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A STUDY OF POSITIVE PERIODIC SOLUTIONS FOR SECOND-ORDER DELAYED DIFFERENTIAL EQUATIONS

YUE LIANG and DI ZHANG

Science College of Gansu
Agricultural University
Lanzhou 730070
P. R. China
e-mail: liangyuegsau@163.com
aqsy0008@163.com

Abstract

This paper deals with the existence of positive δ-periodic solutions for second-order differential equation with delay of the form

$$-x''(t) + Mx(t) = f(t, x(t), x(t-\tau)), \quad t \in \mathbb{R},$$

where $M > 0$ is a constant, $f : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $f(t, u, v)$ is δ-periodic in t, $\tau > 0$ is a constant which denotes the time delay. We first establish the positivity estimation for the linear equation. Then two existence theorems of positive δ-periodic solutions are obtained when f satisfies some inequality conditions. The discussion is based on the fixed point index theory in cones.

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1. Introduction

In this paper, we study the existence of positive δ-periodic solutions for second-order differential equation with delay of the form

$$-x''(t) + Mx(t) = f(t, x(t), x(t-\tau)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $M > 0$ is a constant, $f : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $f(t, u, v)$ is δ-periodic in t, $\tau > 0$ is a constant.

Since the differential equation with delay has extensive physical, biological and engineering background, it has been an important branch of differential equation theory. The existence of periodic solutions for delayed differential equation has been studied in recent years and an increasing number of papers emerged, see [3, 6, 7, 10, 11] and the reference therein. But the positive periodic solutions are more important in applications. The existence of positive periodic solutions of differential equations without delay has been researched, see [4, 5]. Recently, the authors in [6, 8, 9] investigated the existence of positive periodic solutions for delayed differential equations. Motivated by these papers, we consider the existence of positive δ-periodic solutions of Equation (1.1). We first establish the positivity estimation of the linear equation corresponding to Equation (1.1), then by using fixed point index in cones, two existence theorems of positive δ-periodic solutions for Equation (1.1) are proved when nonlinearity f satisfies inequality conditions.

2. Preliminaries

Let $\delta > 0$ be a constant. The set of all continuous δ-periodic function $x(t)$ will be denoted by $C_\delta(\mathbb{R})$. Then $C_\delta(\mathbb{R})$ is a Banach space with norm $\|x\|_C = \max_{t \in [0, \delta]} |u(t)|$. The cone of all nonnegative functions in $C_\delta(\mathbb{R})$ will be denoted by $C_\delta^+(\mathbb{R})$.

For $g \in C_\delta(\mathbb{R})$, we first consider the linear differential equation

$$-x''(t) + Mx(t) = g(t), \quad t \in \mathbb{R}. \tag{2.1}$$

Let $M > 0$ and $\eta = \sqrt{M}$, from Lemma 2.1 of [5], the Equation (2.1) has a unique δ-periodic solution $u(t)$ which is expressed by

$$x(t) = \int_{t-\delta}^{t} \Phi(t-s)g(s)ds := (\Psi g)(t), \quad t \in \mathbb{R}, \tag{2.2}$$

where

$$\Phi(t) = \frac{\cosh \eta(t - \delta/2)}{2\eta \sinh(\eta\delta/2)}, \quad t \in [0, \delta], \tag{2.3}$$

and $\Psi : C_\delta(\mathbb{R}) \to C_\delta(\mathbb{R})$ is a completely continuous linear operator.

By (2.2) and (2.3), we can obtain

$$\|\Psi\| = \int_0^\delta \Phi(s)ds = \frac{1}{M}, \tag{2.4}$$

and

$$\frac{1}{2\eta \sinh(\eta\delta/2)} \le \Phi(t) \le \frac{\cosh(\eta\delta/2)}{2\eta \sinh(\eta\delta/2)}. \tag{2.5}$$

For the sake of brevity, let $\kappa = (\cosh(\eta\delta/2))^{-1}$. Then $\kappa > 0$.

