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ON THE SEMIGROUP OF ORDER-DECREASING PARTIAL ISOMETRIES OF A FINITE CHAIN

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Abstract

Let $\mathcal{I}_n$ be the symmetric inverse semigroup on $X_n = \{1, 2, \ldots, n\}$ and let $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ be its subsemigroups of order-decreasing partial isometries and of order-preserving and order-decreasing partial isometries of X_n, respectively. In this paper, we investigate the cycle structure of an order-decreasing partial isometry and characterize the starred Green's relations on $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$. We show that $\mathcal{DDP}_n$ is an ample semigroup and $\mathcal{ODDP}_n$ is a 0-E-unitary ample semigroup. We also investigate the ranks of $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$.

1. Introduction and Preliminaries

Let $X_n = \{1, 2, \ldots, n\}$ and $\mathcal{I}_n$ be the *symmetric inverse semigroup* (consisting of all partial one-to-one transformations of X_n) under composition of mappings. The symmetric inverse semigroup $\mathcal{I}_n$ is indeed an inverse semigroup (that is, for all $\alpha \in \mathcal{I}_n$ there exists a unique $\alpha' \in \mathcal{I}_n$ such that $\alpha = \alpha\alpha'\alpha$ and $\alpha' = \alpha'\alpha\alpha'$). Inverse semigroups (see [14], Chapter V and [15]) are of interest not only as a naturally occurring special case of semigroups but also for their role in describing partial symmetries. Mathematically this property is expressed by the Vagner-Preston Theorem ([14], Theorem V.I.10), by which every (finite) inverse semigroup is embedded in an appropriate (finite) symmetric inverse semigroup $\mathcal{I}_n$. Every finite inverse semigroup S is embeddable in $\mathcal{I}_n$, the analogue of Cayley's theorem for finite groups. Thus, just as the study of symmetric, alternating and dihedral groups has made a significant contribution to group theory, so has the study of various subsemigroups of $\mathcal{I}_n$, see, for example, [3, 7, 10, 11, 17, 18, 19].

A transformation $\alpha \in \mathcal{I}_n$ is said to be a *partial isometry* if (for all $x, y \in \operatorname{Dom}\alpha$) $|x - y| = |x\alpha - y\alpha|$; *order-preserving* (*order-reversing*) if (for all $x, y \in \operatorname{Dom}\alpha$) $x \leq y \Rightarrow x\alpha \leq y\alpha (x\alpha \geq y\alpha)$; and, *order-decreasing* (*order-increasing*) if (for all $x \in \operatorname{Dom}\alpha$) $x\alpha \leq x (x\alpha \geq x)$. Semigroups of partial isometries on more restrictive but richer mathematical structures have been studied [4, 20]. Al-Kharousi et al. [1] initiated a general study of the semigroup of partial isometries of X_n, $\mathcal{DP}_n$ and its subsemigroup of order-preserving partial isometries $\mathcal{ODP}_n$. Earlier, one of the authors studied the semigroup of partial one-to-one order-decreasing (order-increasing) transformations of X_n, $\mathcal{I}_n^-$ [18] ($\mathcal{I}^+(X)$ [19]), respectively. Analogous to Al-Kharousi et al. [1] this paper investigates the algebraic and rank properties of $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$, the semigroups of order-decreasing partial isometries and of order-preserving and order-decreasing partial isometries of X_n, respectively. Thus, this paper has more or less the same format as Al-Kharousi et al. [1].

In this section, we introduce basic terminology and preliminary results concerning the cycle structure of a partial order-decreasing isometry of X_n. In the next section (Section 2), we characterize the classical Green's relations and their starred analogues, where we show in contrast to the oversemigroups $\mathcal{DP}_n$ and $\mathcal{ODP}_n$ that $\mathcal{ODDP}_n$ and $\mathcal{DDP}_n$ are non-regular. However, we show that $\mathcal{DDP}_n$ is a (non-regular) ample semigroup and $\mathcal{ODDP}_n$ is a (non-regular) 0-E-unitary ample semigroup. Ample semigroups (formerly known as *type A*) were introduced by Fountain [8] as certain generalizations of inverse semigroups.

