STRONG DEVIATION THEOREMS FOR THE SEQUENCE OF CONTINUOUS RANDOM VARIABLES AND THE APPROACH OF LAPLACE TRANSFORM

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Abstract

We use the notion of the log-likelihood ratio to study some limit properties of the sequence of nonnegative continuous random variables in more general conditions. A class of strong deviation theorems represented by inequalities are obtained. The results presented in the paper extend and improve the main results of Liu, Wang and Li.

1. Introduction

In recent years, some results have been obtained in the field of the deviation for the arithmetic means of the random variables. In particular, the strong deviation theory for the discrete random variables has been studied extensively by authors Liu [3] and Wang [5] and have been obtained better results. The main problem, tracing back to Liu [4] is to determine a relationship between the true probability distribution and its

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independent product distribution. Liu [2] discussed the strong deviation theorems for the sequence of nonnegative identical continuous random variables. Wang [6] studied arbitrary stochastically dominated continuous random variables and obtained some deviation theorems and some strong laws of large numbers. Under the condition that the tail probability functions exist an upper bound function, Li et al. [1] provide a lower bound for \( \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) \) and an upper bound for \( \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) \) in terms of some functions of the Laplace transforms of the tails for nonnegative continuous random variables \( X_n(n = 1, 2, \cdots) \).

The main purpose of this paper is to establish the strong deviation theorems represented in the form of the inequality for the sequence of nonnegative continuous random variables, by discussing the superior limit and the inferior limit of \( \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) \) underlying the weaker conditions than the literatures [1, 2, 6]. Moreover, we obtain the estimation of the deviation between \( \frac{1}{n} \sum_{k=1}^{n} X_k \) and \( EX_k \).

Let \( \{X_n, n \geq 1\} \) be a sequence of nonnegative integrable random variables on the probability space \((\Omega, \mathcal{F}, P)\). Assume that the joint distribution density functions of the sequence \( \{X_n, n \geq 1\} \) is

\[
f_n(x_1, x_2, \cdots, x_n), x_k \geq 0, 1 \leq k \leq n, n = 1, 2, \cdots.
\]

Let \( f_k(x_k)(k = 1, 2, \cdots) \) stand for the distribution density functions of the random variable \( X_k \), respectively. Denote the product distribution

\[
\pi_n(x_1, x_2, \cdots, x_n) = \prod_{k=1}^{n} f_k(x_k).
\]

Write

\[
r_n(w) = \frac{f_n(X_1, X_2, \cdots, X_n)}{\pi_n(X_1, X_2, \cdots, X_n)} = \frac{f_n(X_1, X_2, \cdots, X_n)}{\prod_{k=1}^{n} f_k(X_k)}.
\]
where \( r_n(w) \) and \( L_n(w) \) are the likelihood ratio and the log-likelihood ratio, respectively. Here, as usual, the random variable \( r(w) \) is the sample relative entropy rate. Obviously, if \( \{X_n, n \geq 1\} \) are independent, then \( f_n(X_1, X_2, \ldots, X_n) = \prod_{k=1}^{n} f_k(X_k) \), and \( r(w) = 0 \).

**Definition 1.** Let \( \{X_n, n \geq 1\} \) be a sequence of random variables, and \( f_k(x_k), k = 1, 2, \ldots, n \) be the marginal density function of \( f_n(x_1, x_2, \ldots, x_n) \). Let Laplace transform and the tail probability be as follows:

\[
\tilde{f}_k(s) = \int_0^{+\infty} e^{-sx_k} f_k(x_k) dx_k, 
\]

and

\[
\tilde{q}_k(s) = \int_0^{+\infty} e^{-sx} q_k(x) dx, 
\]

where

\[
q_k(x) = \int_{x}^{+\infty} f_k(x_k) dx_k, \quad x > 0. 
\]

Throughout this paper, we assume that there exists \( s_0 \in (0, +\infty) \) such that

\[
\tilde{f}_k(s) < \infty, \quad \tilde{q}_k(s) < \infty, \quad s \in [-s_0, s_0], \quad k = 1, 2, \ldots. 
\]

