Journal of Algebra, Number Theory: Advances and Applications
Volume 19, Number 1, 2018, Pages 1-27
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jantaa_7100121892

$\mathcal{TK}^{\circ}$-NETWORK OF REGULAR SEMIGROUPS WITH INVERSE TRANSVERSALS

YING-YING FENG and LI-MIN WANG

School of Mathematics and Big Data
Foshan University
Foshan, Guangdong, 528000
P. R. China
e-mail: rickyfungyy@fosu.edu.cn

School of Mathematics
South China Normal University
Guangzhou, Guangdong, 510631
P. R. China
e-mail: wanglm@scnu.edu.cn

Abstract

Let S be a regular semigroup with an inverse transversal S° and $\mathcal{C}(S)$ the congruence lattice of S. Two congruences ρ and θ are $\mathcal{K}^{\circ}$-related if $\operatorname{Ker}\rho^{\circ} = \operatorname{Ker}\theta^{\circ}$, where $\rho^{\circ} = \rho|_{S^{\circ}}$. Various characterizations for congruences in the $\mathcal{K}^{\circ}$-class with $\operatorname{Ker}\rho^{\circ} = S^{\circ}$ and in the $\mathcal{T}$-clsaa with $\operatorname{Ker}(\rho T)^{\circ} = S^{\circ}$ are

2010 Mathematics Subject Classification: 20M17.

Keywords and phrases: regular semigroup, inverse transversal, congruence, semilattice transversal, Q-Clifford transversal, $\mathcal{TK}^{\circ}$-network.

This work is supported by the Outstanding Young Teacher Training Programme of Foshan University (fsyq201408).

Received November 19, 2017

obtained. The $\mathcal{TK}^{\circ}$-operator semigroup for regular semigroups with Q-Clifford transversals is determined and is shown to have 19 elements at most.

Introduction

Congruences on a regular semigroup S have attracted considerable attention which they richly deserve. The most effective approach to congruences on regular semigroups is the kernel-trace approach. As a natural development, we have the kernel (respectively, trace) relation $\mathcal{K}$ (respectively, $\mathcal{T}$), which relates two congruences having the same kernel (respectively, trace). Both of these relations have their classes intervals, so one may speak of the least and the greatest elements of $\mathcal{K}$-and $\mathcal{T}$-classes.

Since Blyth and McFadden introduced regular semigroups with inverse transversals in 1982, this type of semigroups have attracted much attention. A regular semigroup S with an inverse transversal S° can be assembled by using three structural ingredients, namely, S°, I, and Λ, where I and Λ are left and right regular bands, respectively. Wang [19] first investigated congruences on regular semigroups with Q-inverse transversals. Later, Tang and Wang [16] characterized congruences on regular semigroups with inverse transversals. Congruences on regular semigroups with inverse transversals are coordinatized by their restrictions to the three structural ingredients. In addition, Tang and Wang [16] mentioned that congruences on this type of semigroups can also be described by admissible triples which consist of a normal subsemigroup of S°, and restrictions of the congruence to I and Λ. Naturally, a $\mathcal{K}^{\circ}$-relation which relates two congruences having the same kernel in S° is introduced in [3].

A further development of the kernel-trace approach is the study of congruence networks. Petrich-Reilly [13] first investigated the $\mathcal{TK}$-min network of the universal relation ω on inverse semigroups in 1982. In 1988, Alimpić-Krgović [1] generalized results on inverse semigroups to regular ones. Pastijn-Trotter [9] considered congruence networks on completely regular semigroups. Congruence network on completely simple semigropus is determined by Petrich [12]. Section 2 leads the reader to the $\mathcal{TK}^{\circ}$-network of the congruence lattice. It turns out that the $\mathcal{K}^{\circ}$-class with $\operatorname{Ker}\rho^{\circ} = S^{\circ}$ consists exactly of $\mathcal{ST}$-congruences, where $\mathcal{ST}$ denotes the class of regular semigroups with semilattice transversals. And on regular semigroups with Q-inverse transversals, the $\mathcal{T}$-class with $\operatorname{Ker}(\rho T)^{\circ} = S^{\circ}$ consists exactly of $\mathcal{CT}$-congruences, where $\mathcal{CT}$ denotes the class of regular semigroups with Clifford transversals.

Operator semigroup is another way of studying congruence networks. $\mathcal{TK}$-operator semigroups for regular semigroups were first investigated by Petrich [11]. Wang [17], [18] determined the $\mathcal{TK}$-operator semigroups of cryptogroups and bisimple ω-semigroups. Wang [19] obtained the $\mathcal{TK}$-operator semigroups for regular semigroups with Q-inverse transversals. In Section 3, we determine the $\mathcal{TK}^{\circ}$-operator semigroup $\Gamma(S)$ for a regular semigroup S with a Q-Clifford transversal. It is shown that in this case $\Gamma(S)$ contains 19 elements at most.

1. Preliminaries

We will follow the notation and terminology of [4] and [16]. We now list only a few of the most frequently used notations.

Let S be a regular semigroup. An inverse subsemigroup S° of S is said to be an *inverse transversal* if $|V(x) \cap S^{\circ}| = 1$ for any $x \in S$, where $V(x)$ is the set of inverses of x. If further, S° is a quasi-ideal of

S (i.e., $S^\circ SS^\circ \subseteq S^\circ$), then S° is called a *Q-inverse transversal* of S. The unique element in $V(x) \cap S^\circ$ is denoted by x° and $(x^\circ)^\circ$ by $x^{\circ\circ}$. We have $x^{\circ\circ\circ} = x^\circ$. The important sets $\{x \in S | x = xx^\circ\} = \{xx^\circ | x \in S\}$ and $\{x \in S | x = x^\circ x\} = \{x^\circ x | x \in S\}$ are denoted by I and Λ, respectively. We have $L = \{x \in S | x = x^\circ x^{\circ\circ}\} = \{xx^\circ x^{\circ\circ} | x \in S\}$ and $R = \{x \in S | x = x^{\circ\circ} x^\circ x\}$ $= \{x^{\circ\circ} x^\circ x | x \in S\}$. A band B is *left* [resp., *right*] *regular* if $efe = ef$ [resp., $efe = fe$] for every $e, f \in B$. A regular semigroup S is said to be *left* [resp., *right*] *inverse* if each $\mathcal{L}$-class [resp., $\mathcal{R}$-class] of S contains a unique idempotent. For a subset X of S, $X^\circ = \{x^\circ \in S^\circ | x \in X\}$ and $E(X) = \{e \in X | e^2 = e\}$. Throughout the paper, *S always denotes a regular semigroup with an inverse transversal S°* if no special mention is made.

The following known results will be used frequently without special reference in this paper.

Result 1.1. Let S be a regular semigroup with an inverse transversal. Then we have

(1) $(xy)^\circ = y^\circ (xyy^\circ)^\circ = (x^\circ xy)^\circ x^\circ = y^\circ (x^\circ xyy^\circ)^\circ x^\circ$ for every $x, y \in S$;

(2) $(xy^\circ)^\circ = y^{\circ\circ} x^\circ$;

(3) ([14]) S is orthodox if and only if $(xy)^\circ = y^\circ x^\circ$ for every $x, y \in S$;

(4) ([15]) I and Λ are left and right regular bands with a common inverse transversal $E(S^\circ)$, respectively.

Let γ be a relation on S. The congruence generated by γ is denoted by γ^*. If γ is an equivalence on S, then $\gamma^\flat$ is the greatest congruence on S contained in γ. It is known that for any $a, b \in S$, $a \gamma^\flat b \Leftrightarrow (xay)\ \gamma\ (xby)$ for all $x, y \in S^1$. Let $X \subseteq S$. We denote by π_X the equivalence that corresponds with the partitioning of S into X and $S \setminus X$. A congruence ρ *saturates* X if X is a union of ρ-classes. In particular, a congruence ρ on S is *idempotent pure* if ρ saturates $E(S)$. It is easy to see that $\pi_X^{\flat}$ is the greatest congruence that saturates X. A congruence ρ on S is *idempotent separating* if $e\rho f$ imply that $e = f$ for every $e, f \in E(S)$.

