of* k. *Then* $d(C_2(k^1)) \geq 3$.

Proof. From Theorem 6 of [2] we know that $C_2(k^1)$ is not cyclic. Therefore from the field theoretical equivalence of Lemma 4 we immediately conclude that $d(C_2(k^1)) \geq 3$.

Remark 1. From Corollary 1 above, Theorem 1 in [4], and the assumption in Corollary 2 that $h_2(k_i) = h_2(k)$ for exactly two k_i we know that given the assumptions of Corollary 2, $h_2(K_0) \geq h_2(k)$ and that $h_2(K_j) \geq h_2(k)$ for the two unramified quadratic extensions K_j of k that contain k_j with $h_2(k_j) > h_2(k)$. Furthermore, from a heuristic investigation (which we thank Abdelkader Zekhnini for) it appears that for fields satisfying the initial assumptions of Corollary 2 (i.e., satisfying all the assumptions of Corollary 2 with the exception that $h_2(K) = h_2(k)/2$ for some unramified quadratic extension K of k), $h_2(K_i) \geq h_2(k)$ for all unramified quadratic extensions K_i of k. However, we have not been able to prove this expected result, and we therefore formulate the following open question.

Open Question 1. Let k satisfy the initial assumptions of Corollary 2. Then does it follow that $h_2(K_i) \geq h_2(k)$ for all unramified quadratic extensions K_i of k?

Remark 2. Even if there are fields k that satisfy all the assumptions of Corollary 2, i.e., if the answer to Open Question 1 is that there exists at least one real quadratic number field k as above such that $h_2(K) = h_2(k)/2$ for some unramified quadratic extension K of k, then we still are not able to distinguish in general between $d(C_2(k^1)) = 2$ and $d(C_2(k^1)) \geq 3$ (see Example 6 below).

We conclude with the following open question.

Open Question 2. Let k satisfy the initial assumptions of Corollary 2, and assume that $h_2(k_i) \geq h_2(k)$ for all unramified quadratic extensions K_i of k. Then is it possible to distinguish between $d(C_2(k^1)) = 2$ and $d(C_2(k^1)) \geq 3$?

4. Examples

Example 1. Although Theorem 1 shows that there do not exist real quadratic number fields satisfying the given conditions with $d(C_2(k^1)) = 2$, there are examples that satisfy the group theoretical translated conditions of Theorem 1 with $d(C_2(k^1)) = 2$. To convey this we utilize Groups 128, 129, 130 of order 64 in Hall & Senior [9] that can be characterized as follows (cf. [13]).

Group 128 $=< a, b \mid a^4 = 1, b^4 = 1, {y_{11}}^2 = 1, [y_{11}, b^2] = 1, [y_{11}, ba b^{-1}] = 1 >$.

Group 129 $=< a, b \mid a^4 = 1, {y_{11}}^2 = 1, y_{12} = b^4, y_{21} = 1 >$.

Group 130 $=< a, b \mid b^4 = 1, {y_{11}}^2 = 1, y_{12} = a^4, y_{21} = 1 >$.

For all three above groups G, we obtain $G/G' \approx (4, 4)$, $G' \approx (2, 2)$, $y_{21} = 1$, $G_3 =< y_{12} >$, $J_0' = J_{11}' = J_{12}' = 1$, and $H_1^{\,2} = H_2^{\,2} = H_3^{\,2} = J_{21}' = J_{22}' = J_{31}' = J_{32}' =< y_{12} >$.

In the following examples, we have made use of calculations available through pari and Keith Mathews' number theory site (www.numbertheory.org) to obtain our 2-class numbers, and we use the notation given in Lemma 2 for our unramified quadratic cyclic extensions K_{ij} of k. For all these examples we obtain that $|C_2(k^1)| = 1$ or 2, or $d(C_2(k^1)) \geq 3$. See Example 3 in [2] for an example with $c = 1$ and $d(C_2(k^1)) = 2$.

Example 2. $k = Q(\sqrt{}(2.17.89))$, $C_2(k) \approx (4, 4)$, $h_2(k_1) = h_2(k_2) = h_2(k_3) = 8$; from Corollary 1 we see that $C_2(k^1)$ is trivial.

Example 3. $k = Q(\sqrt{}(5.61.241))$, $C_2(k) \approx (4, 4)$, $h_2(K_0) = 8$, $h_2(k_1) = h_2(k_2) = h_2(k_3) = 16$ and thus $c = 3$; from Corollary 1 we see that $|C_2(k^1)| = 2$.

Example 4. $k = Q(\sqrt{}(2.41.2833))$, $C_2(k) \approx (4, 8)$, $h_2(K_0) = 64$, $h_2(k_1) = h_2(k_2) = h_2(k_3) = 32$ and thus $c = 3$; from Corollary 1 and Theorem 1 we see that $d(C_2(k^1)) \geq 3$.

Example 5. $k = Q(\sqrt{}(5.401.541))$, $C_2(k) \approx (4, 8)$, $h_2(K_0) = 128$; let $d_1 = 5$, $d_2 = 401$, $d_3 = 541$; then $h_2(k_1) = h_2(k_3) = 32$, $h_2(k_2) = 64$, and thus $c = 2$, $h_2(K_{11}) = h_2(K_{12}) = h_2(K_{31}) = h_2(K_{32}) = 32$, $h_2(K_{21}) = 128$, $h_2(K_{22}) = 512$; from Corollary 1 of [2] we know that $C_2(k^1)$ is not cyclic. From pari we also obtain that $C_2(K_{21}, K_{22}) \approx (4, 8, 8, 8)$ and K_{21}, K_{22} is an unramified degree 8 extension of k. We therefore are able to conclude for this example that $d(C_2(k^1)) \geq 3$.

Example 6. $k = Q(\sqrt{(5.89.1709)})$, $C_2(k) \approx (4, 8)$, $h_2(K_0) = 64$; let $d_1 = 5$, $d_2 = 89$, $d_3 = 1709$; then $h_2(k_2) = h_2(k_3) = 32$, $h_2(k_3) = 64$, and thus $c = 2$, $h_2(K_{21}) = h_2(K_{22}) = h_2(K_{31}) = h_2(K_{32}) = 32$, $h_2(K_{11}) = 64$, $h_2(K_{12}) = 128$. From Theorem 6 in [2], we see that $C_2(k^1)$ is not cyclic, but in this example our methods do not enable us to determine if $d(C_2(k^1)) = 2$ or $d(C_2(k^1)) \geq 3$.

Remark 3. We notice that in Examples 5 and 6, since $h_2(K_0) \geq h_2(k)$ and $h_2(K_{ij}) \geq h_2(k)$ for all i, j, that this is consistent with our conjecture in Remark 1 (see Open Question 1). We also note that in Example 4 above, and in Example 4 in [2] ($k = Q(\sqrt{(17.53.661)})$), we have this same consistency with our conjecture in Remark 1.

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