For standard concepts in semigroup and symmetric inverse semigroup theory, see, for example, [14, 17, 15]. In particular, $E(S)$ denotes the set of idempotents of S and id_A denotes the partial identity of the set A. Let

$$\mathcal{DDP}_n = \{\alpha \in \mathcal{DP}_n : (\forall x \in \operatorname{Dom} \alpha) x\alpha \leq x\}. \tag{1}$$

be the subsemigroup of $\mathcal{I}_n$ consisting of all order-decreasing partial isometries of X_n. Also let

$$\mathcal{ODDP}_n = \{\alpha \in \mathcal{DDP}_n : (\forall x, y \in \operatorname{Dom} \alpha) x \leq y \Rightarrow x\alpha \leq y\alpha\} \tag{2}$$

be the subsemigroup of $\mathcal{DDP}_n$ consisting of all order-preserving and order-decreasing partial isometries of X_n. Then we have the following result.

Lemma 1.1. *$\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ are subsemigroups of $\mathcal{I}_n$.*

Remark 1.2. $\mathcal{DDP}_n = \mathcal{DP}_n \cap \mathcal{I}_n^-$ and $\mathcal{ODDP}_n = \mathcal{ODP}_n \cap \mathcal{I}_n^-$, where $\mathcal{I}_n^-$ is the semigroup of partial one-to-one order-decreasing transformations of X_n [18].

As remarked in [1], it is straightforward that a partial isometry of X_n is determined by its domain and the image of two different points of X_n. Equivalently, it is determined by its domain, the image of a single point and by the knowledge of whether it is order-preserving or order-reversing. As a consequence, elements of any subset of $\mathcal{DP}_n$ fall into two categories: *translations* and *reflections* (about a point or midpoint of two points of X_n). It is also clear that nonidempotent translations do not have fixed points and reflections can have at most one fixed point. Now, let α be an arbitrary element in $\mathcal{I}_n$. The *height* or *rank* of α is $h(\alpha) = |\mathrm{Im}\,\alpha|$, the *right* [*left*] *waist* of α is $w^+(\alpha) = \max(\mathrm{Im}\,\alpha)$, [$w^-(\alpha) = \min(\mathrm{Im}\,\alpha)$], the *right* [*left*] *shoulder* of α is $\varpi^+(\alpha) = \max(\mathrm{Dom}\,\alpha)$ [$\varpi(\alpha) = \min(\mathrm{Dom}\,\alpha)$], and *fix* of α is denoted by $f(\alpha)$, and defined by $f(\alpha) = |F(\alpha)|$, where

$$F(\alpha) = \{x \in X_n : x\alpha = x\}.$$

Next we quote some parts of ([1], Lemma 1.2) that will be needed as well as state an additional observation that will help us understand more the cycle structure of order-decreasing partial isometries.

Lemma 1.3. *Let $\alpha \in \mathcal{DP}_n$. Then we have the following*:

(a) *The map α is either order-preserving or order-reversing. Equivalently, α is either a translation or a reflection.*

(b) *If α is order-preserving and $f(\alpha) \geq 1$, then α is a partial identity.*

(c) *If α is order-decreasing and $F(\alpha) = \{i\}$, then* $\mathrm{Dom}\,\alpha \subseteq \{i, i+1, \ldots, n\}$.

Lemma 1.4 ([18], Lemma 2.1)**.** *Let* α *and* β *be elements in* $\mathcal{I}_n^-$. *Then* $F(\alpha\beta) = F(\alpha) \cap F(\beta)$.

2. Green's Relations and their Starred Analogues

For the definitions of Green's relations, we refer the reader to Howie ([14], Chapter 2). First we have

Theorem 2.1. *Let* $\mathcal{DDP}_n$ *and* $\mathcal{ODDP}_n$ *be as defined in* (1) *and* (2), *respectively. Then* $\mathcal{DDP}_n$ *and* $\mathcal{ODDP}_n$ *are* $\mathcal{J}$*-trivial.*

Proof. It follows from ([18], Lemma 2.2) and Remark 1.2. □

Now since $\mathcal{ODDP}_n$, $(n > 1)$ contains some nonidempotent elements:

$$\begin{pmatrix} x \\ y \end{pmatrix} (x > y \geq 1),$$

it follows immediately that

Corollary 2.2. *For* $n > 1$, $\mathcal{DDP}_n$ *and* $\mathcal{ODDP}_n$ *are non-regular semigroups.*

