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ANOTHER LOOK AT TOPOLOGICAL SUBSPACE PROPERTIES AND EXAMPLES

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Abstract

Within this paper the needed correction for subspace properties is revisited and additional subspace and not subspace properties are given.

1. Introduction and Preliminaries

Subspace properties are deeply rooted in the study of mathematics and have been greatly studied in topology [5].

Definition 1.1. Let P be a topological property. Then P is a subspace property iff a space has property P iff every subspace of the space has property P [5].

However, in the 2016 paper [1], a never before imagined topological property was discovered that created a disconnect in the study of subspace properties. Prior to 2016 paper [1], the existence of the least of all topological properties had not even been imagined and thus not

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consider when defining and investigating subspace properties. In a 2015 paper [2], the need and use of "not-T_0" revealed "not-T_0" as a strong, useful property motivating the addition of "not-T_0" and all other "not-P" properties, where P is a topological property for which "not-P" exists, as important, useful properties in the study of topology, opening a never before imagined fertile topological territory, including the existence of the least topological property.

Theorem 1.1. $L = (T_0$ *or* "*not*-T_0"$)$ *is the least topological property* [1].

Also, L *can be given by* $(P$ *or* "*not*-P"$)$, *where* P *is a topological property for which* "*not*-P" *exists* [1].

Since every topological space has property L [3], then every space and all its subspaces simultaneously share property L and, by the initial definition, L is a subspace property. Compactness is a well-known not subspace property, but a compact space would serve as an example of the subspace property L, creating a disconnect between the intended purpose of subspace properties and the now known reality. However, since the least topological property L had not been considered in the earlier studies of subspace properties, a simple, meaningful solution to ending the disconnect was the exclusion of L from the subspace properties.

Definition 1.2. A topological property P is a subspace property iff $P \neq L$ and a space satisfies property P iff all of its subspaces satisfies P [4].

Within classical topology, there are few examples of a topological property P and its negative "not-P", including "not-P", where P is a subspace property. With the addition of "not-P" into the body of topology, where P is a subspace property, questions naturally arose about a meaningful definition and the resulting properties of "not-P" for a subspace property P. Within the 2016 paper [4], the following definition was given.

Definition 1.3. Let P be a subspace property. Then a space is "not-P" iff each subspace of the space has property P or "not-P" and there exists a subspace with property "not-P" [4].

Within the paper [4], the definitions given above led to "interesting" properties, but, on second thought, appears to be stronger than is required by the definition of a subspace property given in Definition 1.2. Given below is what appears to be weaker than the definition above that perhaps better fits Definition 1.2.

Definition 1.4. Let P be a subspace property. Then a space is "not-P" iff there exists a subspace with property "not-P".

Within this paper, the relationship between the two definitions of "not-P", where P is a subspace property, is investigated and additional properties and examples of subspaces and not subspace properties are given.

2. The Relationship, Properties, and Examples

Unless otherwise stated, Definition 1.4 will be used for "not-P", where P is a subspace property. With the given definition of "not-P", where P is a subspace property, care must be taken to ensure that for each subspace property P, "not-P" exists. The following result given in a 2016 paper [1] is used below to resolve the question.

Theorem 2.1. *Let P be a topological property. Then each of the following are equivalent*: (a) *"not-P" the negation of P, exists*, (b) *"not-P" is a topological property, P is stronger than L, and $P \neq$ "not-P"*, (c) *$P \neq L$ and $P \neq$ "not-P"*, (d) *P is stronger than L, and* (e) *"not-P" is stronger than L* [1].

Theorem 2.2. *Let P be a subspace property. Then "not-P" exists.*

Proof. By Theorem 2.1, L is the only topological property P for which "not-P" does not exist. Since each subspace property P is not L and L is the least topological property, then P is stronger than L, and "not-P" exists.

Also, for the definition of "not-P", where P is a subspace property, to be creditable, "not-("not-P")" would have to be P.

Theorem 2.3. *Let (X, T) be a space. Then (X, T) is "not-P", where P is a subspace property, iff (X, T) is "not-P", where P is a topological property.*

Proof. Suppose (X, T) is "not-P", where P is a subspace property. Then (X, T) has a subspace that is "not-P". Suppose (X, T) is "(not-"not-P")", where P is a topological property. Then (X, T) has property "not-("not-P")" = P and since P is a subspace property, every subspace of (X, T) has property P, which is a contradiction. Thus (X, T) has property "not-P", where P is a topological property.

If (X, T) is "not-P", then (X, T) is a subspace of itself with property "not-P" and (X, T) has property "not-P", where P is a subspace property.

Theorem 2.4. *Definition 1.3 of "not-P" is equivalent to Definition 1.4 for "not-P", where P is a subspace property.*

Proof. Clearly Definition 1.3 implies Definition 1.4.

Suppose a space (X, T) has property "not-P" as defined in Definition 1.4. Then (X, T) has a "not-P" subspace. Since singleton set spaces satisfy all subspace properties [4], then singleton set subspaces of (X, T) have property P. Thus (X, T) has subspaces that are both "not-P" and P. If (Y, T_Y) is a subspace of (X, T) that is "not-P", then (Y, T_Y) is "not-("not-P")" = P. Hence, (X, T) satisfies Definition 1.3.

Theorem 2.5. *Let P be a subspace property. Then "not-P" is not a singleton set property.*

Proof. Suppose "not-P" is a singleton set property. Let (X, T) be a space with property "not-P". Then singleton sets subspaces of (X, T) are both P and "not-P", which is a contradiction.

Within the 2016 paper [4], it was established that for subspace properties P and Q, (P and Q) is a subspace property, "not-P" is not a subspace property, and, if (P and "not-Q") exists, (P and "not-Q") is not a subspace property. In past studies of classic topology, topological properties that are subspace properties and properties that are not subspace properties were studied separately, but now, using the result above, subspace properties and not subspace properties can be studied simultaneously with many new, never before imagined subspace and not subspace properties quickly and easily added to the study.

Also, in classical topology, except for topological properties P and Q where (P or Q) was known to exist or (P and Q) was known to exist, is there inclusion of (P or Q) or (P and Q). By the work above, for subspace properties P and Q, (P and Q) is a subspace property. Thus, a natural question to pose is whether for subspace properties P and Q, is (P or Q) a subspace property? Below this question is addressed.

3. Resolution of the Question Concerning (*P* or *Q*)

Theorem 3.1. *Let P and Q be subspace properties. Then (P or Q) is a subspace property.*

Proof. Within the paper [3], it was shown that for topological properties P and Q, (P or Q) is a topological property. Thus (P or Q) is a topological property. Let (X, T) be a space with property (P or Q). Then (X, T) has property P or (X, T) has property Q. If (X, T) has property P, then every subspace of (X, T) has property P and if (X, T) has property Q, then every subspace of (X, T) has property Q. Hence, every subspace of (X, T) has property (P or Q).

Conversely, suppose every subspace of (X, T) has property (P or Q). Then every subspace of (X, T) has property P or every subspace of (X, T) has property Q. If every subspace of (X, T) has property P, then, since (X, T) is a subspace of itself, (X, T) has property P and if every subspace of (X, T) has property Q, then, as earlier, (X, T) has property Q. Thus (X, T) has property (P or Q).

Mathematical induction can be used to extend the results above for (P or Q) or (P and Q) to finitely many subspace properties. Thus many more subspace properties can be added to the study of subspace properties.

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