For a regular semigroup S, the *lattice of congruences* on S is denoted by $\mathcal{C}(S)$. For any ρ in $\mathcal{C}(S)$, the *kernel* and *trace* of ρ are defined as

$$\operatorname{Ker} \rho = \{x \in S | \, (x, x^2) \in \rho\} \quad \text{and} \quad \operatorname{tr}\rho = \rho|_{E(S)},$$

respectively. The basic fact is that ρ is uniquely determined by the congruence pair $(\operatorname{Ker} \rho, \operatorname{tr}\rho)$. Pastijn and Petrich [7] investigated two equivalence relations $\mathcal{T}$ and $\mathcal{K}$ on $\mathcal{C}(S)$ for a regular semigroup S:

$$\rho\mathcal{T}\theta \Leftrightarrow \operatorname{tr}\rho = \operatorname{tr}\theta, \quad \rho\mathcal{K}\theta \Leftrightarrow \operatorname{Ker} \rho = \operatorname{Ker} \theta.$$

The $\mathcal{T}$-class and $\mathcal{K}$-class are both intervals in $\mathcal{C}(S)$, whose least (resp., greatest) elements are denoted by ρt and ρk (resp., ρT and ρK). Thus, we have four operators T, t, K, k on $\mathcal{C}(S)$. Let $\Gamma = \{T, t, K, k\}$. We denote by Γ^+ the free semigroup generated by Γ. For any $\rho \in \mathcal{C}(S)$, letting Γ^+ act on ρ, we obtain a $\mathcal{TK}$*-network of congruences* $\rho, \rho T, \rho t, \rho K, \rho k, \rho Tk, \ldots$, ordered by inclusion.

In 1988, Pastijn and Petrich [8] defined two more abstract equivalence relations $\mathcal{U}$ and $\mathcal{V}$ on the congruence lattice of a regular semigroup S:

$$\rho\mathcal{U}\theta \Leftrightarrow \rho \cap \leq = \theta \cap \leq, \quad \rho\mathcal{V}\theta \Leftrightarrow \rho\mathcal{U}\theta \text{ and } \rho\mathcal{K}\theta,$$

where $\leq$ denotes the natural partial order on $E(S)$, i.e., $e \leq f \Leftrightarrow ef = fe = e$. Similar to $\mathcal{T}$ and $\mathcal{K}$, for any congruence ρ, the equivalence classes $\rho\mathcal{U}$ and $\rho\mathcal{V}$ are also intervals with least (resp., greatest) elements ρu and ρv (resp., ρU and ρV). We have a set of operators $\Omega = \{T, t, V, v\}$ and the $\mathcal{TV}$-network of congruences $\rho\Omega^+ \cup \{\rho\}$.

Congruences on inverse semigroups are characterized abstractly by the so-called congruence pairs ([10]). Let S be an inverse semigroup. An inverse subsemigroup K of S is *normal* if $E(S) \subseteq K$ and $s^{-1}Ks \subseteq K$ for all $s \in S$. A congruence τ on $E(S)$ is *normal* for S if for any $e, f \in E(S)$ and $s \in S$, $e \tau f$ implies $s^{-1}es \tau s^{-1}fs$. A pair (K, τ) is a *congruence pair* for S if K is a normal subsemigroup of S and τ is a normal congruence on $E(S)$ satisfying:

(1) $ae \in K, e \tau a^{-1}a \Rightarrow a \in K (a \in S, e \in E(S))$;

(2) $k \in K \Rightarrow kk^{-1} \tau k^{-1}k$.

For a congruence pair (K, τ), define a relation $\rho_{(K, \tau)}$ on S by

$$a\rho_{(K, \tau)}b \Leftrightarrow a^{-1}a\tau b^{-1}b, ab^{-1} \in K.$$

Result 1.2 ([10])**.** Let S be an inverse semigroup. If (K, τ) is a congruence pair, then $\rho_{(K, \tau)}$ is the unique congruence ρ on S with $\operatorname{Ker} \rho = K$ and $\operatorname{tr} \rho = \tau$. Conversely, every congruence on S can be so obtained.

Congruences on regular semigroups with inverse transversals were first studied by Wang [19]. In [19], for a regular semigroup with a Q-inverse transversal, the congruences were coordinated by triples consisting of congruences on I, S°, and Λ, and the congruence lattice was investigated as well. Later, Tang and Wang described congruences on regular semigroups with inverse transversals in [16]. We present some notations and results taken from [16] below for our discussions in the sequel.

Let S be a regular semigroup with an inverse transversal S°. For a congruence ρ on S, we denote $\rho|_{S^{\circ}} = \rho^{\circ}$, $\rho|_{I} = \rho_{I}$, and $\rho|_{\Lambda} = \rho_{\Lambda}$.

Result 1.3. For any congruence ρ on S, $x\rho y$ implies $x^\circ \rho y^\circ$.

A congruence τ on I is said to be *normal* if it satisfies the following condition:

$$(\forall e, f \in I, g \in \Lambda, a \in S^\circ) e\tau f \Rightarrow (ge)(ge)^\circ \tau (gf)(gf)^\circ, a^\circ ea \tau a^\circ fa.$$

The normal congruences on Λ may be defined dually.

Let τ_I and τ_Λ be normal congruences on I and Λ, respectively, and let $\pi \in \mathcal{C}(S^\circ)$. The triple $(\tau_I, \pi, \tau_\Lambda)$ is called a *congruence triple* for S, if it satisfies the following conditions:

(CT1) $(\forall e, f \in I, g \in \Lambda)\ e\tau_I f \Rightarrow (ge)^\circ \pi (gf)^\circ, (ge)^\circ (ge)\ \tau_\Lambda (gf)^\circ (gf),$

$(\forall g, h \in \Lambda, e \in I)\ g\tau_\Lambda h \Rightarrow (ge)^\circ \pi (he)^\circ, (ge)(ge)^\circ\ \tau_I (he)(he)^\circ;$

(CT2) $(\forall e \in I, g \in \Lambda, a, b \in S^\circ)\ a\pi b \Rightarrow a^\circ ea \tau_I b^\circ eb, aga^\circ \tau_\Lambda bgb^\circ.$

Define a relation $\rho_{(\tau_I, \pi, \tau_\Lambda)}$ on S by the following rule:

$$x\rho_{(\tau_I, \pi, \tau_\Lambda)} y \Leftrightarrow xx^\circ \tau_I yy^\circ, x^\circ \pi y^\circ, x^\circ x \tau_\Lambda y^\circ y.$$

Result 1.4 ([16])**.** Let S be a regular semigroup with an inverse transversal S°. For each congruence triple $(\tau_I, \pi, \tau_\Lambda)$ for S, the relation $\rho_{(\tau_I, \pi, \tau_\Lambda)}$ is the unique congruence on S whose restrictions on I, S°, Λ are $\tau_I, \pi, \tau_\Lambda$, respectively. Conversely, every congruence on S can be obtained in this way.