For any $g \in C_\delta(\mathbb{R})$, we study the second-order linear delayed differential equation

$$-x''(t) + Mx(t) + Nx(t-\tau) = g(t), \quad t \in \mathbb{R}, \tag{2.6}$$

where $M > 0$ and $N \ge 0$. From [7], we have the following result:

Lemma 1. *For any* $g \in C_\delta(\mathbb{R})$, *if* $0 \le N < \kappa^2 M$, *then the linear delayed differential equation* (2.6) *has a unique* δ*-periodic solution* $x(t)$ *which is given by*

$$x(t) = (I + \Psi \circ B_\tau)^{-1} \int_{t-\delta}^{t} \Phi(t-s)g(s)ds, \quad t \in \mathbb{R}, \tag{2.7}$$

where $B_\tau : C_\delta(\mathbb{R}) \to C_\delta(\mathbb{R})$ *is defined by*

$$B_\tau x(t) = Nx(t-\tau), \quad t \in \mathbb{R},\ \tau > 0.$$

Let $G(x)(t) := f(t, x(t), x(t-\tau)) + Nx(t-\tau)$. An operator $\Lambda : C_\delta(\mathbb{R}) \to C_\delta(\mathbb{R})$ is define by

$$(\Lambda x)(t) = (I + \Psi \circ B_\tau)^{-1} \int_{t-\delta}^{t} \Phi(t-s) G(x)(s) ds, \quad t \in \mathbb{R}. \tag{2.8}$$

Then by Lemma 1, $\Lambda : C_\delta(\mathbb{R}) \to C_\delta(\mathbb{R})$ is continuous and the δ-periodic solution of the second-order delayed differential equation (1.1) is equivalent to the fixed point of Λ.

Choose a subcone of $C_\delta^+(\mathbb{R})$ as

$$\prod = \{x \in C_\delta^+(\mathbb{R}) : x(t) \geq \kappa x(s),\ \forall t,\ s \in \mathbb{R}\}.$$

By Lemma 1, we obtain the following lemma:

Lemma 2. $\Lambda : \prod \to \prod$ *is completely continuous.*

Proof. By the complete continuity of Ψ and the definition of Λ, it is easy to see that $\Lambda : \prod \to C_\delta(\mathbb{R})$ is completely continuous. It remains to prove $\Lambda(\prod) \subset \prod$.

For any $x \in \prod$, it follows from the definition of Λ that

$$(I + \Psi \circ B_\tau)(\Lambda x)(t) = \int_{t-\delta}^{t} \Phi(t-s) G(x)(s) ds, \quad t \in \mathbb{R}. \tag{2.9}$$

From (2.5) and (2.9), for any $t,\ s \in \mathbb{R}$, we have

$$\begin{aligned}(I + \Psi \circ B_\tau)(\Lambda x)(s) &\leq \frac{\cosh(\eta\delta/2)}{2\eta \sinh(\eta\delta/2)} \int_{s-\delta}^{s} G(x)(s) ds \\ &= \frac{\cosh(\eta\delta/2)}{2\eta \sinh(\eta\delta/2)} \int_{0}^{\delta} G(x)(s) ds,\end{aligned}$$

and

$$(I + \Psi \circ B_\tau)(\Lambda x)(t) \geq \frac{1}{2\eta \sinh(\eta\delta/2)} \int_{t-\delta}^{t} G(x)(s)ds$$

$$= \frac{1}{2\eta \sinh(\eta\delta/2)} \int_{0}^{\delta} G(x)(s)ds.$$

This implies that

$$(I + \Psi \circ B_\tau)(\Lambda x)(t) \geq \kappa(I + \Psi \circ B_\tau)(\Lambda x)(s), \quad \forall t, s \in \mathbb{R}. \tag{2.10}$$