Since

\[
e^x \geq 1 + x + \frac{1}{2} x^2 + \cdots + \frac{1}{n!} x^n, \quad x > 0, \quad \forall n \in N. 
\]

It follows from (5), (8) and (9) that \( EX^*_k < \infty, \forall n, k \in N \).
Lemma 1 [1]. Let \( \tilde{f}_k(s) \), \( \tilde{q}_k(s) \) be defined by (5) and (6), respectively. Then we have \( \tilde{q}_k(s) = \left[ 1 - \tilde{f}_k(s) \right] / s \).

Lemma 2 [1, 6]. Let \( f_n(x_1, x_2, \cdots, x_n) \) and \( g_n(x_1, x_2, \cdots, x_n) \) be two probability density functions on \((\Omega, \mathcal{F}, P)\). Suppose that \( t_n(w) = g_n(x_1, x_2, \cdots, x_n) / f_n(x_1, x_2, \cdots, x_n) \), then

\[
\limsup_{n \to \infty} \frac{1}{n} \ln t_n(w) \leq 0, \text{ a.s.} \tag{10}
\]

2. Main Results and Proofs

Theorem 1. Let \( \{X_n, n \geq 1\} \) be a sequence of nonnegative continuous random variables on the probability space \((\Omega, \mathcal{F}, P)\). \( r(w), \tilde{f}_k(s), \tilde{q}_k(s) \) be defined as above (4), (5) and (6), \( D_0 = \{w : r(w) < \infty\} \) and \( P(D_0) = 1 \). Then

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) \leq \alpha(r(w)), \text{ a.s.} \tag{11}
\]

where

\[
\alpha(x) = \inf \{ \varphi(s, x) : -s_0 \leq s < 0 \}, \ x \geq 0, \tag{12}
\]

\[
\varphi(s, x) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [\tilde{q}_k(s) - \tilde{q}_k(0)] - \frac{x}{s}, \ x \geq 0, -s_0 \leq s < 0, \tag{13}
\]

and

\[
\alpha(x) \geq 0, \ \lim_{x \to 0^+} \alpha(x) = \alpha(0) = 0. \tag{14}
\]

Proof. For arbitrary \( s \in [-s_0, s_0] \), let

\[
g_k(s, x_k) = e^{-sx_k} f_k(x_k) / \tilde{f}_k(s), \ x_k \geq 0, \ k \in N. \tag{15}
\]

Obviously, \( \int_{0}^{+\infty} g_k(s, x)dx = 1 \). Suppose further that
\( q_n(s, x_1, \ldots, x_n) = \prod_{k=1}^{n} g_k(s, x_k) = \prod_{k=1}^{n} \left[ e^{-s\eta_k} f_k(x_k) / \tilde{f}_k(s) \right] \), \hspace{1cm} (16)

then \( q_n(s, x_1, \ldots, x_n) \) is a n multivariate probability density function. We assume that

\[
t_n(s, w) = \frac{q_n(s, X_1, \ldots, X_n)}{f_n(X_1, \ldots, X_n)},
\]

where \( X_k = X_k(w), w \in \Omega \). In view of Lemma 2, there exist \( D(s) \in \mathcal{F}, P(D(s)) = 1 \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \ln t_n(s, w) \leq 0, w \in D(s).
\] \hspace{1cm} (18)

From (16), (17) and (18), we have

\[
\limsup_{n \to \infty} \frac{1}{n} \left\{ -s \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} \ln \tilde{f}_k(s) - \ln r_n(w) \right\} \leq 0, w \in D(s).
\] \hspace{1cm} (19)

By (4) and (19), we get

\[
\limsup_{n \to \infty} \frac{s}{n} \sum_{k=1}^{n} X_k \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln \tilde{f}_k(s) + r(w), w \in D(s).
\] \hspace{1cm} (20)

Setting \( s = 0 \) in (20), we obtain

\[
r(w) \geq 0, w \in D(0).
\] \hspace{1cm} (21)

