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# EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR A VARIATIONAL INEQUALITY OF KIRCHHOFF TYPE

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## Abstract

This paper is devoted to prove the existence and the multiplicity of positive solutions for a class of a nonlinear variational inequality of Kirchhoff type. Under more general superlinear assumptions on the nonlinear term, we prove the existence of multiple positive solutions via non-smooth critical point theory for Szulkin-type functional.

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## 1. Introduction and Main Results

Consider the following variational inequality of Kirchhoff type:

$$\begin{cases} -\left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u \geq f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is smooth bounded domain in $\mathbb{R}^N (N = 1, 2, \text{ or } 3)$, $a, b > 0$ are constants and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuos and satisfies:

($\mathrm{f}_1$) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and there exist $c > 0$ and $p \in (1, 2^*)$ $\left(2^* = \frac{2N}{N-2}\right.$ if $N \geq 3$, $2^* = +\infty$ if $N < 3$, is the Sobolev critical exponent) such that

$$|f(x, u)| \leq c(1 + |u|^{p-1}), \quad \forall (x, u) \in \Omega \times \mathbb{R}.$$

($\mathrm{f}_2$) $\frac{F(x, u)}{|u|^4} \to +\infty$ as $|u| \to +\infty$ uniformly in $x \in \Omega$ and there exists $r_0 > 0$ such that

$$F(x, u) \geq 0, \quad \forall (x, u) \in \Omega \times \mathbb{R}, |u| \geq r_0,$$

where $F(x, u) = \int_0^u f(x, s)ds.$

($\mathrm{f}_3$) there exists a constant $\beta < a\lambda_1$ such that

$$4F(x, u) \leq f(x, u)u + \beta|u|^2, \quad \forall (x, u) \in \Omega \times \mathbb{R},$$

where $\lambda_1 > 0$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

($\mathrm{f}_4$) $\widetilde{F}(x, u) \geq 0$ for all $(x, u) \in \Omega \times \mathbb{R}$, and $\widetilde{F}(x, s) \leq \widetilde{F}(x, t)$ whenever $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $s \leq t$, where $\widetilde{F}(x, u) = \frac{1}{4} f(x, u)u - F(x, u)$.

($\mathrm{f}_5$) $f(x, u) = -f(x, u)$ for all $(x, u) \in \Omega \times \mathbb{R}$.

Variational inequalities describe phenomena from mathematical physics. They have applications in physics, mechanics, engineering, optimization, and elliptic inequalities, see, for example, [1-5].

The aim of this work is to study a Kirchhoff type variational inequality which is defined on $\Omega$ by using a non-smooth critical point theory due to Szulkin. In [7], the author has proved a number of existence theorem for critical point of functionals which are not smooth. He has generalized some minimization and minimax methods in critical point theory to a class of functionals which are not necessarily continuous and has introduced a new concept of compactness which is suitable to study these kinds of problems.

In the present paper, by using a minimization principle and the mountain pass theorem of Szulkin-type, we prove existence of positive solutions to a variational inequality of Kirchhoff-type in a closed convex set.

Let $K = \{u \in H_0^1(\Omega) : u \geq 0\}$ be the closed convex set in Sobolev space $H_0^1(\Omega)$ and we consider the problem, denoted by (*P*):

Given $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a continuous function and $a, b > 0$, find $u \in K$ such that

$$\left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u.(\nabla v - \nabla u)dx - \int_\Omega f(x, u)(v - u)dx \geq 0, \quad \forall v \in K.$$

Such kind of problems are called obstacle problems and they have been largely studied due to its physical application. See, for example, the classical books Kinderlehrer and Stampacchia [4], Rodrigues [6], and Troianiello [8] and the references therein.