Result 1.5. Let S be a regular semigroup with an inverse transversal S°. For any congruences ρ and θ,

$$\rho \mathcal{T} \theta \Leftrightarrow \rho_I = \theta_I, \rho_\Lambda = \theta_\Lambda;$$

$$\rho \mathcal{U} \theta \Leftrightarrow \rho|_{E(S^\circ)} = \theta|_{E(S^\circ)}; \quad \rho \mathcal{V} \theta \Leftrightarrow \rho^\circ = \theta^\circ.$$

Substituting the component π in the congruence triple with a normal subsemigroup K° of S°, Tang and Wang arrived at the notion of an admissible triple which is similar to the notion of a congruence triple which was used by Pastijn and Petrich in [7] to describe congruences on regular semigroups. Let S be a regular semigroup with an inverse transversal S°. Assume that τ_I and τ_Λ are normal congruences on I and Λ, respectively, and that K° is a normal subsemigroup of S°. The triple $(\tau_I, K^\circ, \tau_\Lambda)$ is an *admissible triple* for S if it satisfies the following conditions:

(i) (τ_I°, K°) is a congruence pair for S°;

(ii) $(\forall e, f \in I, g \in \Lambda) e\tau_I f \Rightarrow (ge)^\circ(gf)^{\circ\circ} \in K^\circ, (ge)^\circ(ge)\tau_\Lambda(gf)^\circ(gf)$,

$(\forall g, h \in \Lambda, e \in I)\ g\tau_\Lambda h \Rightarrow (ge)^\circ(he)^{\circ\circ} \in K^\circ, (ge)(ge)^\circ\tau_I(he)(he)^\circ$;

(iii) $(\forall e \in I, g \in \Lambda, a, b \in S^\circ)\ a^\circ a\tau_I b^\circ b, ab^\circ \in K^\circ \Rightarrow a^\circ ea\tau_I b^\circ eb$,

$aga^\circ\tau_\Lambda bgb^\circ$.

Define a relation on S by

$$x\eta_{(\tau_I, K^\circ, \tau_\Lambda)}y \Leftrightarrow xx^\circ\tau_I yy^\circ,\ x^\circ x\tau_\Lambda y^\circ y,\ x^\circ y^{\circ\circ} \in K^\circ.$$

By combining Result 1.2 and Result 1.4, congruences on S can be produced in another way.

Result 1.6 ([16]). Let S be a regular semigroup with an inverse transversal S°. For an admissible triple $(\tau_I, K^\circ, \tau_\Lambda)$, the relation $\eta_{(\tau_I, K^\circ, \tau_\Lambda)}$ is the unique congruence ρ on S with $\rho|I = \tau_I$, $\rho|_\Lambda = \tau_\Lambda$ and $\operatorname{Ker}\rho^\circ = K^\circ$. Conversely, every congruence on S can be so obtained.

Notice that $\operatorname{Ker} \rho^\circ = \operatorname{Ker} \rho \cap S^\circ$, Feng and Wang [3] define a relation $\mathcal{K}^\circ$ on $\mathcal{C}(S)$. For any $\rho, \theta \in \mathcal{C}(S)$, let

$$\rho \mathcal{K}^\circ \theta \Leftrightarrow \operatorname{Ker} \rho^\circ = \operatorname{Ker} \theta^\circ.$$

It is also provided expressions for the least and the greatest congruences in the same $\mathcal{K}^\circ$-class as a congruence ρ.

Result 1.7 (1)**.** Let S be a regular semigroup with an inverse transversal S°. For any $\rho \in \mathcal{C}(S)$, $\rho\mathcal{K}^\circ = [\rho k^\circ, \rho K^\circ]$, where

$$\rho k^\circ = \{(x, x^2) \in S \times S | x \in \operatorname{Ker} \rho^\circ\}^*,$$

$$\rho K^\circ = \{(x, y) \in S \times S | (\forall e \in I, g \in \Lambda)(gxe)^\circ \pi^{(S^\circ)}_{\operatorname{Ker} \rho^\circ} (gye)^\circ\},$$

where $\pi^{(S^\circ)}_{\operatorname{Ker} \rho^\circ}$ is the greatest congruence on S° saturating $\operatorname{Ker} \rho^\circ$, i.e.,

$$a \pi^{(S^\circ)}_{\operatorname{Ker} \rho^\circ} b \Leftrightarrow [(\forall x, y \in (S^\circ)^1)\ xay \in \operatorname{Ker} \rho^\circ \Leftrightarrow xby \in \operatorname{Ker} \rho^\circ].$$

(2) Let S be a regular semigroup with a Q-inverse transversal S°. For any $\rho \in \mathcal{C}(S)$, $\rho\mathcal{K}^\circ = [\rho k^\circ, \rho K^\circ]$, where

$$\rho k^\circ = \{(x, x^2) \in S \times S | x \in \operatorname{Ker} \rho^\circ\}^*,$$

$$\rho K^\circ = \{(x, y) \in S \times S | (\forall e \in I, g \in \Lambda) gxe\, \pi^{(S^\circ)}_{\operatorname{Ker} \rho^\circ} gye\},$$

where $\pi^{(S^\circ)}_{\operatorname{Ker} \rho^\circ}$ is the greatest congruence on S° that saturates $\operatorname{Ker} \rho^\circ$.

The greatest congruence in the same $\mathcal{T}$-class as a congruence ρ will be of great use in the sequel.

Result 1.8. Let S be a regular semigroup with an inverse transversal S°. For any $\rho \in \mathcal{C}(S)$,

$$\rho T = \{(x, y) \in S \times S \mid (\forall e \in I, g \in \Lambda)\ (xe)(xe)^\circ \rho_I (ye)(ye)^\circ, (gx)^\circ (gx) \rho_\Lambda (gy)^\circ (gy)\}.$$

Remind that $\mathcal{T} \cap \mathcal{K}^\circ = \varepsilon$.

2. Congruence Classes Containing the Universal Relation and the Related Minimal Congruences

We shall discuss the congruence classes containing the universal relation and the related minimal congruences in this section. As a result we describe semilattice transversal congruences, Q-Clifford transversal congruences and congruences which are idempotent pure in S°. The following relations will be useful.

Definition 2.1. On any regular semigroup S with an inverse transversal S°, define relations $\mathscr{F}$ and $\mathscr{C}$ by

$$a \mathscr{F} b \Leftrightarrow a^{\circ\circ} b^\circ \in E(S^\circ), \quad a \mathscr{C} b \Leftrightarrow a^{\circ\circ} b^\circ, a^\circ b^{\circ\circ} \in E(S^\circ).$$

Definition 2.2. Let S be a regular semigroup with an inverse transversal S°. If further S° is a semilattice, then we call S° is a *semilattice transversal* of S. We denote by $\mathcal{ST}$ this class of semigroups.

Lemma 2.3. *The following conditions on a regular semigroup S with an inverse transversal S° are equivalent*:

(1) S° *is a semilattice transversal of S*;

(2) $S^\circ = E(S^\circ)$;

(3) $L = I$;

(4) $R = \Lambda$.

Proof. (1) $\Rightarrow$ (2). Since transversals are isomorphic, it follows that S° is a semilattice so that $S^{\circ} \subseteq E(S^{\circ})$. Hence $S^{\circ} = E(S^{\circ})$.

(2) $\Rightarrow$ (3). Note that for any $xx^{\circ}x^{\circ\circ} \in L$, we have $x^{\circ} \in S^{\circ} = E(S^{\circ})$ which implies that $x^{\circ\circ} = x^{\circ} = x^{\circ}x^{\circ}$ and that $xx^{\circ}x^{\circ\circ} = xx^{\circ}x^{\circ} = xx^{\circ} \in I$, i.e., $L \subseteq I$ and $L = I$.

(3) $\Rightarrow$ (1). Let $L = I$. Since $S^{\circ} \subseteq L = I \subseteq E(S)$, it follows that S° is both an inverse semigroup and a band, and thus also a semilattice.

(1) $\Leftrightarrow$ (4). The proof is similar to that for (1) $\Leftrightarrow$ (3) and is omitted.