From the proof of Lemma 2.1 in [7], it follows that $(I + \Psi \circ B_\tau)^{-1}$ is a positive operator. Hence by (2.10), we have

$$(\Lambda x)(t) \geq \kappa(\Lambda x)(s), \quad \forall t, s \in \mathbb{R}.$$

Hence, $\Lambda : \prod \to \prod$ is completely continuous. □

Next, we recall some conclusions of fixed point index in cones. Let Ξ be a Banach space and $\Delta \subset \Xi$ a closed convex cone in Ξ. Assume that Θ is a bounded open subset of Ξ with boundary $\partial\Theta$, and $\Delta \bigcap \Theta$ is nonempty. Let $\Lambda : \Delta \bigcap \overline{\Theta} \to \Delta$ be a completely continuous mapping. If $\Lambda u \neq u$ for any $u \in \Delta \bigcap \partial\Theta$, then the fixed point index $i(\Lambda, \Delta \bigcap \Theta, \Delta)$ is well defined. If $i(\Lambda, \Delta \bigcap \Theta, \Delta) \neq 0$, then Λ has a fixed point in $\Delta \bigcap \Theta$. For more details of the concepts and conclusions on the fixed point index, we refer to [1, 2]. The following two lemmas are needed in our arguments.

Lemma 3. *Let Θ be a bounded open subset of Ξ with $\theta \in \Theta$ and $\Lambda : \Delta \bigcap \overline{\Theta} \to \Delta$ be a completely continuous mapping. If*

$$\varrho \Lambda x \neq x, \quad \forall x \in \Delta \bigcap \partial\Theta, \quad 0 < \varrho \leq 1,$$

then $i(\Lambda, \Delta \bigcap \Theta, \Delta) = 1$.

Lemma 4. *Let* Θ *be a bounded open subset of* Ξ *and* $\Lambda : \Delta \bigcap \overline{\Theta} \to \Delta$ *be a completely continuous mapping. If there exists an* $e \in \Delta \setminus \{\theta\}$ *such that*

$$x - \Lambda x \neq \nu e, \quad \forall x \in \Delta \bigcap \partial\Theta, \quad \nu \geq 0,$$

then $i(\Lambda, \Delta \bigcap \Theta, \Delta) = 0$.

3. Main Results

Theorem 1. *Let* $f : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous and* $f(t, u, v)$ δ*-periodic in* t *for every* $u, v \in \mathbb{R}^+$. *If* $M > \kappa^{-2}N \geq 0$ *and* f *satisfies the following conditions*:

(H1) *There exist two positive constants* α_1 *and* α_2 *satisfying* $\alpha_1 + \alpha_2 < M$ *and* $\varepsilon > 0$ *such that*

$$f(t, u, v) \leq \alpha_1 u + \alpha_2 v,$$

for any $t \in \mathbb{R}$ *and* $u, v \in [0, \varepsilon]$.

(H2) *There exist two positive constants* β_1 *and* β_2 *satisfying* $\beta_1 + \beta_2 > M$ *and* $h_0 \in C_\delta^+(\mathbb{R})$ *such that*

$$f(t, u, v) \geq \beta_1 u + \beta_2 v - h_0(t),$$

for any $t \in \mathbb{R}$ *and* $u, v \in \mathbb{R}^+$,

then the Equation (1.1) *has at least one positive* δ*-periodic solution.*

Proof. Define an operator $\Lambda : C_\delta(\mathbb{R}) \to C_\delta(\mathbb{R})$ as in (2.8). By Lemma 2, $\Lambda : \prod \to \prod$ is completely continuous. Let

$$\Delta_r = \{x \in C_\delta(\mathbb{R}) : \|x\|_C < r\}, \quad \Delta_R = \{x \in C_\delta(\mathbb{R}) : \|x\|_C < R\}, \tag{3.1}$$

where $0 < r < R < +\infty$. Then Δ_r and Δ_R are bounded open subsets of $C_\delta(\mathbb{R})$. We will show that Λ has a fixed point in $\prod \bigcap (\Delta_R \setminus \overline{\Delta}_r)$ for some $r > 0$ and $R > 0$.