The main results of this paper are the following:

**Theorem 1.1.** *Assume that* $(\mathrm{f}_1), (\mathrm{f}_2), (\mathrm{f}_3),$ *and* $(\mathrm{f}_5)$ *hold. Then problem* (1.1) *possesses at least one distinct pair of positive solution.*

**Theorem 1.2.** *Assume that* $(\mathrm{f}_1), (\mathrm{f}_2), (\mathrm{f}_4),$ *and* $(\mathrm{f}_5)$ *hold. Then problem* (1.1) *possesses at least one distinct pair of positive solution.*

**Remark 1.3.** Note that conditions $(\mathrm{f}_3)$ and $(\mathrm{f}_4)$ are weaker than the well-known Ambrosetti-Rabinowitz condition (AR for short), which was first introduced in [10], that is,

$$\exists \nu > 4 : \nu F(x, t) \leq tf(x, t),\ t \neq 0.$$

Indeed, there are functionals $f(x, u)$ satisfying conditions $(\mathrm{f}_1)$-$(\mathrm{f}_5)$, and not satisfying the AR condition. For example, let

$$F(x, u) = u^4 \ln(1 + u^4).$$

Then

$$f(x, u) = 4u^3 \ln(1 + u^4) + 4u^4 \frac{u^3}{1 + u^4}.$$

By a simple computation, one can deduces that

$$\nu F(x, u) - f(x, u)u = (\nu - 4)u^4 \ln(1 + u^4) - 4u^4 \frac{u^4}{1 + u^4} > 0,$$

for $|u|$ large enough. Thus, $f$ does not satisfy the AR-condition. Moreover, it is easy to check that $f$ satisfies all the assumptions of Theorems 1.1 and 1.2.

## 2. Variational Framework and Technical Lemmas

Let $H := H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and the norm

$$\langle u, v\rangle = \int_\Omega \nabla u.\nabla v dx, \quad \|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

We denote by $|.|_p$ the usual $L^p$-norm. Since $\Omega$ is a bounded domain, then $H \hookrightarrow L^p(\Omega)$ continuously for $p \in [1, 2^*]$, and compactly for $p \in [1, 2^*]$, and there exists $\gamma_p > 0$ such that

$$|u|_p \leq \gamma_p\|u\|, \quad \forall u \in H. \tag{2.1}$$

Let $K = \{u \in H, u \geq 0\}$ be the closed convex set in the space $H$. Recall that a function $u \in H$ is called weak solution of (1.1) if

$$(a + b\int_{\Omega}|\nabla u|^2 dx)\int_{\Omega}\nabla u.(\nabla v - \nabla u)dx - \int_{\Omega}f(x, u)(v - u)dx \geq 0, \ \forall v \in H. \tag{2.2}$$

Now we give some preliminaries about Szulkin-type function (see [7]). Let $X$ be a real Banach space and $X^*$ its dual. Let $\phi$ be a functional which is of class $C^1$ and let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper (i.e., $\psi \neq +\infty$), convex and lower semicontinuous functional. We say that $I = \phi + \psi$ is a Szulkin-type functional. An element $u \in X$ is called a critical point of $I = \phi + \psi$ if

$$\phi'(u)(v - u) + \psi(v) - \psi(u) \geq 0, \quad \text{for all} \quad v \in X,$$

which is equivalent to

$$0 \in \phi'(u) + \partial\psi(u) \quad \text{in} \quad X^*,$$

where $\partial\psi(u)$ is the subdifferential of the convex functional $\psi$ at $u \in X$ defined by

$$\partial\psi(u) = \{\phi \in X^* : \psi(v) - \psi(u) \geq \langle\phi, v - u\rangle, \quad \forall v \in X\}.$$

**Definition 2.1.** The functional $I = \phi + \psi$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, denoted by $(PSZ)_c$ if every sequence $\{u_n\} \subset X$ such that $\lim_{n\to\infty} I(u_n) = c$ and

$$\langle\phi'(u_n), v - u_n\rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v - u_n\| \quad \text{for all } v \in X, \tag{2.3}$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

The following theorem which was proved by Szulkin, is the main tool to prove the main results of this paper.

**Theorem 2.2.** *Let* $X$ *be a Banach space,* $I : X \to (-\infty, +\infty]$ *a Szulkin-type functional satisfies* $(PSZ)_c$, $I(0) = 0$ *and* $\phi, \psi$ *are even. Assume also that*

(i) *there exists a subspace* $X_1$ *of* $X$ *of finite codimension and numbers* $\alpha, \rho > 0$ *such that* $I|_{\partial B\rho \cap X_1} \geq \alpha$;

(ii) *there is a finite dimensional subspace* $X_2$ *of* $X$, $\dim X_2 > \operatorname{codim} X_1$, *such that* $I(u) \to -\infty$ *as* $\|u\| \to \infty$, $u \in X_2$.