□

Proposition 2.4. *The homomorphic image of a regular semigroup with a semilattice transversal is still a regular semigroup with a semilattice transversal.*

Proof. Let S be a regular semigroup with a semilattice transversal, and $\rho \in \mathcal{C}(S)$. By ([16], Theorem 5.2), S/ρ is a regular semigroup with an inverse transversal. For any $a\rho \in (S/\rho)^{\circ}$, by ([2], Lemma 2.2), there exists $s^{\circ} \in S^{\circ}$ such that $a\rho = s^{\circ}\rho$. Since S° is a semilattice, we have $E(S^{\circ}) = S^{\circ}$, which yields $(a\rho)^2 = (s^{\circ}\rho)^2 = (s^{\circ})^2\rho = s^{\circ}\rho = a\rho$. Therefore $(S/\rho)^{\circ} = E((S/\rho)^{\circ})$, which establishes that S/ρ is a regular semigroup with a semilattice transversal by Lemma 2.3. □

Proposition 2.5. *The following statements concerning a congruence* ρ *on S are equivalent*:

(1) ρ *is an* $\mathcal{ST}$*-congruence*;

(2) $\operatorname{Ker} \rho^{\circ} = S^{\circ}$;

(3) $\mathcal{L}|_{S^{\circ}} \subseteq \rho^{\circ}$;

(4) $\rho^\circ \mathscr{C} \rho^\circ = \omega_{S^\circ}$;

(5) $\rho^\circ \mathscr{F} \rho^\circ = \omega_{S^\circ}$;

(6) ρ° *is a semilattice congruence on* S°.

Proof. The equivalence of (2) and (6) follows from ([10], III.6).

(1) $\Rightarrow$ (2). Let $a \in S^\circ$. From the hypothesis it follows, in view of Lemma 2.3, that $a\rho \in E((S/\rho)^\circ)$. We now infer from ([2], Proposition 2.2) that $a\rho = e^\circ\rho$ for some $e^\circ \in E(S^\circ)$. We further remark that $a \in \operatorname{Ker}\rho^\circ$ with $\operatorname{Ker}\rho^\circ = S^\circ$.

(2) $\Rightarrow$ (3). Notice first that if $\operatorname{Ker}\rho^\circ = S^\circ$ and $a \in \operatorname{Ker}\rho^\circ$, then $a\rho^\circ e^\circ$ for some $e^\circ \in E(S^\circ)$. Thus $a^\circ\rho^\circ(e^\circ)^\circ = e^\circ$ and so $a^\circ a\rho e^\circ\rho a$. Next let $a, b \in S^\circ$ be such that $a\mathcal{L}b$. Then $a^\circ a = b^\circ b$ by ([2], Proposition 2.1). Further $\operatorname{Ker}\rho^\circ = S^\circ$ gives $a\rho^\circ a^\circ a = b^\circ b\rho^\circ b$ which implies that $\mathcal{L}|_{S^\circ} \subseteq \rho^\circ$.

(3) $\Rightarrow$ (1). Suppose that $\mathcal{L}|_{S^\circ} \subseteq \rho^\circ$. In view of Lemma 2.3, to show that $S/\rho \in \mathcal{ST}$, it will be sufficient to show that $a\rho \in E((S/\rho)^\circ)$ if $a \in S^\circ$. Let $a \in S^\circ$, then $a\mathcal{L}|_{S^\circ} a^\circ a$. From the hypothesis we have $a\rho^\circ a^\circ a$. Again, since $a^\circ a \in E(S^\circ)$, this implies that $a\rho \in E((S/\rho)^\circ)$. In view of Lemma 2.3, we know that $S/\rho \in \mathcal{ST}$.

(2) $\Rightarrow$ (4). Let $a, b \in S^\circ$. Since $\operatorname{Ker}\rho^\circ = S^\circ$, we have $a\rho^\circ aa^\circ$ and $b\rho^\circ bb^\circ$. Also, $aa^\circ, bb^\circ \in E(S^\circ)$, while $aa^\circ \mathscr{C} bb^\circ$. Thus $a\rho^\circ \mathscr{C}^\circ\rho^\circ b$ and the claim follows.

(4) $\Rightarrow$ (5). This is obvious since $\mathscr{C} \subseteq \mathscr{F}$.

(5) $\Rightarrow$ (2). For any $a \in S^\circ$, we have $a\rho^\circ x \mathscr{F} y\rho^\circ a^\circ a$ for some $x, y \in S^\circ$. Hence $y^\circ\rho^\circ(a^\circ a)^\circ = a^\circ a^{\circ\circ} = a^\circ a$. Again, since $x \mathscr{F} y$, this implies that $x^{\circ\circ}y^\circ \in E(S^\circ)$ and therefore $xy^\circ \in E(S^\circ)$, which in conjunction with the foregoing gives $a = a(a^\circ a)\rho xy^\circ \in E(S^\circ)$. Thus $a \in \operatorname{Ker}\rho^\circ$ and so $\operatorname{Ker}\rho^\circ = S^\circ$. □

Notation 2.6. For any regular semigroup S with an inverse transversal S°, let $\tilde{\eta}$ denote the least $\mathcal{ST}$-congruence on S. We clearly have $\tilde{\eta} = \omega k^\circ = (\mathcal{L}|_{S^\circ})^*$ in view of $\mathcal{L}|_{S^\circ} \subseteq \rho^\circ \Leftrightarrow \mathcal{L}|_{S^\circ} \subseteq \rho$.

Definition 2.7. Let S be a regular semigroup with an inverse transversal S°. If S° is a Clifford semigroup, then S° is called a *Clifford transversal*. We denote by $\mathcal{CT}$ this class of semigroups.

In order to describe $\mathcal{CT}$-congruences on regular semigroups with Q-inverse transversals we first need an auxiliary result.

Lemma 2.8. *Let ρ be a congruence on a regular semigroup S with a Q-inverse transversal S°. Then $(\rho T)^\circ = (\rho^\circ)^{T(S^\circ)}$, where $(\rho^\circ)^{T(S^\circ)}$ is the greatest congruence on S° having the same trace as ρ°.*

Proof. Notice first that if $e^\circ, f^\circ \in E(S^\circ)$ and $e^\circ(\rho T)^\circ f^\circ$, then $e^\circ\rho T\, f^\circ$. But $\rho T|_{E(S^\circ)} = \rho|_{E(S^\circ)}$ and so $e^\circ\, \rho\, f^\circ$. Thus $e^\circ\, \rho^\circ\, f^\circ$ and so $\operatorname{tr}(\rho T)^\circ \subseteq \operatorname{tr}\rho^\circ$. Again, since $\rho \subseteq \rho T$, this implies that $\rho^\circ \subseteq (\rho T)^\circ$, and therefore $\rho^\circ \subseteq \operatorname{tr}(\rho T)^\circ$, which together with the foregoing easily implies that $\operatorname{tr}(\rho T)^\circ, = \operatorname{tr}\rho^\circ$. Next, consider $\xi \in \mathcal{C}(S^\circ)$ with $\operatorname{tr}\xi = \operatorname{tr}\rho^\circ$. By ([19], Lemma 4.2), we can assume that $\lambda^\circ = \xi$ for some $\lambda \in \mathcal{C}(S)$. If $a, b \in S^\circ$

are such that $a\xi b$, then $a\lambda b$. Thus $(ae)(ae)^{\circ}\lambda_I(be)(be)^{\circ}$, $(ga)^{\circ}(ga)\lambda_{\Lambda}$ $(gb)^{\circ}(gb)$ for all $e \in I$ and $g \in \Lambda$. But $(ae)(ae)^{\circ} = aea^{\circ} \in S^{\circ}IS^{\circ} \subseteq S^{\circ}$, $(ga)^{\circ}(ga) = a^{\circ}ga \in S^{\circ}\Lambda S^{\circ} \subseteq S^{\circ}$ and so $(ae)(ae)^{\circ}, (ga)^{\circ}(ga) \in E(S^{\circ})$. Also, $\lambda|_{E(S^{\circ})} = \xi|_{E(S^{\circ})} = \rho^{\circ}|_{E(S^{\circ})} = \rho|_{E(S^{\circ})}$ and hence $(ae)(ae)^{\circ}\rho_I(be)(be)^{\circ}$, $(ga)^{\circ}(ga)\rho_{\Lambda}(gb)^{\circ}(gb)$, which implies that $a\rho Tb$, by ([20], Theorem 2.3). Hence $a(\rho T)^{\circ}b$ and $\xi \subseteq (\rho T)^{\circ}$, which proves that $(\rho T)^{\circ} = (\rho^{\circ})^{T(S^{\circ})}$.