Let \( -s_0 \leq s < 0 \), dividing the two sided of (20) by \( -s \), we get

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\ln \tilde{f}_k(s)}{s} - \frac{r(w)}{s}, w \in D(s).
\] \hspace{1cm} (22)

By (22), the property of the superior limit: \( \limsup_{n \to \infty} (a_n - b_n) \leq d_n \Rightarrow \limsup_{n \to \infty} (a_n - c_n) \leq \limsup_{n \to \infty} (b_n - c_n) + d_n \) and the inequality \( \ln x \leq x - 1(x > 0) \), and Lemma 1, for \( w \in D(s) \), we have
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_n - EX_k] \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[ - \frac{\ln \tilde{f}_k(s)}{s} - EX_k \right] - \frac{r(w)}{s} \\
\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[ - \frac{\tilde{f}_k(s) - 1}{s} - EX_k \right] - \frac{r(w)}{s} \\
\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [\tilde{q}_k(s) - \tilde{q}_k(0)] - \frac{r(w)}{s}. \quad (23)
\]

Let \( Q^- \) be the set of rational numbers in the interval \([-s_0, 0)\), and let \( D = \bigcap_{s \in Q^-} D(s) \), then \( P(D) = 1 \). We have

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - EX_k] \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [\tilde{q}_k(s) - \tilde{q}_k(0)] - \frac{r(w)}{s}, \quad w \in D, \forall s \in Q^- . \quad (24)
\]

Let

\[
g(s) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [\tilde{q}_k(s) - \tilde{q}_k(0)] , -s_0 \leq s < 0. \quad (25)
\]

By virtue of (12), (13) and (25), we get

\[
\alpha(x) = \inf \{ g(s) - \frac{x}{s} : -s_0 \leq s < 0 \}, \quad x \geq 0; \quad (26)
\]

\[
\phi(s, x) = g(s) - \frac{x}{s}, \quad x \geq 0, -s_0 \leq s < 0. \quad (27)
\]

Clearly, \( g(s) \geq 0 \) and \( \phi(s, x) \geq 0 \). This implies \( \alpha(x) \geq 0 \). Now, we prove that \( g(s) \) is a continuous function in interval \([-s_0, 0] \). In fact, for any \( \varepsilon > 0 \), if \( -s_0 \leq s + \Delta s < s \leq 0 \), it follows from (8) that for each \( k \), there exists a constant \( M > 0 \) such that \( \int_{M}^{+\infty} e^{-(s+\Delta s)x} q_k(x)dx < \varepsilon \). Since
\[
\int_0^{+\infty} e^{-sx} q_k(x)dx - \int_0^{+\infty} e^{-(s+\Delta s)x} q_k(x)dx \geq \int_0^M \left[ e^{-sx} - e^{-(s+\Delta s)x} \right] q_k(x)dx - \varepsilon
\]
\[
\geq (1 - e^{-\Delta s M}) \int_0^M e^{-sx} q_k(x)dx - \varepsilon
\]
\[
\geq (1 - e^{-\Delta s M}) \bar{q}_k(s) - \varepsilon. \quad (28)
\]

It is easy to see that \(M\) satisfying \(\int_0^{+\infty} e^{-(s+\Delta s)x} q_k(x)dx < \varepsilon\) is no increase when \(\Delta s \to 0\). This implies that
\[
\int_0^{+\infty} e^{-sx} q_k(x)dx - \int_0^{+\infty} e^{-(s+\Delta s)x} q_k(x)dx \geq -\varepsilon, \text{ as } \Delta s \to 0, \forall k \in N. \quad (29)
\]