*Then* $I$ *has at least* $\dim X_2 - \operatorname{codim} X_1$ *distinct pairs of nontrivial critical points.*

We define the functional $\phi : H \to \mathbb{R}$ by

$$\phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_{\Omega} F(x, u(x))dx. \tag{2.4}$$

Using $(f_1)$ and the Sobolev embedding theorem, we can prove easily that $\phi \in C^1(H, \mathbb{R})$. We define the indicator functional of the set $K$ by

$$\psi_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{if } u \notin K. \end{cases}$$

We remark that the functional $\psi_K$ is convex, proper and lower semicontinuous. So, $I = \phi + \psi_K$ is a Szulkin-type functional.

**Lemma 2.3.** *If* $u \in H$ *is a critical point of* $I = \phi + \psi_K$, *then* $u$ *is a solution of problem* (1.1).

**Proof.** Let $u \in H$ be a critical point of $I = \phi + \psi_K$. Then, we have

$$\phi'(u)(v - u) + \psi_K(v) - \psi_K(u) \geq 0, \quad \forall v \in H.$$

We first prove that $u \in K$. If this were not true, we have $\psi_K(u) = +\infty$, and taking $v = 0 \in K$ in the above inequality, we obtain a contradiction. Next, for a fixed $v \in K$, since

$$0 \leq \phi'(u)(v - u) = (a + b\|u\|^2)\int_\Omega \nabla u(\nabla v - \nabla u)dx$$

$$- \int_\Omega f(x, u)(v - u)dx,$$

the inequality is proved. □

**Lemma 2.4.** *Assume that f satisfies* $(\mathrm{f}_1)$ *and* $(\mathrm{f}_3)$, *then* $I = \phi + \psi_K$ *satisfies the* $(PSZ)_c$ *for every* $c \in \mathbb{R}$.

**Proof.** Let $\{u_n\} \subset H$ be such that

$$I(u_n) = \phi(u_n) + \psi_K(u_n) \to c, \quad (c \in \mathbb{R}), \tag{2.5}$$

and

$$\phi'(u_n)(v - u_n) + \psi_K(v) - \psi_K(u_n) \geq -\varepsilon_n\|v - u_n\|, \quad \forall v \in H, \tag{2.6}$$

where $\{\varepsilon_n\} \subset [0, \infty\}$ is a sequence with $\varepsilon_n \to 0$. By (2.5), we have $\{u_n\}$ is in *K*. Next, we prove that $\{u_n\}$ is bounded in *H*. It follows from $(\mathrm{f}_3)$ and (2.5) that

$$\begin{aligned}
c + 1 + \|u_n\| &\geq \phi(u_n) - \frac{1}{4}\langle \phi'(u_n), u_n \rangle \\
&= \frac{a}{4}\|u_n\|^2 + \int_\Omega \left(\frac{1}{4} f(x, u_n)u_n - F(x, u_n)\right)dx \\
&\geq \frac{a}{4}\|u\|^2 - \frac{\beta}{4}\int_\Omega |u|^2 dx \\
&\geq \frac{1}{4}(a - \frac{\beta}{\lambda_1})\|u_n\|^2.
\end{aligned}$$

Since $\beta < a\lambda_1$, then $(a - \frac{\beta}{\lambda_1}) > 0$. Thus $\{u_n\}$ is bounded in $H$. Because the sequence $\{u_n\}$ is bounded in $H$, going if necessary to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } H; \\ &u_n \to u \quad \text{in } L^p(\Omega), \quad \text{for } 1 \le p < 2^*; \\ &u_n(x) \to u(x) \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{2.7}$$

As $K$ is weakly closed, then $u \in K$. Setting $v = u$ in (2.6), we obtain that

$$\begin{aligned} &(a + b\|u_n\|^2)\int_\Omega \nabla u_n(\nabla u - \nabla u_n)dx \\ &+ \int_\Omega f(x, u_n)(u - u_n)dx \ge -\varepsilon_n\|u - u_n\|. \end{aligned}$$

Therefore, for large $n \in \mathbb{N}$, we have

$$\begin{aligned} (a + b\|u_n\|^2)\|u - u_n\|^2 \le\ &(a + b\|u_n\|^2)\int_\Omega \nabla u(\nabla u - \nabla u_n)dx \\ &+ \int_\Omega f(x, u_n)(u - u_n)dx + \varepsilon_n\|u - u_n\|. \end{aligned} \tag{2.8}$$