□

Proposition 2.9. *The following statements concerning a congruence* ρ *on a regular semigroup with a Q-inverse transversal are equivalent*:

(1) ρ *is a* $\mathcal{CT}$*-congruence*;

(2) ρT *is an* $\mathcal{ST}$*-congruence*;

(3) $\rho T = \rho \vee \tilde{\eta}$, *where* $\tilde{\eta}$ *is the least* $\mathcal{ST}$*-congruence*;

(4) $\operatorname{tr} \rho \supseteq \operatorname{tr} \tilde{\eta}$.

Proof. (1) $\Rightarrow$ (2). Since ρ is a $\mathcal{CT}$-congruence, we have that S°/ρ° is a Clifford semigroup. In view of ([10], Proposition III.6.5) we know that $(\rho^{\circ})^{T(S^{\circ})}$ is a semilattice congruence on S° and so $S^{\circ}/(\rho T)^{\circ}$ is a semilattice by Lemma 2.8. From ([16], Theorem 5.2), we infer that $S^{\circ}/(\rho T)^{\circ}$ is an inverse transversal of $S/\rho T$ and it follows that $S/\rho T$ has a semilattice transversal, which gives that ρT is an $\mathcal{ST}$-congruence.

(2) $\Rightarrow$ (3). That ρT is an $\mathcal{ST}$-congruence yields $\rho T \supseteq \tilde{\eta}$. Further, $\rho T \supseteq \rho$ gives $\rho T \supseteq \rho \vee \tilde{\eta}$. Also $\operatorname{Ker}(\rho T)^{\circ} \subseteq S^{\circ} = \operatorname{Ker}(\tilde{\eta})^{\circ} \subseteq \operatorname{Ker}(\rho \vee \tilde{\eta})^{\circ}$, $\operatorname{tr} \rho T = \operatorname{tr} \rho \subseteq \operatorname{tr}(\rho \vee \tilde{\eta})$, and thus by ([16], Theorem 4.5), we have $\rho T \subseteq \rho \vee \tilde{\eta}$, and whence $\rho T = \rho \vee \tilde{\eta}$.

(3) $\Rightarrow$ (4). Clearly, $\operatorname{tr}\rho = \operatorname{tr}\rho T = \operatorname{tr}(\rho \vee \tilde{\eta}) \supseteq \operatorname{tr}\tilde{\eta}$.

(4) $\Rightarrow$ (1). The hypothesis gives $\rho T \supseteq (\tilde{\eta})T \supseteq \tilde{\eta}$ whence ρT is an $\mathcal{ST}$-congruence by Proposition 2.4. Therefore $a\,\rho T\, aa^{\circ}\rho T\, a^{\circ}a$ for all $a \in S^{\circ}$. Remark further that for any $e^{\circ} \in E(S^{\circ})$, since $a\,\rho T\, aa^{\circ}$, we have

$$(ae^{\circ})(ae^{\circ})^{\circ}\rho(aa^{\circ}e^{\circ})(aa^{\circ}e^{\circ})^{\circ} = aa^{\circ}e^{\circ}aa^{\circ} = e^{\circ}aa^{\circ},$$

the last equality holds from the fact that $aa^{\circ}, e^{\circ} \in E(S^{\circ})$ and that $E(S^{\circ})$ is a semilattice. Furthermore,

$$ae^{\circ} = (ae^{\circ})(ae^{\circ})^{\circ}(ae^{\circ})\rho(e^{\circ}aa^{\circ})(ae^{\circ}) = e^{\circ}ae^{\circ}.$$

Again, since $a\rho Ta^{\circ}a$, we have

$$(e^{\circ}a)^{\circ}(e^{\circ}a)\rho(e^{\circ}a^{\circ}a)^{\circ}(e^{\circ}a^{\circ}a) = a^{\circ}ae^{\circ}a^{\circ}a = a^{\circ}ae^{\circ}.$$

Then

$$e^{\circ}a = (e^{\circ}a)^{\circ}(e^{\circ}a)^{\circ}(e^{\circ}a)\rho(e^{\circ}a)(a^{\circ}ae^{\circ}) = e^{\circ}ae^{\circ},$$

and therefore also $ae^{\circ}\rho e^{\circ}a$ which implies that ρ is a $\mathcal{CT}$-congruence. $\square$

Notation 2.10. For any regular semigroup S with an inverse transversal S°, let $\tilde{\nu}$ denote the least $\mathcal{CT}$-congruence on S. Clearly, on regular semigroups with Q-inverse transversals, $\tilde{\nu} = (\tilde{\eta})t$.

If the inverse transversal S° of S is both a Q-inverse transversal and a Clifford transversal, we call it a *Q-Clifford transversal*. The next result provides a number of different characterizations of regular semigroups with Q-Clifford transversals.

Proposition 2.11. *The following conditions on a regular semigroup S with a Q-inverse transversal S° are equivalent*:

(1) *S is a regular semigroup with a Q-Clifford transversal S°*;

(2) *for every $\rho \in \mathcal{C}(S)$, ρT is an $\mathcal{ST}$-congruence*;

(3) *for every $\rho \in \mathcal{C}(S)$, ρk° is idempotent separating*;

(4) $\mu = \tilde{\eta}$;

(5) *for every $\rho \in \mathcal{C}(S)$, $\rho T = \rho \vee \tilde{\eta}$*;

(6) *for every $\rho \in \mathcal{C}(S)$, $\rho k^\circ = \rho \cap \mu$.*

Proof. The equivalence of (2) and (5) follows from Proposition 2.9.

(1) $\Rightarrow$ (2). Notice that both regular semigroups with Q-inverse transversals and Clifford semigroups preserve homomorphic images. It follows that regular semigroups with Q-Clifford transversals preserve homomorphic images. Thus for every $\rho \in \mathcal{C}(S)$, S/ρ has a Q-Clifford transversal and so, by Proposition 2.9, ρT is an $\mathcal{ST}$-congruence.

(2) $\Rightarrow$ (4). It follows from the hypothesis that $\mu = \varepsilon T \supseteq \tilde{\eta}$. Now $\operatorname{tr} \mu = \varepsilon_{E(S)} \subseteq \operatorname{tr} \tilde{\eta}$, $\operatorname{Ker} \mu^\circ \subseteq S^\circ = \operatorname{Ker} (\tilde{\eta})^\circ$, which implies that $\mu \subseteq \tilde{\eta}$ and hence $\mu = \tilde{\eta}$.

(4) $\Rightarrow$ (6). For every $\rho \in \mathcal{C}(S)$, $\rho k^\circ \subseteq \omega k^\circ = \tilde{\eta} = \mu$ and thereby, $\rho k^\circ \subseteq \rho \cap \mu$. We have $\operatorname{tr}(\rho \cap \mu) \subseteq \operatorname{tr} \mu = \varepsilon_{E(S)} \subseteq \operatorname{tr} \rho k^\circ$, $\operatorname{Ker} (\rho \cap \mu)^\circ \subseteq \operatorname{Ker} \rho^\circ = \operatorname{Ker} (\rho k^\circ)^\circ$ and thus $\rho \cap \mu \subseteq \rho k^\circ$ whence $\rho k^\circ = \rho \cap \mu$.

(6) $\Rightarrow$ (3). It is obvious, since $\rho k^\circ = \rho \cap \mu \subseteq \mu$ for every $\rho \in \mathcal{C}(S)$.

(3) $\Rightarrow$ (1). Our hypothesis claims that $\tilde{\eta} = \omega k^\circ \subseteq \mu$. Now $\mathrm{tr}\mu = \varepsilon_{E(S)} \subseteq \mathrm{tr}\,\tilde{\eta}$ and $\mathrm{Ker}\,\mu^\circ \subseteq S^\circ = \mathrm{Ker}\,(\tilde{\eta})^\circ$. Thus $\mu \subseteq \tilde{\eta}$ and so $\mu = \tilde{\eta}$. As a result we see that $\tilde{\nu} = (\tilde{\eta})t = \mu t = \varepsilon$ and that S has a Q-Clifford transversal.