Let $r \in (0, \varepsilon)$, where ε is the positive constant in condition (H1). We prove that Λ satisfies the condition of Lemma 3 in $\prod \cap \Delta_r$. If there exist $x_0 \in \prod \cap \partial\Delta_r$ and $0 < \varrho_0 \leq 1$ such that

$$\varrho_0 \Lambda x_0 = x_0,$$

by the definition of Λ and Lemma 1, we have

$$-x_0''(t) + Mx_0(t) + Nx_0(t-\tau) = \varrho_0 f(t, x_0(t), x_0(t-\tau)) + \varrho_0 N x_0(t-\tau), \; t \in \mathbb{R}.$$

Since $0 < \varrho_0 \leq 1$, deduced from the above inequality that

$$-x_0''(t) + Mx_0(t) \leq f(t, x_0(t), x_0(t-\tau)), \quad t \in \mathbb{R}. \tag{3.2}$$

By $x_0 \in \prod \cap \partial\Delta_r$, we have

$$0 \leq x_0(t) \leq \|x_0\|_C = r < \varepsilon,$$

and

$$0 \leq x_0(t-\tau) \leq \|x_0\|_C = r < \varepsilon.$$

Hence, by condition (H1), we can obtain

$$f(t, x_0(t), x_0(t-\tau)) \leq \alpha_1 x_0(t) + \alpha_2 x_0(t-\tau), \quad t \in \mathbb{R}.$$

Combining this inequality with (3.2), we have

$$-x_0''(t) + Mx_0(t) \leq \alpha_1 x_0(t) + \alpha_2 x_0(t-\tau), \quad t \in \mathbb{R}.$$

Integrating both sides of this inequality from 0 to δ and using the periodicity of x_0, we have

$$M\int_0^\delta x_0(t)dt \leq \alpha_1 \int_0^\delta x_0(t)dt + \alpha_2 \int_0^\delta x_0(t-\tau)dt$$

$$= (\alpha_1 + \alpha_2)\int_0^\delta x_0(t)dt.$$

Since $x_0 \in \prod$, by the definition of $\prod$, we have

$$x_0(t) \geq \kappa x_0(s), \quad \forall t, s \in \mathbb{R}.$$

Hence, we can obtain

$$(M - \alpha_1 - \alpha_2)\kappa\delta x_0(s) \leq (M - \alpha_1 - \alpha_2)\int_0^\delta x_0(t)dt \leq 0, \quad \forall s \in \mathbb{R}.$$

Since $M > \alpha_1 + \alpha_2$, $\kappa > 0$ and $\delta > 0$, it follows that $x_0(s) \leq 0$ for any $s \in \mathbb{R}$, which contracts with $x_0 \in \prod \cap \partial\Delta_r$. This means that Λ satisfies the condition of Lemma 3. By Lemma 3,

$$i(\Lambda, \prod \cap \Delta_r \prod) = 1. \tag{3.3}$$

On the other hand, we show that Λ satisfies the condition of Lemma 4 in $\prod \cap \Delta_R$. Let $e(t) \equiv 1$ for any $t \in \mathbb{R}$. Then $e \in \prod \setminus \{\theta\}$. We show that if R is large enough, then $x - \Lambda x \neq \nu e$ for any $x \in \prod \cap \partial\Delta_R$ and $\nu \geq 0$. In fact, if there exist $x_1 \in \prod \cap \partial\Delta_R$ and $\nu_1 \geq 0$ such that

$$x_1 - \Lambda x_1 = \nu_1 e,$$

namely, $x_1 - \nu_1 e = \Lambda x_1$. By the definition of Λ, we have

$$-x_1''(t) + Mx_1(t) - \nu_1(M + N) = f(t, x_1(t), x_1(t - \tau)), \quad t \in \mathbb{R}.$$

By condition (H2), it follows that

$$-x_1''(t) + Mx_1(t) \geq \beta_1 x_1(t) + \beta_2 x_1(t - \tau) - h_0(t), \quad t \in \mathbb{R}.$$

Integrating both sides of this inequality from 0 to δ and using the periodicity of x_1, we have

$$M\int_0^\delta x_1(t)dt \geq \beta_1\int_0^\delta x_1(t)dt + \beta_2\int_0^\delta x_1(t - \tau)dt - \int_0^\delta h_0(t)dt$$

$$= (\beta_1 + \beta_2)\int_0^\delta x_1(t)dt - \int_0^\delta h_0(t)dt.$$

This implies that

$$(\beta_1 + \beta_2 - M)\int_0^{\delta} x_1(t)dt \leq \int_0^{\delta} h_0(t)dt \leq \delta\|h_0\|_C.$$