In one hand, by $(f_1)$, (2.7) and the Hölder inequality, one has

$$\begin{aligned} \int_\Omega f(x, u_n)(u - u_n)dx &\le \int_\Omega c(1 + |u_n|^{p-1})(u - u_n)dx \\ &\le c\int_\Omega |u - u_n|dx + c\int_\Omega |u_n|^{p-1}|u - u_n|dx \\ &\le c\|u - u_n\|_1 + c\|u - u_n\|_p\|u_n\|_p^{p-1} \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.9}$$

In the other hand, by (2.7) and the fact that $\{u_n\}$ is bounded in $H$, we have

$$\lim_{n} \langle a + b\|u_n\|^2 \rangle \langle u, u - u_n \rangle = 0. \tag{2.10}$$

Since $\varepsilon_n \to 0^+$, combine (2.9) and (2.10), we conclude that the second term in (2.8) converges to 0. Hence, $\{u_n\}$ converges strongly to $u$ in $H$. The proof is completed. □

**Lemma 2.5.** *Assume that* $(\mathrm{f}_1)$ *and* $(\mathrm{f}_4)$ *hold. Then* $I$ *satisfies the* $(PSZ)_c$ *for every* $c \in \mathbb{R}$.

**Proof.** Let $\{u_n\} \subset H$ satisfies (2.5) and (2.6). It is clear that $\{u_n\}$ is in $K$. Next, we prove that $\{u_n\}$ is bounded in $H$. Suppose to the contrary that $\|u_n\| \to \infty$. Setting $v_n = \dfrac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. So, up to a subsequence, we may assume that

$$v_n \rightharpoonup v \quad \text{in } H;$$

$$v_n \to v \quad \text{in } L^p(\Omega), \quad \text{for } 1 \le p < 2^*;$$

$$v_n(x) \to v(x) \quad \text{a.e. } x \in \Omega.$$

There are two case need to be considered: $v \neq 0$ or $v = 0$. We first consider the case $v \neq 0$. Set

$$\Lambda_n(r_1, r_2) = \{x \in \Omega : r_1 \le |u_n(x)| < r_2\},$$

and

$$A := \{x \in \Omega : v(x) \neq 0\}.$$

Obviously, $\mathrm{meas}(A) > 0$. For $x \in A$, we have $|u_n(x)| \to +\infty$ as $n \to \infty$. Hence, $A \subset \Lambda_n(r_0, \infty)$ for large $n \in \mathbb{N}$, where $r_0$ is given in $(\mathrm{f}_2)$. By $(\mathrm{f}_2)$, we have

$$\frac{F(x, u_n)}{|u_n|^4} |v_n|^4 dx \to +\infty \quad \text{as} \quad n \to \infty.$$

Hence, using Fatou's lemma, we have

$$\int_A \frac{F(x, u_n)}{|u_n|^4} |v_n|^4 dx \to +\infty \quad \text{as} \quad n \to \infty. \tag{2.11}$$

It follows from (2.5) and (2.11) that

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \frac{c + o(1)}{\|u_n\|^4} = \lim_{n\to+\infty} \frac{\phi(u_n)}{\|u_n\|^4} \\
&= \lim_{n\to+\infty} \frac{1}{\|u_n\|^4} \left( \frac{a}{2} \|u_n\|^2 + \frac{b}{4} \|u_n\|^4 - \int_\Omega F(x, u_n) dx \right) \\
&= \frac{b}{4} - \lim_{n\to+\infty} \frac{1}{\|u_n\|^4} \left( \int_{\Lambda_n(0,\, r_0)} F(x, u_n) dx + \int_{\Lambda_n(r_0,\, \infty)} F(x, u_n) dx \right) \\
&\le \frac{b}{4} + \limsup_{n\to\infty} \left[ \frac{c}{\|u_n\|^3} \left( 1 + \frac{r_0^{p-1}}{p} \right) \int_\Omega |v_n| dx \right] \\
&\quad - \lim_{n\to+\infty} \int_A \frac{F(x, u_n)}{|u_n|^4} |v_n|^4 dx \\
&\le \frac{b}{4} - \liminf_{n\to+\infty} \int_A \frac{F(x, u_n)}{|u_n|^4} |v_n|^4 dx = -\infty,
\end{aligned}$$

which is a contradiction, thus $\{u_n\}$ is bounded. By a similar argument as the proof of Lemma 2.4, we can conclude that $\{u_n\}$ converge strongly in $H$.