Finally, we consider congruences which are idempotent pure in S°.

Proposition 2.12. *The following statements concerning a congruence* ρ *on a regular semigroup* S *with an inverse transversal* S° *are equivalent.*

(1) ρ° *is idempotent pure in* S°;

(2) $\rho^\circ \cap \mathcal{L} = \varepsilon$;

(3) $\rho \subseteq \mathscr{C}$;

(4) $\rho \subseteq \mathscr{F}$;

(5) ρ° *is over semilattices.*

Proof. The equivalence of (1), (2) and (5) is a consequence of the remark after ([10], Definition III.4.1) and ([10], Proposition III.4.2). (3) trivially implies (4).

(2) $\Rightarrow$ (3). Let $a\rho b$. Then $a^{\circ\circ}b^\circ = a^{\circ\circ}b^\circ b^{\circ\circ}b^\circ \rho b^{\circ\circ}a^\circ a^{\circ\circ}b^\circ = (a^{\circ\circ}b^\circ)^\circ (a^{\circ\circ}b^\circ)$ and since also $a^{\circ\circ}b^\circ \mathcal{L} (a^{\circ\circ}b^\circ)^\circ (a^{\circ\circ}b^\circ)$, the hypothesis yields $a^{\circ\circ}b^\circ = (a^{\circ\circ}b^\circ)^\circ (a^{\circ\circ}b^\circ)$ and thus $a^{\circ\circ}b^\circ \in E(S^\circ)$. One shows similarly that $a^\circ b^{\circ\circ} \in E(S^\circ)$, and hence $\rho \subseteq \mathscr{C}$.

(4) $\Rightarrow$ (1). Let $a \in \mathrm{Ker}\,\rho^\circ$. Then $a \in S^\circ$ and $a\rho^\circ e^\circ$ for some $e^\circ \in E(S^\circ)$. Thus $a^\circ \rho e^\circ$ and $a^\circ a \rho e^\circ \rho a$. By hypothesis, we have $a \mathscr{F} a^\circ a$ so that $a^{\circ\circ}(a^\circ a)^\circ = a^{\circ\circ}a^\circ a^{\circ\circ} = a^{\circ\circ} = a \in E(S^\circ)$ which implies that ρ° is idempotent pure in S°.

3. $\mathcal{TK}^{\circ}$-Operator Semigroups for Regular Semigroups with *Q*-Clifford Transversals

For a regular semigroup S with an inverse transversal S° and its congruence ρ, it is natural to consider the free semigroup Γ^{+} generated by the four operators

$$T : \lambda \mapsto \lambda T, \quad t : \lambda \mapsto \lambda t, \quad K^{\circ} : \lambda \mapsto \lambda K^{\circ}, \quad k^{\circ} : \lambda \mapsto \lambda k^{\circ}.$$

Here $\lambda T, \lambda t, \lambda K^{\circ}, \lambda k^{\circ}$ denote, respectively, the greatest and the least congruence on S in the same $\mathcal{T}$-and $\mathcal{K}^{\circ}$-class as λ and $\Gamma = \{T, t, K^{\circ}, k^{\circ}\}$. For any $\rho \in \mathcal{C}(S)$, letting Γ^{+} act on ρ, we obtain a $\mathcal{TK}^{\circ}$-*network of congruences* $\rho, \rho T, \rho t, \rho K^{\circ}, \rho k^{\circ}, \rho TK^{\circ}, \rho Tk^{\circ}, \ldots$, ordered by inclusion. In this section, we determine the semigroup $\Gamma^{+} / \sum^{*}$ generated by Γ with relations valid in all networks of congruences on regular semigroups with Q-Clifford transversals. We provide a system of representatives Ω for the congruence $\sum^{*}$ on Γ^{+}, and prove that $\Gamma^{+} / \sum^{*}$ is just the $\mathcal{TK}^{\circ}$-operator semigroup of this class of semigroups. Throughout this section, *S denotes a regular semigroup with a Q-Clifford transversal*.

First we need an example to illustrate that a regular semigroup with a Q-Clifford transversal is not necessarily a Clifford semigroup, and hence our study of the $\mathcal{TK}^{\circ}$-network of congruences on regular semigroups with Q-Clifford transversals is not trivial comparing with the $\mathcal{TK}$-network of congruences on Clifford semigroups. It is well known that any completely simple semigroup has a Q-inverse transversal. For a completely simple semigroup $S = \mathcal{M}(I, G, \Lambda; P)$, we may assume that I and Λ have a common element 1. Let $S^{\circ} = \{(1, a, 1) \mid a \in G\}$. Then $S^{\circ} \simeq G$ and S° is a Q-inverse transversal of S. Hence S° is a Q-Clifford transversal of S. For $(i, p_{\lambda i}^{-1}, \lambda) \in E(S)$ and $(j, b, \mu) \in S$, $(i, p_{\lambda i}^{-1}, \lambda)$ $(j, b, \mu) = (i, p_{\lambda i}^{-1} p_{\lambda j} b, \mu)$, $(j, b, \mu)(i, p_{\lambda i}^{-1}, \lambda) = (j, bp_{\mu i} p_{\lambda i}^{-1}, \lambda)$. $(i, p_{\lambda i}^{-1}, \lambda)$ $(j, b, \mu) \neq (j, b, \mu)(i, p_{\lambda i}^{-1}, \lambda)$ if $i \neq j$. So S is not a Clifford semigroup.

Lemma 3.1. *Let S be a regular semigroup with an inverse transversal.*

(1) *For any group congruence ρ on S, $\rho K^\circ = \rho$ and so $\sigma K^\circ = \sigma$;*

(2) *for any $\mathcal{ST}$-congruence ρ on S, $\rho T = \rho$ and so $\tilde{\eta} T = \tilde{\eta}$;*

(3) *for any idempotent pure congruence ρ° on S°, $\rho t = \rho$ and hence $\tilde{\tau} t = \tilde{\tau}$.*

Proof. (1) If ρ is a group congruence, then $\operatorname{tr} \rho K^\circ \subseteq \omega_{E(S)} = \operatorname{tr} \rho$. But $\operatorname{Ker} (\rho K^\circ)^\circ = \operatorname{Ker} \rho^\circ$ and so $\rho K^\circ \subseteq \rho$. Finally $\rho \subseteq \rho K^\circ$ proves $\rho K^\circ = \rho$.

(2) If ρ is an $\mathcal{ST}$-congruence, then $\operatorname{Ker} (\rho T)^\circ \subseteq S^\circ = \operatorname{Ker} \rho^\circ$. Since $\operatorname{tr} \rho T = \operatorname{tr} \rho$, we have $\rho T \subseteq \rho$. Thus $\rho T = \rho$ follows by $\rho \subseteq \rho T$.

(3) If ρ° is an idempotent pure congruence on S°, then $\operatorname{Ker} \rho^\circ = E(S^\circ) \subseteq \operatorname{Ker} (\rho t)^\circ$. Thus, with $\operatorname{tr} \rho = \operatorname{tr} \rho t$, we have $\rho \subseteq \rho t$ and $\rho t = \rho$ by $\rho t \subseteq \rho$. □

Constant operators can be obtained from the characterizations of regular semigroups with Q-Clifford transversals in terms of congruences in Proposition 2.11.

Lemma 3.2. *Let S be a regular semigroup with an inverse transversal S°. Then S° is a Q-Clifford transversal if and only if $Tk^\circ = k^\circ T$ and Tk° is a constant operator.*

Proof. First suppose that S° is a Q-Clifford transversal of S. Then by Proposition 2.11, for any $\rho \in \mathcal{C}(S)$, $\rho Tk^\circ = \widetilde{\eta}$ and $\rho k^\circ T = \mu = \widetilde{\eta}$. Hence $Tk^\circ = k^\circ T$ and they are constant operators. Conversely, assume that $Tk^\circ = k^\circ T$ is a constant operator. Then for any $\rho \in \mathcal{C}(S)$, $\rho Tk^\circ = \omega Tk^\circ = \widetilde{\eta}$. Thus $\operatorname{Ker}(\rho T)^\circ = \operatorname{Ker}(\widetilde{\eta})^\circ = S^\circ$ whence ρT is an $\mathcal{ST}$-congruence. It follows from Proposition 2.11 that S has a Q-Clifford transversal. □

Lemma 3.3. *Let* $\rho \in \mathcal{C}(S)$.