On the semigroup S the relation $\mathcal{L}^*(\mathcal{R}^*)$ is defined by the rule that $(a, b) \in \mathcal{L}^*(\mathcal{R}^*)$ if and only if the elements a, b are related by the Green's relation $\mathcal{L}(\mathcal{R})$ in some oversemigroup of S. The join of the equivalences $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. For the definition of the starred analogue of the Green's relation $\mathcal{J}$, see [9] or [18]. Note that on any regular (inverse) semigroup the starred relations coincide with the (corresponding) usual Green's relations [9].

A semigroup S in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent is called *abundant* [9].

By ([6], Lemma 1.6) and ([14], Proposition 2.4.2 & Ex. 5.11.2), we deduce the following lemma:

Lemma 2.3. *Let* $\alpha, \beta \in \mathcal{DDP}_n$ *or* $\mathcal{ODDP}_n$. *Then*

(1) $\alpha \leq_{\mathcal{R}^*} \beta$ *if and only if* $\operatorname{Dom} \alpha \subseteq \operatorname{Dom} \beta$;

(2) $\alpha \leq_{\mathcal{L}^*} \beta$ *if and only if* $\operatorname{Im} \alpha \subseteq \operatorname{Im} \beta$;

(3) $\alpha \leq_{\mathcal{H}^*} \beta$ *if and only if* $\operatorname{Dom} \alpha \subseteq \operatorname{Dom} \beta$ *and* $\operatorname{Im} \alpha \subseteq \operatorname{Im} \beta$.

Proof. It is enough to observe that $\mathcal{ODDP}_n$ and $\mathcal{DDP}_n$ are full subsemigroups of $\mathcal{I}_n$ in the sense that $E(\mathcal{ODDP}_n) = E(\mathcal{DDP}_n) = E(\mathcal{I}_n)$.

□

As in Al-Kharousi et al. [1] to characterize the analogue of Green's relation $\mathcal{D}$, first we define the *gap* and *reverse gap of the image set of* α *as ordered* $(p-1)$*-tuples* as follows:

$$g(\operatorname{Im} \alpha) = (|a_2\alpha - a_1\alpha|, |a_3\alpha - a_2\alpha|, \ldots, |a_p\alpha - a_{p-1}\alpha|),$$

and

$$g^R(\operatorname{Im} \alpha) = (|a_p\alpha - a_{p-1}\alpha|), \ldots, |a_3\alpha - a_2\alpha|, |a_2\alpha - a_1\alpha|),$$

where $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1\alpha & a_2\alpha & \cdots & a_p\alpha \end{pmatrix}$ with $1 \leq a_1 < a_2 < \cdots < a_p \leq n$.

For example, if

$$\alpha = \begin{pmatrix} 1 & 2 & 4 & 7 & 8 \\ 3 & 4 & 6 & 9 & 10 \end{pmatrix}, \beta = \begin{pmatrix} 2 & 4 & 7 & 8 \\ 10 & 8 & 5 & 4 \end{pmatrix} \in \mathcal{DP}_{10},$$

then $g(\operatorname{Im} \alpha) = (1, 2, 3, 1)$, $g(\operatorname{Im} \beta) = (2, 3, 1)$, $g^R(\operatorname{Im} \alpha) = (1, 3, 2, 1)$, and $g^R(\operatorname{Im} \beta) = (1, 3, 2)$. From [1], we have

Lemma 2.4 ([1], Lemma 2.3)**.** *Let* $\alpha, \beta \in \mathcal{DP}_n$. *Then* $g(\operatorname{Im} \alpha) = g(\operatorname{Im} \beta)$ *or* $g(\operatorname{Im} \alpha) = g^R(\operatorname{Im} \beta)$ *if and only if there is an isometry between* $\operatorname{Im} \alpha$ *and* $\operatorname{Im} \beta$.

However, in $\mathcal{DDP}_n$ things are not as smooth as in $\mathcal{DP}_n$. The next theorem gives a characterization of $\mathcal{D}^*$ in $\mathcal{DDP}_n$.