Next we consider the case $v = 0$. We define

$$\phi(t_n u_n) = \max_{t \in [0, 1]} \phi(t u_n).$$

For any $M > 0$, set $\tilde{v}_n = \sqrt{\frac{4M}{a}} \frac{u_n}{\|u_n\|} = \sqrt{\frac{4M}{a}} v_n$. By $(\mathrm{f}_1)$ and he Sobolev embedding theorem, we have

$$\left| \int_\Omega F(x, \tilde{v}_n) dx \right| \le c \int_\Omega |\tilde{v}_n| dx + \frac{c}{p} \int_\Omega |\tilde{v}_n|^p \, dx \to 0,$$

as $n \to \infty$. Consequently, for $n$ sufficiently large such that

$$\phi(t_n u_n) \geq \phi(\tilde{v}_n) \geq \frac{a}{2}\|\tilde{v}_n\|^2 + \frac{b}{4}\|\tilde{v}_n\|^4 - \int_\Omega F(x, \tilde{v}_n) \geq M.$$

This means that

$$\lim_{n\to\infty} \phi(t_n u_n) = \infty.$$

In view of the choice of $t_n$ we know that $\langle \phi'(t_n u_n), t_n u_n \rangle = 0$. Hence, by $(\mathrm{f}_4)$, we have

$$\infty \leftarrow \phi(t_n u_n) - \frac{1}{4}\langle \phi'(t_n u_n), t_n u_n \rangle = \frac{a}{4}\|t_n u_n\|^2 + \int_\Omega \widetilde{F}(x, t_n u_n)dx$$

$$\leq \frac{a}{4}\|u_n\|^2 + \int_\Omega \widetilde{F}(x, u_n)dx = \phi(u_n) - \frac{1}{4}\langle \phi'(u_n), u_n \rangle,$$

which contradicts (2.5), thus $\{u_n\}$ is bounded in $H$. Since $\{u_n\} \subset H$ is bounded, using a similar arguments as (2.8), (2.9), and (2.10), we can conclude that $u_n \to u$ in $H$, as $n \to \infty$. This completes the proof. □

## 3. Proof of Main Results

Let $\{e_j\}$ is an orthonormal basis of $H$ and define $x_j = \mathbb{R}e_j$,

$$Y_l = \bigoplus_{j=1}^{l} X_j, \quad Z_k = \bigoplus_{j=k}^{\infty} X_j, \quad l, k \in \mathbb{Z}. \tag{3.1}$$

Therefore, we have the following lemma from [9].

**Lemma 3.1** ([9], Lemma 3.8)**.** *If* $1 \leq p < 2^*$, *then we have*

$$\beta_k(p) := \sup_{u \in Z_k, \|u\|=1} |u|_p \to 0, \quad k \to \infty.$$

**Lemma 3.2.** *Suppose that* $(\mathrm{f}_1)$ *is satisfied. Then there exist constants* $\rho, \alpha > 0$ *and* $m \in \mathbb{Z}$ *such that* $I|_{\partial B_\rho \cap Z_m} \geq \alpha$.

**Proof.** By Lemma 3.1, we can choose an integer $m \geq 1$ such that $0 < \beta_m(1) << 1$, $0 < \beta_m(p) << 1$ and

$$\|u\|_1 \leq \beta_m(1)\|u\|, \quad \|u\|_p \leq \beta_m(p)\|u\|, \quad \forall u \in X_1.$$

For any $u \in Z_m$ with $\|u\| = \rho < 1$, by $(\mathrm{f}_1)$, we have

$$\begin{aligned} I(u) = \phi(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x, u)dx \\ &\geq \frac{a}{2}\|u\|^2 - c\int_\Omega |u|dx - \frac{c}{p}\int_\Omega |u|^p dx \\ &\geq \frac{a}{2}\|u\|^2 - c\beta_m(1)\|u\| - \frac{c}{p}\beta_m^p(p)\|u\|^p \\ &\geq \frac{a}{2}\|u\|^2 - c\beta_m(1)\|u\| - c_p\|u\|^p \\ &= \rho\left(\frac{a}{2}\rho - c\beta_m(1) - c_p\rho^{p-1}\right) > 0. \end{aligned}$$

