Journal of Mathematical Sciences: Advances and Applications
Volume 50, 2018, Pages 1-12
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100121935

NECESSARY AND SUFFICIENT CONDITIONS FOR A SPACE TO HAVE A MAXIMAL, PROPER, DENSE, $T_{3\frac{1}{2}}$ SUBSPACE

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Abstract

Within this paper necessary and sufficient conditions for a space to have a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace are given and subspaces properties of such spaces are further investigated.

1. Introduction and Preliminaries

In the 1936 paper [9], for a space (X, T), an externally generated, strongly (X, T) related space, called the T_0-identification space of (X, T), was introduced and used to jointly characterize metrizable and pseudometrizable.

2010 Mathematics Subject Classification: 54B05, 54D15.
Keywords and phrases: proper subspaces, dense subspaces, maximal $T_{3\frac{1}{2}}$ subspaces.

Received February 28, 2018

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, Y))$ is the T_0-identification space of (X, T).

Theorem 1.1. *A space* (X, T) *is pseudometrizable iff* $(X_0, Q(X, T))$ *is metrizable.* T_0*-identification spaces were cleverly created to add* T_0 *to the externally generated, strongly* (X, T) *related* T_0*-identification space of* (X, T) [9], *which, when combined with the fact that metrizable and (pseudometrizable and* T_0*) are equivalent, was used to establish the result above.*

The characterization of metrizable and pseudometrizable given above using T_0-identification spaces raised the question: What other properties P not necessarily T_0 and (P and T_0), if any, would behave in the same manner as pseudometrizable and metrizable = (pseudometrizable and T_0)?, which led to the introduction and investigation of weakly *Po* properties in 2015 [1].

Definition 1.2. Let P be topological properties such that $Po = (P \text{ and } T_0)$ exists. Then a space (X, T) is weakly *Po* iff its T_0-identification space $(X_0, Q(X, T))$ has property P. A topological property *Po* for which weakly *Po* exists is called a weakly *Po* property [1].

Since for a space, its T_0-identification space has property T_0, then, for a topological property P for which *Po* exists, a space is weakly *Po* iff its T_0-identification space has property *Po*.

Within the paper [1], it was shown that for a weakly *Po* property *Qo*, a space is weakly *Qo* iff its T_0-identification space is weakly *Qo*, which led to the introduction and investigation of T_0-identification *P* properties [2].

Definition 1.3. Let *S* be a topological property. Then *S* is a T_0-identification *P* property iff both a space and its T_0-identification space simultaneously shares property *S*.

In the introductory weakly *Po* property paper [1], it was shown that weakly *Po* is neither T_0 nor "not-T_0", where "not-T_0" is the negation of T_0. The need and use of "not-T_0" revealed "not-T_0" as a useful, powerful topological property and tool, motivating the inclusion of the long-neglected properties "not-*P*", where *P* is a topological property for which "not-*P*" exists, as important properties for investigation and use in the study of topology. As a result, within a short time period, many new, important, fundamental, foundational, never before imagined properties and examples have been discovered, expanding and changing the study of topology forever.

In past studies of weakly *Po* spaces and properties, for a classical topological property *Qo*, a special topological property *W* was sought such that for a space with property *W*, its T_0-identification space has property *Qo*, which then implies the initial space has property *W*, with no certainty that such a topological property *W* exists. As given above, the study of weakly *Po* spaces and related properties has been a productive study, but, if the past search process continued, the study of weakly *Po* spaces and properties would continue to be uncertain, tedious, and never ending. To make the process more certain, the question of precisely which topological properties are weakly *Po* properties arose leading to answers in a 2017 paper [3].

Answer 1.1. Let Q be a topological property for which both Qo and (Q and "not-T_0") exist. Then Q is a T_0-identification P property that is weakly Po and Q = weakly Qo = (Qo or (Q and "not-T_0")) [3].

Answer 1.2. $\{Qo|$ Q is a T_0-identification P property$\}$ = $\{Qo|$ Qo is a weakly Po property$\}$ = $\{Qo|$ Q is a topological property and Qo exists$\}$ [3].

Thus, major progress was achieved in the study of topology, weakly Po, and related properties. If Q is a topological property for which both Qo and (Q and "not-T_0") exist, Answer 1.1 quickly and easily gives weakly Qo. If Q is a topological property for which $Q = Qo$, then $Q = Qo$ is a weakly Po property, but $Q = Qo$ is not a T_0-identification P property or weakly Po. Within the paper [3], a topological property W that can be both T_0 and "not-T_0" was shown to exist that is a T_0-identification P property that is weakly Po such that W = weakly Qo, again making the search process certain, but, just knowing such a W exists, gave little insight into the precise, needed topological property W, raising the question of whether the known information could somehow be used to more precisely determine W for the fixed Qo.