Theorem 2.5. *Let* $S = \mathcal{DDP}_n$. *Suppose*

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1\alpha & a_2\alpha & \cdots & a_p\alpha \end{pmatrix} \text{ and } \beta = \begin{pmatrix} b_1 & b_2 & \cdots & b_p \\ b_1\alpha & b_2\alpha & \cdots & b_p\alpha \end{pmatrix}$$

are elements in $\mathcal{DDP}_n$ *with* $1 \le a_1 < a_2 < \cdots < a_p \le n$ *and* $1 \le b_1 < b_2 < \cdots < b_p \le n$. *Then* $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if there exists an order-preserving isometry* $\theta : \operatorname{Dom}\alpha \to \operatorname{Dom}\beta$; *or there exists an order-reversing isometry* $\theta' : \operatorname{Im}\alpha \to \operatorname{Im}\beta$ *and* $(n-1)/2 \ge a_p - a_1 = b_p - b_1$ *for* $n \ge 7$. *Moreover, for* $2 \le n \le 6$ *we have* $\mathcal{D}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$, *and for* $n \ge 7$ *we have* $\mathcal{D}^* = (\mathcal{R}^* \circ \mathcal{L}^*)^2 = (\mathcal{L}^* \circ \mathcal{R}^*)^2$.

Proof. Case 1. Define $\theta : \operatorname{Dom}\alpha \to \operatorname{Dom}\beta$ by $(a_i)\theta = b_i$. Then θ is an order-preserving isometry, and we may without loss of generality suppose $a_1 \ge b_1$. Then by isometry and order-preserving properties we must have $a_i \ge b_i$ for all $i \in \{1, 2, \ldots, p\}$. It is now not difficult to see that $\theta\beta \in \mathcal{DDP}_n$ and $\alpha\mathcal{R}^*\theta\beta\mathcal{L}^*\beta$ whence $(\alpha, \beta) \in \mathcal{D}^*$. Notice that $id_{\{1\}}\mathcal{L}\binom{2}{1}\mathcal{R}id_{\{2\}}$ but there is no $\alpha \in \mathcal{DDP}_n$ such that $id_{\{1\}}\mathcal{R}\alpha\mathcal{L}\, id_{\{2\}}$. Thus, $\mathcal{R}^* \circ \mathcal{L}^* \ne \mathcal{L}^* \circ \mathcal{R}^*$.

Case 2. Define $\theta : \operatorname{Dom}\alpha \to \operatorname{Dom}\beta$ by $(a_{p-i+1})\theta' = b_i$. Then θ' is an order-reversing isometry, and suppose also that $(n-1)/2 \ge a_p - a_1$. Further, for $i \in \{1, 2, \ldots, p\}$ define the following maps:

$$\alpha' = \begin{pmatrix} n - a_p + a_i \\ a_i\alpha \end{pmatrix}, \delta = \begin{pmatrix} n - a_p + a_i \\ a_p - a_i + 1 \end{pmatrix}, \text{ and } \beta' = \begin{pmatrix} b_i \\ a_p - a_{p-i+1} + 1 \end{pmatrix}.$$

(i) Since $n - a_p + a_i \geq a_i \geq a_i\alpha$, it follows that α' is order-decreasing. Moreover, $|(n - a_p + a_i)\alpha' - (n - a_p + a_j)\alpha'| = |a_i\alpha - a_j\alpha| = |a_i - a_j| = |(n - a_p + a_i) - (n - a_p + a_j)|$, and so α' is an isometry.

(ii) First observe that $b_p - b_1 = a_p - a_1$, and since $n - a_p + a_1 = n - (a_p - a_1) = n - (b_p - b_1) = n - b_p + b_1 \geq a_p - a_1 + 1 \Rightarrow n - (a_p - a_i) \geq a_p - a_i + 1$ (for all i), it follows that δ is order-decreasing. Moreover, $|(n - a_p + a_i)\delta - (n - a_p + a_j)\delta| = |(a_p - a_i + 1) - (a_p - a_j + 1)| = |a_i - a_j| = |(n - a_p + a_i) - (n - a_p + a_j)|$, and so δ is an isometry.