Here we use the fact that $0 < \beta_m(1) < \frac{a}{2c}\rho - \frac{c_p}{c}\rho^{p-1}$ if $m$ suitable large. Thus, this completes the proof. □

**Lemma 3.3.** *Suppose that* $(\mathrm{f}_1)$ *and* $(\mathrm{f}_2)$ *are satisfied. Then, for any finite dimensional subspace* $\widetilde{H} \subset H$, *there is* $R = R(\widetilde{H}) > 0$ *such that*

$$\phi(u) \leq 0, \quad \forall u \in \widetilde{H} \setminus B_R.$$

**Proof.** For any finite dimensional subspace $\widetilde{H} \subset H$, there is a positive integral number $n > m$ (where $m$ is given by Lemma 3.2) such that $\widetilde{H} \subset Y_n$. Since all norms are equivalent in a finite dimensional space, there is a constant $c_4 > 0$ such that

$$\|u\|_4 \geq c_4\|u\|, \quad \forall Y_n. \tag{3.2}$$

By $(\mathrm{f}_1)$ and $(\mathrm{f}_2)$, we know that for any $M > \frac{b}{4c_4^4}$ there is a constant $C_M > 0$ such that

$$F(x, u) \geq M|u|^4 - C_M|u|^2, \quad \forall (x, u) \in \Omega \times \mathbb{R}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} I(u) = \phi(u) &\leq \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - M\|u\|_4^4 + C_M\|u\|_2^2 \\ &\leq \frac{1}{2}\|u\|^2 - \left(Mc_4^4 - \frac{b}{4}\right)\|u\|^4 + C_M C_2^2\|u\|^2, \end{aligned}$$

for all $u \in Y_{n.}$ Consequently, there is a large $R = R(\widetilde{H}) > 0$ such that $\phi(u) \leq 0$ on $\widetilde{H} \setminus B_R$. Thus, the proof is complete. □

**Proof of Theorem 1.1.** Let $X = H, X_1 = Z_m$, and $X_2 = Y_n$. Obviously, $I(0) = 0$ and $(\mathrm{f}_5)$ implies that $I$ is even. By Lemmas 2.4, 3.1, and 3.2, all conditions of Theorem 2.2 are satisfied. Thus, problem (2.5) possesses $\dim X_2 - \operatorname{codim} X_1 = n - m + 1 > 1$ distinct pairs of positive solutions. □

**Proof of Theorem 1.2.** Let $X = H, X_1 = Z_m$, and $X_2 = Y_n$. Obviously, $I(0) = 0$ and $(\mathrm{f}_5)$ implies that $I$ is even. By Lemmas 2.5, 3.1, and 3.2, all conditions of Theorem 2.2 are satisfied. Thus, problem (2.5) possesses $\dim X_2 - \operatorname{codim} X_1 = n - m + 1 > 1$ distinct pairs of positive solutions. □

## References

[1] A. Friedman, Variational Principles and Free Boundary Value Problems, Wiley-Interscience, New York, 1983.

[2] J. Heinonen, T. Kilpelainen and O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford Univ. Press, Oxford, 1993.

[3] S. Karamardian, Generalized complementarity problem, J. Opim. Theory Appl. 8(3) (1971), 161-168.

DOI: https://doi.org/10.1007/BF00932464

[4] D. Kinderlehrer and G. Stampacchia, Convex Programming and Variational Inequalities and their Applications, Academic Press, New York, 1980.

[5] O. Mancino and G. Stampacchia, Convex programming and variational inequalities, J. Optim. Theory Appl. 9(1) (1972), 3-23.

DOI: https://doi.org/10.1007/BF00932801

[6] J. F. Rodrigues, Obstacle problems in mathematical physics, In: Mathematics Studies, 134, Elsevier (1987).

[7] A. Szulkin, Minimax principles for lower semicountinuous functions and application to nonlinear boundary value problems, Ann. Inst. H. Poincaré Anal. Nonlinéaire 3(2) (1986), 77-109.

[8] G. M. Troianiello, Elliptic Differential Equations and Obstacle Problems, The University Series in Mathematics, 1987.

[9] M. Willem, Minimax Theorems, Birkhäuser, Berlin, 1996.

[10] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14(4) (1973), 349-381.

DOI: https://doi.org/10.1016/0022-1236(73)90051-7

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