Since $x_1 \in \prod$, $x_1(t) \geq \kappa x_1(s)$ for any $t, s \in \mathbb{R}$. So,

$$(\beta_1 + \beta_2 - M)\kappa\delta x_1(s) \leq \delta\|h_0\|_C, \quad \forall s \in \mathbb{R}.$$

Hence, we can obtain

$$\|x_1\|_C \leq \frac{\|h_0\|_C}{(\beta_1 + \beta_2 - M)\kappa}.$$

Let $R > \max\{\frac{\|h_0\|_C}{(\beta_1 + \beta_2 - M)\kappa}, r\}$. Then the condition of Lemma 4 holds. By Lemma 4, we have

$$i(\Lambda, \prod \cap \Delta_R, \prod) = 0. \tag{3.4}$$

Combining (3.3) and (3.4), using the additivity of fixed point index, we have

$$i(\Lambda, \prod \cap (\Delta_R \setminus \overline{\Delta}_r), \prod) = i(\Lambda, \prod \cap \Delta_R, \prod) - i(\Lambda, \prod \cap \Delta_r, \prod) = -1 \neq 0.$$

Hence, Λ has at least one fixed point in $\prod \cap (\Delta_R \setminus \overline{\Delta}_r)$, which is a positive δ-periodic solution of the second-order delayed differential equation (1.1). □

Theorem 2. *Let* $f : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous and* $f(t, u, v)$ δ*-periodic in* t *for every* $u, v \in \mathbb{R}^+$. *If* $M > \kappa^{-2}N \geq 0$ *and* f *satisfies the following conditions*:

(H3) *There exist two positive constants* β_1 *and* β_2 *satisfying* $\beta_1 + \beta_2 > M$ *and* $\varepsilon > 0$ *such that*

$$f(t, u, v) \geq \beta_1 u + \beta_2 v,$$

for any $t \in \mathbb{R}$ *and* $u, v \in [0, \varepsilon]$.

(H4) *There exist two positive constants* α_1 *and* α_2 *satisfying* $\alpha_1 + \alpha_2 < M$ *and* $h_1 \in C_\delta^+(\mathbb{R})$ *such that*

$$f(t, u, v) \leq \alpha_1 u + \alpha_2 v + h_1(t),$$

for any $t \in \mathbb{R}$ *and* $u, v \in \mathbb{R}^+$,

then the Equation (1.1) *has at least one positive* δ*-periodic solution.*

Proof. Define an operator $\Lambda : C_\delta(\mathbb{R}) \to C_\delta(\mathbb{R})$ as in (2.8). Then $\Lambda : \prod \to \prod$ is completely continuous. Let Δ_r and Δ_R be defined by (3.1) for any $0 < r < R < +\infty$. We show that Λ has a fixed point in $\prod \cap (\Delta_R \setminus \overline{\Delta}_r)$ for some positive numbers r and R.

Let $r \in (0, \varepsilon)$, where ε is the positive number in condition (H3). Choose $e(t) \equiv 1$ for any $t \in \mathbb{R}$. Then $e \in \prod \setminus \{\theta\}$. We show that Λ satisfies the condition of Lemma 4 in $\prod \cap \Delta_r$. In fact, if there exist $x_0 \in \prod \cap \partial\Delta_r$ and $\nu_0 \geq 0$ such that

$$x_0 - \nu_0 e = \Lambda x_0,$$

then by the definition of Λ, we can obtain

$$-x_0''(t) + Mx_0(t) \geq f(t, x_0(t), x_0(t-\tau)), \quad t \in \mathbb{R}. \tag{3.5}$$

Since $x_0 \in \prod \cap \partial\Delta_r$, it follows that $x_0(t), x_0(t-\tau) \in [0, \varepsilon]$. By the condition (H3), we have

$$f(t, x_0(t), x_0(t-\tau)) \geq \beta_1 x_0(t) + \beta_2 x_0(t-\tau), \quad t \in R.$$