(iii) Since $b_1 \geq 1 = a_p - a_{p-1+1} + 1 \Rightarrow b_i \geq a_p - a_{p-i+1} + 1$ (for all i), it follows that β' is order-decreasing. Moreover, $|b_i\beta' - b_j\beta'| = |(a_p - a_{p-i+1} + 1) - (a_p - a_{p-j+1} + 1)| = |a_{p-i+1} - a_{p-j+1}| = |(b_i)(\theta')^{-1} - (b_j)(\theta')^{-1}| = |b_i - b_j|$, and so β' is an isometry.

Now observe that $\alpha\mathcal{L}^*\alpha'\mathcal{R}^*\delta\mathcal{L}^*\beta'\mathcal{R}^*\beta$ by Lemma 2.3, and so $\alpha\mathcal{D}^*\beta$.

Similarly, we can show that $\alpha\mathcal{R}^*\alpha''\mathcal{L}^*\delta'\mathcal{R}^*\beta''\mathcal{L}^*\beta$ for some α'', δ', and β'' in $\mathcal{DDP}_n$.

Finally, to show that $\mathcal{D}^* = (\mathcal{R}^* \circ \mathcal{L}^*)^2 = (\mathcal{L}^* \circ \mathcal{R}^*)^2$, it is enough to show that $\mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* \neq \mathcal{D}^* \neq \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*$. To see this, for $n \geq 7$, let

$$\alpha = \begin{pmatrix} n-3 & n-2 & n \\ n-3 & n-2 & n \end{pmatrix} \text{ and } \beta = \begin{pmatrix} n-3 & n-1 & n \\ n-3 & n-1 & n \end{pmatrix}.$$

Now if $\alpha\mathcal{L}^*\alpha'\mathcal{R}^*\beta'\mathcal{L}^*\beta$, then

$$\operatorname{Im}\alpha = \operatorname{Im}\alpha' = \{n-3, n-2, n\}, \operatorname{Im}\beta' = \operatorname{Im}\beta = \{n-3, n-1, n\},$$

and

$$\operatorname{Dom}\alpha' = \operatorname{Dom}\beta' \subseteq \{n-3, n-2, n-1, n\}.$$

It is not difficult to see that it is impossible to have a domain that will admit the two possible image sets (for α' and β'). Thus, $(\alpha, \beta) \notin \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$.

Similarly, we can show that $(\alpha, \beta) \notin \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*$. □

An immediate consequence of Lemma 2.4 and Theorem 2.5 is that for $\alpha, \beta \in \mathcal{DDP}_n$, we have $(\alpha, \beta) \in \mathcal{D}^*$ if and only if

$$g(\operatorname{Im}\alpha) = \begin{cases} g(\operatorname{Im}\beta); \text{ or} \\ g^R(\operatorname{Im}\beta), \; a_p - a_1 \leq (n-1)/2. \end{cases} \tag{3}$$

The corresponding result for $\mathcal{ODDP}_n$ can be proved similarly, and is in fact easier.

Theorem 2.6. *Let* $S = \mathcal{ODDP}_n$. *Then* $\alpha \leq_{\mathcal{D}^*} \beta$ *if and only if there exists an order-preserving isometry* $\theta : \operatorname{Im}\alpha \to \operatorname{Im}\beta$. *Moreover,* $\mathcal{D}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$.

Similarly, an immediate consequence of Lemma 2.4 and Theorem 2.6 is that for $\alpha, \beta \in \mathcal{DDP}_n$

$$(\alpha, \beta) \in \mathcal{D}^* \text{ if and only if } g(\operatorname{Im}\alpha) = g(\operatorname{Im}\beta). \tag{4}$$

An abundant semigroup S in which $E(S)$ is a semilattice is called *adequate* [8]. Of course inverse semigroups are adequate since in this case $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$. As in [8], for an element a of an adequate semigroup S, the (unique) idempotent in the $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) containing a will be denoted by $a^*(a^+)$. An adequate semigroup S is said to be *ample* if $ea = a(ea)^*$ and $ae = (ae)^+ a$ for all elements a in S and all idempotents e in S. Ample semigroups were known as *type A* semigroups [13].