(1) $\rho TK^\circ = \omega$;

(2) $\rho TK^\circ t = \sigma$;

(3) $\rho Tk^\circ = \rho k^\circ T = \widetilde{\eta}$;

(4) $\rho TK^\circ tk^\circ = \sigma \cap \widetilde{\eta}$;

(5) $\rho k^\circ t = \varepsilon$;

(6) $\rho k^\circ tK^\circ = \widetilde{\tau}$;

(7) $\rho k^\circ tK^\circ T = \widetilde{\tau}T = \widetilde{\tau} \vee \widetilde{\eta}$.

Proof. Note that by Proposition 2.11, ρT is an $\mathcal{ST}$-congruence and ρk° is idempotent separating. These facts and Lemma 3.2 imply (1) to (7).

□

Lemma 3.4. *Let* $\rho \in \mathcal{C}(S)$.

(1) *If* $\lambda \in [\rho t, \rho]$, $\theta \in [\rho k^\circ, \rho]$, *then* $\lambda \vee \theta = \rho$;

(2) *If* $\lambda \in [\rho, \rho T]$, $\theta \in [\rho, \rho K^\circ]$, *then* $\lambda \cap \theta = \rho$.

Proof. (1) Since $\rho t \subseteq \lambda \subseteq \rho$ and since $\rho k^\circ \subseteq \theta \subseteq \rho$, we must have $\rho = \rho t \vee \rho k^\circ \subseteq \lambda \vee \theta \subseteq \rho$ so that $\lambda \vee \theta = \rho$. Similar arguments apply to (2).

□

Lemma 3.5. *The following statements are valid*:

(1) $Tk^\circ T = Tk^\circ$;

(2) $k^\circ Tk^\circ = k^\circ T$;

(3) $tK^\circ t = tK^\circ$.

Proof. We will use freely Lemma 3.4. Let $\rho \in \mathcal{C}(S)$.

(1) First $\rho Tk^\circ \subseteq \rho T \subseteq \rho TK^\circ$, then by Lemma 3.4, $\rho Tk^\circ = \rho Tk^\circ T \cap \rho T$. Note that $\rho Tk^\circ \subseteq \rho T$, we have $\rho Tk^\circ T \subseteq \rho TT = \rho T$ whence $\rho Tk^\circ = \rho Tk^\circ T \cap \rho T = \rho Tk^\circ T$.

(2) From $\rho k^\circ t \subseteq \rho k^\circ \subseteq \rho k^\circ T$ and Lemma 3.4, we derive that $\rho k^\circ T = \rho k^\circ \vee \rho k^\circ Tk^\circ$. Since we also have $\rho k^\circ \subseteq \rho k^\circ T$, it follows that $\rho k^\circ = \rho k^\circ k^\circ \subseteq \rho k^\circ Tk^\circ$ and so $\rho k^\circ T = \rho k^\circ Tk^\circ$.

(3) From $\rho tk^\circ \subseteq \rho t \subseteq \rho tK^\circ$ and Lemma 3.4, we find that $\rho tK^\circ = \rho t \vee \rho tK^\circ t$. This together with $\rho t \subseteq \rho tK^\circ$ gives $\rho t = \rho tt \subseteq \rho tK^\circ t$ and $\rho tK^\circ = \rho tK^\circ t$. □

In view of Lemma 3.3, we may drop ρ in those expressions and use the following notations:

$$\varepsilon = k^\circ t, \qquad \tilde{\tau} = k^\circ tK^\circ, \qquad \tilde{\tau} \vee \tilde{\eta} = k^\circ tK^\circ T, \qquad \tilde{\eta} = k^\circ T,$$

$$\omega = TK^\circ, \qquad \sigma = TK^\circ t, \qquad \sigma \cap \tilde{\eta} = TK^\circ tk^\circ.$$

Also let

$$\Delta = \{\varepsilon, \sigma, \tilde{\eta}, \tilde{\tau}, \sigma \cap \tilde{\eta}, \tilde{\tau}, \vee \tilde{\eta}, \omega\}.$$

The relations $\sum$ satisfied in all networks of congruences on S are described in the following lemma.

Lemma 3.6. *Operators* Γ *satisfy the following relations* $\sum$.

(1) $(K^{\circ})^2 = k^{\circ}K^{\circ} = K^{\circ}$, $\quad (k^{\circ})^2 = K^{\circ}k^{\circ} = k^{\circ}$,

$t^2 = Tt = t$, $\quad T^2 = tT = T$;

(2) $K^{\circ}TK^{\circ} = TK^{\circ}T = TK^{\circ}$, $\quad tk^{\circ}t = k^{\circ}tk^{\circ} = k^{\circ}t$;

(3) $tK^{\circ}t = tK^{\circ}$;

(4) $k^{\circ}T = Tk^{\circ}$.

Proof. (1) This follows immediately from the definition of operators Γ.

(2) Lemma 3.3 yields $TK^{\circ} = \omega$ and $k^{\circ}t = \varepsilon$ whence equalities prevail.

(3) This follows from Lemma 3.5.

(4) This follows from Lemma 3.2. □

For convenience, we regard $\sum$ as relations on Γ^+ in an obvious way. So $\sum^*$ is a congruence on Γ^+ generated by $\sum$. We are now ready for the principal result of this section.

Theorem 3.7. *Let* S *be a regular semigroup with a Q-Clifford transversal. The set*

$$\Omega = \{K^{\circ}, K^{\circ}T, K^{\circ}t, K^{\circ}tK^{\circ}, K^{\circ}tk^{\circ}, K^{\circ}tK^{\circ}T, k^{\circ}, t, tk^{\circ}, tK^{\circ}, tK^{\circ}T, T\} \cup \Delta$$

is a system of representatives for the congruence on Γ^+ *generated by the relations* $\sum$. *The multiplication table of* $\Gamma^+ / \sum^*$ *can be completed from the following partial table* (*Table* 1). *The elements in* Δ *act as right zeros of* Ω, *and* Δ *is a minimal ideal of* $\Gamma^+ / \sum^*$.

Proof. The multiplication of elements of Δ by elements of Γ follows from Lemma 3.3 in a straightforward manner. This also shows that $\Delta\Gamma = \Delta.$

For each w in Γ^{+} we consider four words wK°, wk°, wT, and wt. If the last letter of w is K° or k°, because of relations $(K^{\circ})^2 = k^{\circ}K^{\circ} = K^{\circ}$ and $(k^{\circ})^2 = K^{\circ}k^{\circ} = k^{\circ}$, wK° and wk° have representatives of the same length as w, so it suffices to consider the words wT and wt. The case when the last letter of w is either T or t follows symmetrically. In the remaining cases, we search among the relations $\sum$ whether the resulting word can be replaced by a shorter one and replace Tk° by $k^{\circ}T$. If in this process we reach an element of Δ, we may stop because of the first paragraph of the proof. We now apply this procedure successively to K°, k°, T, and t, thereby obtaining the words in Ω. That no two elements of Ω are related by our congruence will follow from ([11], Example 1) since $\mathcal{K} = \mathcal{K}^{\circ}$ on Clifford semigroups. $\square$

We have a similar definition to $\mathcal{TK}$-operator semigroup of a class of regular semigroups.