The investigation of that question led to the introduction and investigation of *OXTO* subsets and the corresponding subspace for each space (X, T) [4].

Definition 1.4. Let (X, T) be a space and for each $x \in X$, let C_x be the T_0-identification space equivalence class containing x. Then Y is an *OXTO* subset of X iff Y contains exactly one element from each equivalence class C_x.

Within the paper [4], it was shown that for a space (X, T), for each *OXTO* subset Y of X, (Y, T_Y) is homeomorphic to $(X_0, Q(X, T))$. Since, as stated earlier, the T_0-identification space of each space is T_0 and T_0 is a topological property, then for each *OXTO* subset Y of X in a space (X, T), (Y, T_Y) is T_0.

Also, within the paper [4], it was shown that for each topological property Q for which Qo exists, a space (X, T) is weakly Qo iff for each *OXTO* subset Y of (X, T), (Y, T_Y) has property Qo, which can be, and has been, used to precisely determine weakly Qo [4]. Thus, the study of weakly Po and weakly Po properties has been completely internalized and greatly simplified by the use of *OXTO* subsets and the corresponding subspaces, and the earlier results, replacing many, earlier uncertainties with certainties.

The continued investigation of weakly Po properties, "not-T_0", and *OXTO* subsets and subspaces of a space led to the definition and unexpected results below.

Theorem 1.2. *Let (X, T) be a space. Then the following are equivalent*: (a) (X, T) *is "not-T_0"*, (b) (X, T) *has a maximal*, *proper*, *dense T_0 subspace*, *and* (c) *for each OXTO subset Y of X*, (Y, T_Y) *is a maximal*, *proper*, *dense T_0 subspace of* (X, T) [5].

Definition 1.5. Let (X, T) be a space, let Y be a subspace of (X, T), and let Q be a topological property for which Qo exists. Then (Y, T_Y) is a maximal, proper, dense, Qo subspace of (X, T) iff (Y, T_Y) is a proper, dense, Qo subspace of (X, T) such that for each subspace (Z, T_W) of (X, T), where Z properly containing Y, (Z, T_Z) is "not-Qo" [6].

Theorem 1.3. *Let* (X, T) *be a space and let* Q *be a topological property for which* Qo *exists. Then* (a) *for each OXTO subset* Y *of* X, (Y, T_Y) *is a maximal, proper, dense,* Qo *subspace of* (X, T) *iff* (b) (X, T) *is weakly* Qo *and "not-*T_0*"* [6].

The results above raised questions about necessary and sufficient conditions, if any, for a space to have a maximal, proper, dense, T_i subspace, $i = 1, 2,$ which were resolved in the paper [7], raising the question of necessary and sufficient conditions, if any, for a space to have a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace.

The completely regular and $T_{3\frac{1}{2}}$ separation axioms were introduced in 1925 [10].

Definition 1.6. A space (X, T) is completely regular iff for each closed set C and each $x \notin C$, there exists a continuous function $f : (X, T) \to ([0, 1], U)$, where U is the usual relative metric topology on $[0, 1]$, such that $f(x) = 0$ and $f(C) = 1$. A completely regular T_1 space is denoted by $T_{3\frac{1}{2}}$.

The work in this paper focuses attention on "not-$T_{3\frac{1}{2}}$", whose definition is given below.

Definition 1.7. A space is "not-$T_{3\frac{1}{2}}$" iff it is "not-T_1" or "not-completely regular".

In the 2017 paper [8], it was shown that $T_{3\frac{1}{2}}$ is a weakly Po property with completely regular = weakly (completely regular)o = weakly $T_{3\frac{1}{2}}$. Below Theorem 1.3 is applied to completely regular and $T_{3\frac{1}{2}}$, giving necessary and sufficient conditions in "not-T_0" spaces for a space to have a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace, and the question above for maximal, proper, dense, $T_{3\frac{1}{2}}$ subspaces is addressed.

2. Weakly $T_{3\frac{1}{2}}$ and "not-T_0" Spaces and Necessary and Sufficient Conditions on a Space for the Space to Have a Maximal, Proper, Dense, $T_{3\frac{1}{2}}$ Subspace

Corollary 2.1. *Let* (X, T) *be a space. Then* (a) *for each OXTO subset* Y *of* X, (Y, T_Y) *is a maximal, proper, dense,* $T_{3\frac{1}{2}}$ *subspace of* (X, T) *iff* (b) (X, T) *is* (*completely regular = weakly* (*completely regular*)*o*) *and "not-*T_0*".*