Theorem 2.7. *$\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ are non-regular ample semigroups.*

Proof. The proofs are similar to that of ([18], Theorem 2.6). □

Let $E' = E \setminus \{0\}$. A semigroup S is said to be 0-E-unitary if $(\forall e \in E')(\forall s \in S) es \in E' \Rightarrow s \in E$. That is, a *full* subsemigroup of $\mathcal{I}_n$ is 0-E-unitary if and only if only idempotents have fixed points. The structure theorem for 0-E-unitary inverse semigroups was given by Lawson [16] and, Gomes and Howie [12].

Remark 2.8. $\mathcal{ODDP}_n$ is a 0-E-unitary ample subsemigroup of $\mathcal{I}_n$.

3. Rank Properties

Let S be a semigroup and let A be a subset of S. We say that A is a *generating* set if $\langle A \rangle = S$. The *rank* of a finite semigroup S is usually denoted and defined by

$$rank\, S = \min\{|A| : A \subset S, \langle A \rangle = S\}.$$

Rank properties of various semigroups of transformations have been investigated by various authors in recent years, and we draw particular attention to Gomes and Howie [12], Garba [11], Umar [18], Ganyushkin and Mazorchuk [10] and, more recently Al-Kharousi et al. [1].

3.1. Rank of $\mathcal{ODDP}_n$

It has already been observed (Lemma 1.3(a)) that $\mathcal{ODP}_n$ consists of translations. Let

$$\eta = \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix}.$$

Then it is easy to see that non-identity translations are restrictions of the map η^i. Next, we observe that $S = \mathcal{DDP}_n$ or $S = \mathcal{ODDP}_n$ is the union of $K_0, K_1, \cdots, K_n$; where $K_p = \{\alpha \in S : h(\alpha) = p\}$. Then clearly

$$K_{n-1} = \{\eta, id_{X_n \setminus \{i\}} : 1 \leq i \leq n\},$$

for $S = \mathcal{ODDP}_n$ and it generates $\mathcal{ODDP}_n \setminus \{id_{X_n}\}$.

It is also not difficult to see that η is the unique nonidempotent in K_{n-1}. Moreover, we have

Lemma 3.1 ([18], Lemma 3.5)**.** *Let* $\alpha, \beta, \alpha\beta \in \mathcal{ODDP}_n$, *be each of height* $n-1$. *Then* $\alpha\beta$ *is a partial identity if and only if* $\alpha = \beta = \alpha\beta$.

Thus we now have

Theorem 3.2. *For* $n \geq 2$, *we have*

(a) *rank* $(\mathcal{ODDP}_n \setminus \{id_{X_n}\}) = n+1$. *Moreover,* K_{n-1} *is the unique minimum generating set for* $\mathcal{ODDP}_n \setminus \{id_{X_n}\}$.

(b) *rank* $\mathcal{ODDP}_n = n+2$. *Moreover,* $K_{n-1} \cup \{id_{X_n}\}$ *is the unique minimum generating set for* $\mathcal{ODDP}_n$.

Proof. (a) The minimality of K_{n-1} as a generating set for $\mathcal{ODDP}_n$ follows from Lemma 3.1 and the remarks preceding it. The main result in Doyen [5] states that: "Any finite $\mathcal{J}$-trivial monoid has a unique minimum generating system." Thus, by Theorem 2.1, we deduce that K_{n-1} is the unique minimum generating set for $\mathcal{ODDP}_n$.

(b) It follows directly from (a) above. □

3.2. Rank of $\mathcal{DDP}_n$

To find the rank of $\mathcal{DDP}_n$, $(n > 2)$, we note the following:

(a) the product of two order-reversing transformations is order-preserving;

(b) the product of order-reversing and order-preserving transformation is order-reversing;

(c) order-decreasing and order-reversing partial isometries exist only for heights less than or equal to $n/2$.

Thus since order-preserving maps generate themselves only, we need some order-reversing maps to generate $\mathcal{DDP}_n$.