Therefore, it follows from (25) and (29) that
\[
0 > g(s) - g(s + \Delta s)
\]
\[
= \lim sup_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \left[ \bar{q}_k(s) - \bar{q}_k(0) \right] - \lim sup_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \left[ \bar{q}_k(s + \Delta s) - \bar{q}_k(0) \right]
\]
\[
\geq \lim inf_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \left[ \bar{q}_k(s) - \bar{q}_k(0) \right] + \lim inf_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \left[ \bar{q}_k(0) - \bar{q}_k(s + \Delta s) \right]
\]
\[
\geq \lim inf_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \left[ \bar{q}_k(s) - \bar{q}_k(s + \Delta s) \right]
\]
\[
= \lim inf_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \left[ \int_0^{+\infty} e^{-sx} q_k(x)dx - \int_0^{+\infty} e^{-(s+\Delta s)x} q_k(x)dx \right]
\]
\[
\geq -\varepsilon, \text{ as } \Delta s \to 0. \quad (30)
\]

By (30), we know that \(g(s)\) is a continuous function with respect to \(s\) on the interval \([-s_0, 0]\). Hence, \(q(s, x)\) is also a continuous function with respect to \(s\) on the interval \([-s_0, 0]\). From (26), for each \(w \in D \cap D_0\), take \(s_n(w) \in Q^-\), \(n = 1, 2, \ldots\), such that
\[
\lim_{n \to \infty} \varphi(s_n(w), r(w)) = \alpha(r(w)).
\] (31)

From (24)-(27) and (31), we obtain

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) \leq \varphi(s_n(w), r(w)), w \in D \cap D_0.
\] (32)

We also have

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) \leq \alpha(r(w)), w \in D \cap D_0.
\] (33)

By \(P(D \cap D_0) = 1\) and (33), (12) holds.

Following, we prove that \(\lim_{x \to 0^+} \alpha(x) = 0\).

When \(-\sqrt{x} \in [-s_0, 0]\), taking \(s = -\sqrt{x}\), It follows from (26) that

\[
0 \leq \alpha(x) \leq g(-\sqrt{x}) + \sqrt{-x}.
\] (34)

By virtue of the continuous property of \(g(s)\) on interval \([-s_0, 0]\) and \(g(0) = 0\), we have

\[
\lim_{x \to 0^+} \alpha(x) = \alpha(0) = 0.
\] (35)

The proof of Theorem 1 is completed.

Imitating the proof of Theorem 1 in this paper and the proof of Theorem 1 in [1], we can establish the following theorem:

**Theorem 2.** Let \(\{X_n, n \geq 1\}\) be a sequence of nonnegative continuous random variables on the probability space \((\Omega, \mathcal{F}, P)\). \(r(w), \tilde{f}_k(s), \tilde{q}_k(s)\) be defined as above (4), (5) and (6), \(D_0 = \{w : r(w) < \infty\}\) and \(P(D_0) = 1\). Then

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) \geq \beta(r(w)), a.s.,
\] (36)

where
\[
\beta(x) = \sup\{\psi(s, x) : 0 < s \leq s_0\}, \quad x \geq 0,
\]
(37)

\[
\psi(s, x) = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[ \bar{q}_k(s) - \bar{q}_k(0) \right] - \frac{x}{s}, \quad x \geq 0, \quad 0 < s \leq s_0,
\]
(38)

and

\[
\beta(x) \leq 0, \quad \lim_{x \to 0^+} \beta(x) = \beta(0) = 0.
\]
(39)

**Theorem 3.** Let \( \{X_n, n \geq 1\} \) be a sequence of nonnegative continuous random variables on the probability space \((\Omega, \mathcal{F}, P)\). \( r(w) \), \( \bar{f}_k(s) \) be defined as above (4) and (5), \( D_0 = \{w : r(w) < \infty\} \) and \( P(D_0) = 1 \). Let

\[
m = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} EX_k^2.
\]

Then

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (X_i - EX_i) \geq \delta(m, r(w)), \text{ a.s.},
\]
(40)

where

\[
\delta(m, r(w)) = \sup\{ -\frac{sm}{2} - \frac{r(w)}{s} : 0 < s \leq s_0\},
\]
(41)

and

\[
\delta(m, r(w)) \leq 0, \quad \lim_{r(w) \to 0^+} \delta(m, r(w)) = 0.
\]