Theorem 2.1. *Let* (X, T) *be a space and let* Y *be a proper subset of* X. *Then the following are equivalent*: (a) (X, T) *has a maximal, proper, dense,* $T_{3\frac{1}{2}}$ *subspace* (Y, T_Y), (b) (X, T) *is "not-*$T_{3\frac{1}{2}}$*",* (Y, T_Y) *is a proper,* $T_{3\frac{1}{2}}$ *subspace of* (X, T), *and for each* $x \in X \setminus Y$, $W = (\{x\} \cup Y)$, *and* $C_W(x) = \{y \in W \mid Cl_W(\{x\}) = Cl_W(\{y\})\}$, *every* T*-open set containing* x *intersects* Y, $C_W(x)$ *contains at most two elements, and if* $C_W(x) = \{x\}$ *and* $\{x\}$ *is not* T_W*-closed,* (W, T_W) *is "not-*T_1*",* (*or*($\{x\}$ *is* T_W*-closed,* (W, T_W) *is* T_1, *and there exists a* T_W*-closed set* C *and a* $z \in (W \setminus C)$ *such that there does not exist a continuous function* $h : (W, T_W) \to ([0, 1], U)$ *such that* $h(z) = 0$ *and* $h(C) = 1$, *and* (c) (Y, T_Y) *is a proper,* $T_{3\frac{1}{2}}$ *subspace of* (X, T), *and for each* $x \in X \setminus Y$, $W = (\{x\} \cup Y)$, *and* $C_W(x) = \{y \in W \mid Cl_W(\{x\}) = Cl_W(\{y\})\}$, *every* T*-open set containing* x *intersects* Y, $C_W(x)$ *contains at most two elements, and if* $C_W(x) = \{x\}$ *and* $\{x\}$ *is not* T_W*-closed,* (W, T_W) *is* *"not-T_0", (or($\{x\}$ is T_W-closed, (W, T_W) is T_1, and there exists a T_W-closed set C and a $z \in (W \setminus C)$ such that there does not exist a continuous function $h : (W, T_W) \to ([0, 1], U)$ such that $h(z) = 0$ and $h(C) = 1$.*

Proof. (a) implies (b): Since (Y, T_Y) is a maximal, proper, $T_{3\frac{1}{2}}$ subspace of (X, T), then, by definition, (X, T) is "not-$T_{3\frac{1}{2}}$", and (Y, T_Y) is a proper, $T_{3\frac{1}{2}}$ subspace of (X, T). If $\{x\}$ is T_W-open, then there exists a T-open set O containing x such that $O \cap Y = \phi$ and (Y, T_Y) is not dense in (X, T). Thus $\{x\}$ is not T_W-open. Suppose $C_W(x)$ contains three or more elements. Let x, a, and b be distinct elements of $C_W(x)$. Then $Cl_W(\{a\}) = Cl_W(\{x\}) = Cl_W(\{b\})$. Thus $Cl_Y(\{a\}) = Cl_Y(\{b\})$ and (Y, T_Y) is "not-T_0" and hence "not-T_1", which is a contradiction. Suppose $C_W(x) = \{x, a\}$, $a \neq x$. Then $Cl_W(\{x\}) = Cl_W(\{a\})$ and (W, T_W) is "not-T_0", which implies (W, T_W) is "not-T_1" and thus, "not-$T_{3\frac{1}{2}}$". Suppose $C_W(x) = \{x\}$ and $\{x\}$ is not T_W-closed. Then (there exists a $y \in Y$ such $y \in Cl_W(\{x\})$ and $x \notin Cl_W(\{y\})$ or $y \notin Cl_W(\{x\})$ and $x \in Cl_W(\{y\})$) or for all $y \in Y$, $x \notin Cl_W(\{y\})$ and $y \notin Cl_W(\{x\})$. In either of the first two cases, (W, T_W) is "not-T_1" and, thus, (W, T_W) is "not-$T_{3\frac{1}{2}}$". Thus, consider the third case. Since for each $y \in Y$, $y \notin Cl_W(\{x\})$, then $Cl_W(\{x\}) = \{x\}$. Thus $Y = W \setminus \{x\}$ is T_W-, T_Y-open and each T_Y-open set is T_W-open. Since (Y, T_Y) is T_1, $x \notin Cl_W(\{y\})$ for all $y \in Y$, and $y \notin Cl_W(\{x\})$ for all $y \in Y$, then (W, T_W) is T_1. Since (W, T_W) is "not-$T_{3\frac{1}{2}}$" and T_1, then (W, T_W) is "not-completely regular" and there exists a T_W-closed set C and a $z \notin C$ such that there does not exist a continuous function $f : (W, T_W) \to ([0, 1], U)$, where U is the usual relative metric topology on $[0, 1]$, such that $f(z) = 0$ and $f(C) = 1$.

Clearly (b) implies (c).