Now, for $1 < i < n,$ define

$$\eta_i = \begin{cases} \begin{pmatrix} i & i+1 & \cdots & 2i-1 \\ i & i-1 & \cdots & 1 \end{pmatrix}, & i \le \left\lceil \frac{n}{2} \right\rceil; \\ \begin{pmatrix} i & i+1 & \cdots & n \\ i & i-1 & \cdots & 2i-n \end{pmatrix}, & i > \left\lceil \frac{n}{2} \right\rceil. \end{cases}$$

It is clear that each η_i is an order-reversing nonidempotent (for $n > 2$) partial isometry and

$$h(\eta_i) = \begin{cases} i, & i \le \left\lceil \frac{n}{2} \right\rceil; \\ n - i + 1, & i > \left\lceil \frac{n}{2} \right\rceil. \end{cases}$$

The next lemma is evident.

Lemma 3.3. *For* $S = \mathcal{ODDP}_n,$ *we have*

$$\{\alpha \in \mathcal{DDP}_n : h(\alpha) > \lceil n/2 \rceil\} = \{\alpha \in \mathcal{ODDP}_n : h(\alpha) > \lceil n/2 \rceil\}.$$

The next two lemmas are slightly less evident.

Lemma 3.4. *Let* $\alpha \in \mathcal{DDP}_n$ *with* $F(\alpha) = \{i\}.$ *Then* $h(\alpha) \le h(\eta_i).$

Proof. This follows directly from Lemma 1.3(c). □

Lemma 3.5. *Let* $\alpha, \beta \in \mathcal{DDP}_n$ *be such that* $\alpha\beta = \eta_i.$ *Then* $\alpha = \eta_i$ *or* $\beta = \eta_i.$

Proof. Let $\alpha\beta = \eta_i$ then $i \in F(\alpha) \cap F(\beta),$ by Lemma 1.4. Suppose α is order-preserving and β is order-reversing then α is a partial identity by Lemma 1.3(b). Moreover, $h(\beta) \le h(\eta_i) = h(\alpha\beta) \le h(\beta)$ by Lemma 3.4, hence $\beta = \eta_i$ by Lemma 1.3(c). The case when α is order-reversing and β is order-preserving, is similar. □

From Lemmas 3.3, 3.4, and 3.5, we deduce the next lemma.

Lemma 3.6. *Any generating set for* $\mathcal{DDP}_n$ *must contain* $K_{n-1} \cup \{\eta_i : 1 < i < n\}$.

While the above lemma shows the necessity of $K_{n-1} \cup \{\eta_i : 1 < i < n\}$ as part of any generating set for $\mathcal{DDP}_n$, the next lemma shows its sufficiency.

Lemma 3.7. $\mathcal{DDP}_n \setminus \{id_{X_n}\} = \langle K_{n-1} \cup \{\eta_i : 1 < i < n\}\rangle$.

Proof. The result follows from the facts that nonidempotent translations are restrictions of η^i and reflections are restrictions of $\eta^i\eta_j$ for $i \geq 0$. □

Theorem 3.8. *For* $n \geq 2$, *we have*

(a) *rank* $(\mathcal{DDP}_n \setminus \{id_{X_n}\}) = 2n - 1$. *Moreover*, $K_{n-1} \cup \{\eta_i : 1 < i < n\}$ *is the unique minimum generating set for* $\mathcal{ODDP}_n \setminus \{id_{X_n}\}$.

(b) *rank* $\mathcal{DDP}_n = 2n$. *Moreover*, $K_{n-1} \cup \{\eta_i : 1 < i < n\} \cup \{id_{X_n}\}$ *is the unique minimum generating set for* $\mathcal{ODDP}_n$.

Proof. (a) It follows from Lemmas 3.5, 3.6, and 3.7, the fact that $|K_{n-1}| = n + 1$ and that there are $n - 2\eta_i s$. For uniqueness, see the proof of Theorem 3.2 above.

(b) It follows directly from (a) above. □

We conclude with a table of summary of the rank results proved in the paper.

Semigroup	$\mathcal{ODDP}_n \setminus \{id_{X_n}\}$	$\mathcal{ODDP}_n$	$\mathcal{DDP}_n \setminus \{id_{X_n}\}$	$\mathcal{DDP}_n n + 1$
Rank	$n + 1$	$n + 2$	$2n - 1$	$2n$

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