Definition 3.8. Let S be a regular semigroup with an inverse transversal, the semigroup $\Gamma(S)$ generated by the operators Γ on $\mathcal{C}(S)$ is called the *$\mathcal{TK}^{\circ}$-operator semigroup of S*. For a class $\mathcal{A}$ of regular semigroups with inverse transversals, if there exists a semigroup A in $\mathcal{A}$ such that for any semigroup S in $\mathcal{A}$, $\Gamma(S)$ is a homomorphic image of $\Gamma(A)$, then $\Gamma(A)$ is called the *$\mathcal{TK}^{\circ}$-operator semigroup of $\mathcal{A}$*.

Table 1. Partial multiplication table of $\Gamma^{+} / \sum^{*}$

	K°	k°	T	t
K°	K°	k°	$K^\circ T$	$K^\circ t$
k°	K°	k°	$k^\circ T$	$k^\circ t$
T	TK°	$k^\circ T$	T	t
t	tK°	tk°	T	t
$K^\circ T$	TK°	$k^\circ T$	$K^\circ T$	$K^\circ t$
$K^\circ t$	$K^\circ tK^\circ$	$K^\circ tk^\circ$	$K^\circ T$	$K^\circ t$
$K^\circ tK^\circ$	$K^\circ tK^\circ$	$K^\circ tk^\circ$	$K^\circ tK^\circ T$	$K^\circ tK^\circ$
$K^\circ tk^\circ$	$K^\circ tK^\circ$	$K^\circ tk^\circ$	$k^\circ T$	$k^\circ t$
$K^\circ tK^\circ T$	TK°	$k^\circ T$	$K^\circ tK^\circ T$	$K^\circ tK^\circ$
tk°	tK°	tk°	$k^\circ T$	$k^\circ t$
tK°	tK°	tk°	$tK^\circ T$	tK°
$tK^\circ T$	TK°	$k^\circ T$	$tK^\circ T$	tK°
$k^\circ t$	K°	$k^\circ t$	$k^\circ T$	$k^\circ t$
$k^\circ tK^\circ$	$k^\circ tK^\circ$	$k^\circ t$	$k^\circ tK^\circ T$	$k^\circ tK^\circ$
$k^\circ tK^\circ T$	TK°	$k^\circ T$	$k^\circ tK^\circ T$	$k^\circ tK^\circ$
$k^\circ T$	TK°	$k^\circ T$	$k^\circ T$	$k^\circ t$
TK°	TK°	$k^\circ T$	TK°	$TK^\circ t$
$TK^\circ t$	$TK^\circ t$	$TK^\circ tk^\circ$	TK°	$TK^\circ t$
$TK^\circ tk^\circ$	$TK^\circ t$	$TK^\circ tk^\circ$	$k^\circ T$	$k^\circ t$

From [11], there exists a Clifford semigroup S such that $\Gamma(S) \simeq \Gamma^{+} / \sum^{*}$. Notice that $\mathcal{K} = \mathcal{K}^\circ$ on Clifford semigroups. Thus from Theorem 3.7, we have

Theorem 3.9. *The* $\mathcal{TK}^{\circ}$*-operator semigroup of the class of regular semigroups with Q-Clifford transversals is* $\Gamma^{+} / \sum^{*}$.

Routine analysis of the multiplication table of Ω leads to the following result.

Proposition 3.10. *The* $\mathcal{D}$*-structure of* Ω *has the form of Diagram* 2 *with the egg-box picture of each* $\mathcal{D}$*-class. The* $\mathcal{D}$*-class* $\{K^{\circ}T, K^{\circ}t\}$ *is the only irregular* $\mathcal{D}$*-class.* $K^{\circ}T$, $K^{\circ}t$, $K^{\circ}tK^{\circ}T$, tk° *are the only non-idempotents in* Ω.

K°	k°

t	T

$K^{\circ}T$	$K^{\circ}t$

$K^{\circ}tK^{\circ}$	$K^{\circ}tK^{\circ}T$	$K^{\circ}tk^{\circ}$
tK°	$tK^{\circ}T$	tk°

$TK^{\circ}t$	TK°	$k^{\circ}T$	$k^{\circ}t$	$k^{\circ}tK^{\circ}$	$k^{\circ}tK^{\circ}T$	$TK^{\circ}tk^{\circ}$

Diagram 2. $\mathcal{D}$-structure of Ω.

References

[1] B. P. Alimpié and D. N. Krgović, Some congruences on regular semigroups, Lecture Notes in Math. 1320 (1988), 1-10.

[2] Y. Y. Feng and L. M. Wang, Congruences on regular semigroups associated with Green's relations, Submitted.

[3] Y. Y. Feng and L. M. Wang, The $\mathcal{K}^{\circ}$-relation on the congruence lattice of a regular semigroup with an inverse transversal, Acta Mathematica Sinica, English Series 28(5) (2012), 1061-1074.

DOI: https://doi.org/10.1007/s10114-011-9112-0

[4] J. M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, London, 1995.

[5] Y. F. Luo, The *VT*-operator semigroup for two kinds of regular semigroups, Semigroup Forum 59(3) (1999), 374-388.

DOI: https://doi.org/10.1007/PL00022747

[6] Y. F. Luo and L. M. Wang, Semigroups of operators associated with the kernel and trace of congruences on *E*-unitary completely regular semigroups, Comm. Algebra 29(1) (2001), 99-108.

DOI: https://doi.org/10.1081/AGB-100000787

[7] F. Pastijn and M. Petrich, Congruences on regular semigroups, Trans. Amer. Math. Soc. 295(2) (1986), 607-633.

DOI: https://doi.org/10.1090/S0002-9947-1986-0833699-3

[8] F. Pastijn and M. Petrich, The congruence lattice of a regular semigroup, J. Pure and Appl. Algebra 53(1-2) (1988), 93-123.

DOI: https://doi.org/10.1016/0022-4049(88)90015-1

[9] F. Pastijn and P. G. Trotter, Lattices of completely regular semigroup varieties, Pacific J. Math. 119(1) (1985), 191-214.

DOI: https://doi.org/10.2140/pjm.1985.119.191

[10] M. Petrich, Inverse Semigroups, Wiley, New York, 1984.

[11] M. Petrich, The semigroup generated by certain operators on the congruence lattice of a Clifford semigroup, Semigroup Forum 45(1) (1992), 332-341.

DOI: https://doi.org/10.1007/BF03025774

[12] M. Petrich, Congruence networks for completely simple semigroups, J. Austral. Math. Soc. 56(2) (1994), 243-266.

DOI: https://doi.org/10.1017/S1446788700034868

[13] M. Petrich and N. R. Reilly, A network of congruences on an inverse semigroup, Trans. Amer. Math. Soc. 270(1) (1982), 309-325.

DOI: https://doi.org/10.1090/S0002-9947-1982-0642343-6

[14] T. Saito, Construction of regular semigroups with inverse transversals, Proc. Edinburgh Math. Soc. 32(1) (1989), 41-51.

DOI: https://doi.org/10.1017/S0013091500006891

[15] X. L. Tang, Regular semigroups with inverse transversals, Semigroup Forum 55(1) (1997), 24-32.

DOI: https://doi.org/10.1007/PL00005909

[16] X. L. Tang and L.-M. Wang, Congruences on regular semigroups with inverse transversals, Comm. Algebra 23(11) (1995), 4157-4171.

DOI: https://doi.org/10.1080/00927879508825455

[17] L.-M. Wang, *TK*-operator semigroups for cryptogroups, Semigroup Forum 60(3) (2000), 368-384.

DOI: https://doi.org/10.1007/s002339910030

[18] L.-M. Wang, Trace-kernel-operator semigroups of bisimple ω-semigroups, Semigroup Forum 60(3) (2000), 424-435.

DOI: https://doi.org/10.1007/s002339910034

[19] L.-M. Wang, On congruence lattices of regular semigroups with *Q*-inverse transversals, Semigroup Forum 50(1) (1995), 141-160.

DOI: https://doi.org/10.1007/BF02573513

[20] L.-M. Wang and X. L. Tang, Congruence lattices of regular semigroups with inverse transversals, Comm. Algebra 26(4) (1998), 1243-1255.

DOI: https://doi.org/10.1080/00927879808826196

■