**Proof.** By the proof of Theorem 1, (20) holds, that is

\[
\limsup_{n \to \infty} \frac{-s}{n} \sum_{k=1}^{n} X_k \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln \frac{\bar{f}_k(s)}{s} + r(w), \quad w \in D(s).
\]
(42)

Let \( 0 < s \leq s_0 \), dividing the two sided of (42) by \(-s\), we get

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k \geq \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} - \frac{\ln \frac{\bar{f}_k(s)}{s}}{s} - \frac{r(w)}{s}, \quad w \in D(s).
\]
(43)

By (43), the property of the inferior limit: \( \liminf_{n \to \infty} (a_n - b_n) \geq d_n \Rightarrow \liminf_{n \to \infty} (a_n - c_n) \geq \liminf_{n \to \infty} (b_n - c_n) + d_n \) we have
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - EX_k] \geq \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[ -\frac{\ln \tilde{f}_k(s)}{s} - EX_k \right] - \frac{r(w)}{s}, \quad w \in D(s).
\] (44)

By \( e^x \leq 1 + x + \frac{1}{2} x^2 \) as \( x < 0 \) and \( \ln(1 + x) \leq x \) as \( x > -1 \), we obtain

\[
\ln \tilde{f}_k(s) = \ln E e^{-sX_k} \leq \ln E (1 - sX_k + \frac{1}{2} (sX_k)^2) \leq -sEX_k + \frac{s^2}{2} EX_k^2. \quad (45)
\]

It follows from (44) and (45) that

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - EX_k] \geq -\frac{s}{2} \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} EX_k^2 - \frac{r(w)}{s}, \quad w \in D(s). \quad (46)
\]

Let

\[
\delta(m, r(w)) = \sup \left\{ -\frac{sm}{2} - \frac{r(w)}{s} : 0 < s \leq s_0 \right\}, \quad (47)
\]

where

\[
m = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} EX_k^2 \geq 0. \quad (48)
\]

Obviously, \( \delta(m, r(w)) \leq 0 \). we take \( s = \sqrt{r(w)} \) when \( \sqrt{r(w)} \in (0, s_0] \). We obtain

\[
\delta(m, r(w)) \geq -\frac{\sqrt{r(w)}m}{2} - \sqrt{r(w)} = -\left( \frac{m}{2} + 1 \right) \sqrt{r(w)}. \quad (49)
\]

Therefore, \( \lim_{r(w) \to 0^+} \delta(m, r(w)) = 0 \). From (47), for each \( w \in D_0 \), take \( s_n(w) \in (0, s_0] \), \( n = 1, 2, \ldots \), such that

\[
\lim_{n \to \infty} \left( -\frac{s_n m}{2} - \frac{r(w)}{s_n} \right) = \delta(m, r(w)). \quad (50)
\]

Let \( D = \bigcap_{n \in N} D(s_n) \), then \( P(D) = 1 \). We have by (46)
\[ \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - EX_k] \geq - \frac{s_n m}{2} - \frac{r(w)}{s_n}, \quad w \in D \cap D_0, \quad s_n \in (0, s_0]. \] (51)

Since \( P(D \cap D_0) = 1 \), it follows from (47), (50) and (51) that (40) holds. This completes the proof of Theorem 3.

**Theorem 4.** Let \( \{X_n, n \geq 1\} \) be a sequence of nonnegative continuous random variables on the probability space \((\Omega, \mathcal{F}, P)\). There exists a constant \( M \) such that \( |X_k(w)| \leq M \), for all \( k \in N \) and \( w \in D_0 \) and \( P(D_0) = 1 \). \( r(w), \tilde{f}_k(s) \) be defined as above (4) and (5), \( D_1 = \{w : r(w) < \infty\} \) and \( P(D_1) = 1 \). Let \( m = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} EX_k^2 \). Then