(c) implies (a): Since for each $x \in X \setminus Y$, every T-open set containing x intersects Y, then (Y, T_Y) is dense in (X, T), and (Y, T_Y) is a proper, dense, $T_{3\frac{1}{2}}$ subspace of (X, T). If $C_W(x) = \{x, a\}$, $a \neq x$, then $Cl_W(\{x\}) = Cl_W(\{a\})$ and (W, T_W) is "not-T_0" and, thus "not-T_1" and "not-$T_{3\frac{1}{2}}$". If $C_W(x) = \{x\}$ and $\{x\}$ is not T_W-closed, (W, T_W) is "not-T_1" and, thus, "not-$T_{3\frac{1}{2}}$", and if $\{x\}$ is T_W-closed, (W, T_W) is T_1, and there exists a T_W-closed set C and a $z \in (W \setminus C)$ such that there does not exist a continuous function $h : (W, T_W) \to ([0, 1], U)$ such that $h(z) = 0$ and $h(C) = 1$, which implies (W, T_W) is "not-$T_{3\frac{1}{2}}$".

3. An Additional Characterization and Properties of Such Spaces

Theorem 3.1. *Let (X, T) be a space and let Y be a proper subset of X. Then* (a) *(Y, T_Y) is a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace of (X, T) iff* (b) *for each subset Z of X that properly contains Y, (Z, T_Z) is "not-$T_{3\frac{1}{2}}$", and each subspace of (Y, T_Y) is $T_{3\frac{1}{2}}$.*

Proof. (a) implies (b): Since (Y, T_Y) is $T_{3\frac{1}{2}}$ and $T_{3\frac{1}{2}}$ is a subspace property, then each subspace of (Y, T_Y) is $T_{3\frac{1}{2}}$. Since (Y, T_Y) is a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace of (X, T), then, by definition, for each subset Z of X that properly contains Y, (Z, T_Z) is "not-$T_{3\frac{1}{2}}$".

(b) implies (a): Since Y is a subspace of itself, then (Y, T_Y) is a proper, $T_{3\frac{1}{2}}$ subspace of (X, T) and since for each subset Z of X that properly contains Y, (Z, T_Z) is "not-$T_{3\frac{1}{2}}$", then, by definition, (Y, T_Y) is a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace of (X, T).

Definition 3.1. A space (X, T) has the maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace property iff (X, T) has a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace (Y, T_Y).

Theorem 3.2. *The maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace property is a topological property.*

Proof. Let (X, T) have the maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace property and let $f : (X, T) \to (Z, S)$ be a homeomorphism. Let (Y, T_Y) be a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace of (X, T). Then $f(Y)$ is a proper subset of Z and since $T_{3\frac{1}{2}}$ is a topological property and $f_{f(Y)} : (Y, T_Y) \to (f(Y), S_{f(Y)})$ is a homeomorphism, then $(f(Y), S_{f(Y)})$ is a proper, $T_{3\frac{1}{2}}$ subspace of (Z, S). Let $z \in Z \setminus f(Y)$ and let $U \in S$ such that $z \in U$. Let x be the unique element of X such that $f(x) = z$. Then $x \notin Y$ and $x \in f^{-1}(U) \in T$ and, since (Y, T_Y) is dense in (X, T), $f^{-1}(U) \cap Y \neq \phi$. Thus $U \cap f(Y) \neq \phi$ and $(f(Y), S_{f(Y)})$ is dense in (Z, S). Suppose there exists a subset W of Z that properly contains $f(Y)$ that is $T_{3\frac{1}{2}}$. Then $f^{-1}(W)$ is a subset of X that properly contains Y and $(f^{-1}(W), T_{f^{-1}(W)})$ is $T_{3\frac{1}{2}}$, which is a contradiction. Thus for each subset W of Z that properly contains $f(Y)$, (W, S_W) is "not-$T_{3\frac{1}{2}}$". Thus $(f(Y), S_{f(Y)})$ is a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace of (Z, S).

Theorem 3.3. *The maximal, proper, dense,* $T_{3\frac{1}{2}}$ *subspace property is not a subspace property.*

Proof. Let (X, T) have the maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace property with maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace (Y, T_Y) having two or more elements. Let a and b be distinct elements of Y, let $Z = \{a\}$, and let $W = \{a, b\}$. Then (Z, T_Z) is a proper, $T_{3\frac{1}{2}}$ subspace of (W, T_W), where (W, T_W) is $T_{3\frac{1}{2}}$ and "not-"not-$T_{3\frac{1}{2}}$"". Thus the maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace property is not a subspace property.

The following example shows that for a space (X, T), a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace of (X, T) need not be unique.

Example 3.1. Let $X = \{a, b\}$, where a and b are distinct, and let T be the indiscrete topology on X. Then each singleton set subspace of (X, T) is a maximal, proper, dense, $T_{3\frac{1}{2}}$ subspace of (X, T) and maximal, proper, dense, $T_{3\frac{1}{2}}$ subspaces are not unique.

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DOI: https://doi.org/10.1007/BF01208659

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