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A CONSEQUENCE OF ESSENTIAL SPECTRUM OF AREA ZERO OF ESSENTIALLY $\mathcal{A}_{1,1}$ OPERATORS

To the memory of my grandmother, Maria

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Abstract

We define the class that essentially $\mathcal{A}_{1,1}$ operators and prove that for such operators with zero area of the essential spectrum are essentially normal.

1. Introduction

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all linear bounded operators on $\mathcal{H}$, and let $\mathbb{K}$ denote the two-sided ideal of all compact operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *essentially normal* if $D_T := T^*T - TT^*$ is a compact operator, and T is called α-*hyponormal* (*essentially* α-*hyponormal*) if $D_T^{\alpha} := (T^*T)^{\alpha} - (TT^*)^{\alpha}$ is a positive definite operator

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(D_T^{α} is the sum of a positive definite operator and a compact operator, or equivalently, $\pi(T)$ is an α-hyponormal operator in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathbb{K}$, where $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbb{K}$ is the canonical map), respectively. An operator $T \in \mathcal{L}(\mathcal{H})$ is of class $\mathcal{A}_{s,t}$ (notation $T \in \mathcal{A}_{s,t}(\mathcal{H})$) if $C_{s,t}^T := |T_{s,t}|^{\frac{t}{s+t}} - |T|^{2t}$ is a positive semidefinite operator and *essentially* $\mathcal{A}_{s,t}$ operator (notation $T \in \mathcal{A}_{1,1}^e(\mathcal{H})$) if $C_{s,t}^T$ is the sum of a positive semidefinite operator and a compact operator, where $T_{s,t} = |T|^s U |T|^t$, $s, t > 0$, is the generalized Aluthge transform. Furthermore, for T in $\mathcal{L}(\mathcal{H})$, let $\sigma(T)$ and $\sigma_e(T)$ denote the spectrum and the essential spectrum of T, $\|T\|_e$ the norm of $\pi(T)$ in the Calkin algebra, and $\sigma_w(T)$ the Weyl spectrum of T, that is the union of $\sigma_e(T)$ and the bounded components of $\mathbb{C}\backslash\sigma_e(T)$ that are associated with nonzero Fredholm index.

It is well-known that for a fixed $\alpha > 0$, α-hyponormal operators are class $\mathcal{A}_{s,t}$ operators for all $s, t > 0$ and for $\alpha \leq 1$, Putnam's inequality holds, that is

$$\|D_T^{\alpha}\| \leq \frac{1}{\pi} \mu_{2\alpha}(\sigma(T)),$$

where $\mu_{\beta}(\sigma) = \frac{\beta}{2} \int\int_{\sigma} \rho^{\beta-1} d\rho d\theta$. In particular, for $\alpha = 1$, one obtains the classical Putnam's inequality [5] that involves $\mu_2(\sigma(T))$, that is the area measure of the spectrum of T, and which holds the nice consequence that a hyponormal operator with spectrum or area zero must be a normal operator.

In [2], the following extension of Putnam's inequality was proved.

Theorem A ([2], Theorem 3.1)**.** *If* $T \in \mathcal{A}_{s,t}(\mathcal{H})$ *and* $s, t \leq 1$, *then*

$$\| |T_{s,t}|^{\frac{2\min\{s,t\}}{s+t}} - |T|^{2\min\{s,t\}} \| \leq \| |T_{s,t}|^{\frac{2\min\{s,t\}}{s+t}} - |T_{s,t}^*|^{\frac{2\min\{s,t\}}{s+t}} \|$$

$$\leq \frac{\min\{s,t\}}{\pi} \iint_{\sigma(T)} r^{2\min\{s,t\}-1} dr d\theta.$$

Since $|T_{1,1}| = |T^2|$, and if $T \in \mathcal{A}_{1,1}$, then

$$\| |T^2| - |T|^2 \| \leq \| |T_{1,1}| - |T_{1,1}^*| \| \leq \frac{1}{\pi} \mu_2(\sigma(T)).$$

According to ([3], Corollary 2.2), if $T \in \mathcal{A}_{s,t}(\mathcal{H})$ and $T_{s,t}$ is a normal operator, then T is also a normal operator. Thus, a remarkable consequence of the Theorem A results.

Corollary B. *If* $T \in \mathcal{A}_{1,1}(\mathcal{H})$ *and* $\mu_2(\sigma(T)) = 0$, *then the operator* T *must be normal.*

It is the purpose of this note to obtain a further consequence concerning essentially $\mathcal{A}_{1,1}$ operators.

Theorem. *If* $T \in \mathcal{A}_{1,1}^e(\mathcal{H})$ *and* $\mu_2(\sigma_e(T)) = 0$, *then the operator* T *is essentially normal.*

Proof. Let $T \in \mathcal{A}_{1,1}^e(\mathcal{H})$ and let $\mathcal{A}_T$ be the unital C^*-algebra generated by $\pi(T)$ in the Calkin algebra. Let $\rho : \mathcal{A}_T \to \mathcal{L}(\mathcal{K})$ be a faithful representation and denote $\rho(\pi(T))$ by Q_T. Since $\mathcal{A}_T$ is a unital algebra, $\sigma(Q_T) = \sigma_e(T)$ (according to [4], Proposition 1.3) and since ρ is an isometric *-isomorphism, $C_{1,1}^{Q_T} = \rho(C_{1,1}^{\pi(T)})$, $Q_T \in \mathcal{A}_{1,1}(\mathcal{K})$. According to Corollary B, the operator Q_T is normal and which concludes the proof.

Since almost α-hyponormal operators (that is operators for which D_T^{α} can be written as the sum of a positive semidefinite operator and a trace class operator) with $\alpha \leq 1$ and whose Weyl spectrum has area zero must be almost normal, that is its selfcommutator is trace class (see [1]), the following is a natural question to ask. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *almost* $\mathcal{A}_{1,1}$ operator (notation $T \in \mathcal{A}_{1,1}^{al}(\mathcal{H})$) if $|T^2| - |T|^2$ is the sum of a positive semidefinite operator and a trace class operator.

Question. If $T \in \mathcal{A}_{1,1}^{al}(\mathcal{H})$ and $\mu_2(\sigma_w(T)) = 0$, then is $|T^2| - |T|^2$ a trace class operator? If so, is the trace zero?

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