\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) \leq h(r(w), m), \text{ a.s.,} \] (52)

where

\[ h(m, r(w)) = \inf \{ - \frac{esm}{2} - \frac{r(w)}{s} : \max \{-s_0, -\frac{1}{M}\} \leq s \leq 0\}, \] (53)

and

\[ h(m, r(w)) \geq 0, \quad \lim_{r(w) \to 0^+} h(m, r(w)) = 0. \]

**Proof.** By the proof of Theorem 1, (20) holds, that is

\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln \tilde{f}_k(s) + r(w), \quad w \in D(s). \] (54)

Let \( -s_0 \leq s < 0 \), dividing the two sided of (54) by \( -s \), we get

\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} - \frac{\ln \tilde{f}_k(s)}{s} - \frac{r(w)}{s}, \quad w \in D(s). \] (55)

By (55), the property of the superior limit: \( \limsup_{n \to \infty} (a_n - b_n) \leq d_n \Rightarrow \limsup_{n \to \infty} (a_n - c_n) \leq \limsup_{n \to \infty} (b_n - c_n) + d_n \), we have
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - EX_k] \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[ -\frac{\ln f_k(s)}{s} - EX_k \right] - \frac{r(w)}{s}, \; w \in D(s). \]

(56)

By \( e^x \leq 1 + x + \frac{1}{2} e x^2 \) as \( x \leq 1 \) and \( \ln(1 + x) \leq x \) as \( x > -1 \), we obtain when \( -sX_k \leq 1 \).

\[
\ln f_k(s) = \ln E e^{-sX_k} \leq \ln E(1 - sX_k + \frac{1}{2} e(sX_k)^2) \leq -sEX_k + \frac{es^2}{2} EX_k^2. \quad (57)
\]

when \( s \in \{ \max\{ -s_0, -\frac{1}{M} \}, 0 \} \), it follows from (56) and (57) that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [X_k - EX_k] \leq -\frac{es}{2} \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} EX_k^2 - \frac{r(w)}{s}, \; w \in D(s) \cap D_0. \quad (58)
\]

Let

\[
k(m, r(w)) = \inf \left\{ -\frac{em}{2} - \frac{r(w)}{s} : \max\{-s_0, -\frac{1}{M}\} \leq s < 0 \right\}, \quad (59)
\]

where

\[
m = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} EX_k^2 \geq 0. \quad (60)
\]

Obviously, \( k(m, r(w)) \geq 0 \). By putting \( s = -\sqrt{r(w)} \) when \( -\sqrt{r(w)} \in \{ \max\{ -s_0, -\frac{1}{M} \}, 0 \} \), we obtain

\[
k(m, r(w)) \leq \frac{em\sqrt{r(w)}}{2} + \sqrt{r(w)} = (\frac{em}{2} + 1)\sqrt{r(w)}, \; \forall w \in D_1. \quad (61)
\]

Therefore, \( \lim_{r(w) \to 0^+} \delta(m, r(w)) = 0 \). Similar to the proof of Theorem 3, we have by (58) and (59)

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) \leq k(m, r(w)), \; a.s. \quad (62)
\]

This completes the proof of Theorem 4.
Remark 1. Theorems (1-4) show that the smaller \( r(w) \) is, the smaller the deviation is. Therefore, \( r(w) \) measures the deviation between the reality distribution \( f_n(x_1, \ldots, x_n) \) and the independent product distribution \( \prod_{k=1}^{n} f_k(x_k) \).

Remark 2. Theorems 1 and 2 in this paper remove the condition that the tail probability functions exist an upper bound function in Theorems (1-4) in [1] and drop the condition that the sequence exists a stochastically dominated by a nonnegative random variable in Theorems (1-4) in [6]. Moreover, we extend the main results in [2] to the nonnegative continuous random variables with different distributions.

Remark 3. Theorems 3 and 4 in this paper not only show that the smaller \( r(w) \) is, the smaller the deviation is, but also obtain the deviation estimation between \( \frac{1}{n} \sum_{k=1}^{n} X_k \) and \( \frac{1}{n} \sum_{k=1}^{n} EX_k \).

References