Combining this inequality with (3.5), we have

$$-x_0''(t) + Mx_0(t) \geq \beta_1 x_0(t) + \beta_2 x_0(t-\tau), \quad t \in R.$$

Integrating both sides of this inequality from 0 to δ and using the periodicity of x_0, we have

$$M\int_0^\delta x_0(t)dt \geq \beta_1\int_0^\delta x_1(t)dt + \beta_2\int_0^\delta x_0(t-\tau)dt$$

$$= (\beta_1 + \beta_2)\int_0^\delta x_0(t)dt.$$

This implies that

$$(\beta_1 + \beta_2 - M)\int_0^\delta x_0(t)dt \leq 0.$$

Since $x_0 \in \prod$, it follows that

$$(\beta_1 + \beta_2 - M)\kappa\delta x_0(s) \leq 0, \quad \forall\ s \in \mathbb{R}.$$

This implies $x_0(t) \leq 0$ for any $t \in \mathbb{R}$ because of $\beta_1 + \beta_2 > M$, $\kappa > 0$ and $\delta > 0$, which contracts with $x_0 \in \prod \cap \partial\Delta_r$. Hence, by Lemma 4, we have

$$i(\Lambda, \prod \cap \Delta_r, \prod) = 0. \tag{3.6}$$

On the other hand, we show that Λ satisfies the condition of Lemma 3 in $\prod \cap \Delta_R$ when R is large enough. In face, if there exist $x_1 \in \prod \cap \partial\Delta_R$ and $0 < \varrho_1 \leq 1$ such that

$$x_1 = \varrho_1\Lambda x_1.$$

By the definition of Λ, we have

$$-x_1''(t) + Mx_1(t) \leq f(t, x_1(t), x_1(t-\tau)), \quad t \in R.$$

Since $x_1 \in \prod \cap \partial\Delta_r$, by the condition (H4), we have

$$-x_1''(t) + Mx_1(t) \leq \alpha_1 x_1(t) + \alpha_2 x_1(t-\tau) + h_1(t), \quad t \in R.$$

Integrating both sides of this inequality from 0 to δ and using the periodicity of x_1, we have

$$M\int_0^\delta x_1(t)dt \le \alpha_1\int_0^\delta x_1(t)dt + \alpha_2\int_0^\delta x_0(t-\tau)dt + \int_0^\delta h_1(t)dt$$

$$= (\alpha_1 + \alpha_2)\int_0^\delta x_1(t)dt + \int_0^\delta h_1(t)dt.$$

This implies that

$$(M - \alpha_1 - \alpha_2)\int_0^\delta x_1(t)dt \le \delta\|h_1\|_C.$$

Since $u_0 \in \prod$, it follows that

$$(M - \alpha_1 - \alpha_2)\kappa\delta x_1(s) \le \delta\|h_1\|_C, \quad \forall s \in \mathbb{R}.$$

This implies

$$\|x_1\|_C \le \frac{\|h_1\|_C}{(M - \alpha_1 - \alpha_2)\kappa}.$$

Let $R > \max\{\frac{\|h_1\|_C}{(M - \alpha_1 - \alpha_2)\kappa}, r\}$. Then the condition of Lemma 3 holds. Hence by Lemma 3, we have

$$i(\Lambda, \prod \cap \Delta_R, \prod) = 1. \tag{3.7}$$

Combining (3.6) and (3.7), utilizing the additivity of fixed point index, we have

$$i(\Lambda, \prod \cap (\Delta_R \setminus \overline{\Delta}_r), \prod) = i(\Lambda, \prod \cap \Delta_R, \prod) - i(\Lambda, \prod \cap \Delta_r, \prod) = 1 \ne 0.$$

Hence, Λ has at least one fixed point in $\prod \cap (\Delta_R \setminus \overline{\Delta}_r)$, which is a positive δ-periodic solution of the second-order delayed differential equation (1.1).

Competing Interests

None of the authors have any competing interests in the manuscript. All authors contributed equally in writing this paper. All authors read and approved the final manuscript.

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