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HOMOGENIZATION PROBLEM WITH OSCILLATING BOUNDARY

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Abstract

In this paper, we study the asymptotic behaviour of solutions for homogenization problems of partial differential equation with a very rapidly oscillating locally periodic boundary. As a consequence, we show that u_ε converges to u_0 in the H^1-norm and the error is $O(\varepsilon^{2\alpha-1})$(if $1 < \alpha < 2$) or $O(\varepsilon^{\alpha+1})$(if $\alpha \geq 2$), where u_ε and u_0 are solutions of oscillating and homogenized boundary problems, respectively. Our method is based on correctors that can be used to obtain effective approximation.

1. Introduction

Homogenization of periodically oscillating boundaries, such problems arises in the context of fluid flows over a rough surface [3, 4], of reinforcement by a thin layer [11], or of electromagnetic scattering by

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an obstacle with a periodic coating [1, 15]. Indeed, boundary layers results can be applied to the homogenization of neutronic diffusion or transport equation [2, 8, 12]. Studying boundary layers is also the key for determining interface transmission conditions in various mechanical problems [7, 13, 17].

In this paper, we are interested in the following boundary-value problem for the second order elliptic equation with a very rapidly oscillating locally periodic boundary:

$$\begin{cases} -\Delta u_\varepsilon = f(x) & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + \varepsilon^{\alpha-1} p(x', \dfrac{x'}{\varepsilon^\alpha}) u_\varepsilon = \varepsilon^{\alpha-1} g(x', \dfrac{x'}{\varepsilon^\alpha}) & \text{on } \Gamma_1^\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

where $x' = (x_1, x_2, \cdots, x_{n-1})$ and ν_ε is the outward normal.

Let Ω_ε be a bounded domain in R^n and $\partial\Omega_\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2$, where a curve

$$\Gamma_1^\varepsilon : x_n = \varepsilon F(x', \frac{x'}{\varepsilon^\alpha}), \ \alpha > 1. \tag{1.2}$$

Without loss of generality, we assume that Ω_ε is a domain bounded by $|x'| \le (0, 1)^{n-1}$, $x_n = B$ and curve Γ_1^ε, where B is a positive absolutely constant.

We suppose that

$p(x', \xi') > 0$ and $F(x', \xi')$ has compactly supported on Γ_1 uniformly in ξ'. (1.3)

We also impose the smoothness condition and periodicity condition

$p(x', \xi')$, $g(x', \xi')$ and $F(x', \xi')$ are smooth functions in (x', ξ') and 1-periodic in ξ', (1.4)

where $\xi' = \dfrac{x'}{\varepsilon^\alpha}$.

As the author know, there are many papers about results of convergence rates for the elliptic homogenization problems with oscillating boundary data. In 1997, Friedman et al. [16] had studied two dimensional domain, whose boundary is oscillating according to three scales. They extended the results of Belyaev's [9, 10] to the three-scale oscillating boundary. In 1999, Chechkin et al. [14] considered the problem (1.1). They obtained the error estimate of solutions as

$$\|u_\varepsilon - u_0\|_{H^1(\Omega_\varepsilon)} \leq C(\varepsilon^{1/2} + \varepsilon^{\alpha-1}),$$

where u_0 is the solution of homogenized problem according to problem (1.1). Recently, Aleksanyan et al. ([5, 6]) studied the boundary homogenization for Dirichlet problem. In particular, they proved point wise and L^p convergence results.

The main purpose of this paper is to improve the results of Chechkin et al. (see [14], Theorem 2) and obtain better error estimate up to the order of $O(\varepsilon^{2\alpha-1})$(if $1 < \alpha < 2$) or $O(\varepsilon^{\alpha+1})$(if $\alpha > 2$) by using the correctors.

We now describe the outline of this paper as well as some key ideas used in the proof of its main results. In Section 2, we establish the corrector u_1 that plays an important role in improving the power of ε. In Section 3, we show that the solution u_ε of problem (1.1) converges to the solution u_0 of the corresponding homogenized problem in the H^1-norm with error estimate up to the order of $O(\varepsilon^{\alpha-1})$. This can be obtained via the corrector u_1. Following the same line of research, by using the second order expansion, we improve the error estimate up to the order of $O(\varepsilon^{\alpha})$ in Section 4. Finally, in Section 5, we obtain the approximation u_ε to the higher power of $\varepsilon^{2\alpha-1}$(if $1 < \alpha < 2$) or $\varepsilon^{\alpha+1}$(if $\alpha \geq 2$) by using correctors u_1 and ϕ.

2. Preliminaries

In order to improve the power of ε, in this section, we shall construct the corrector u_1. Firstly, as a preliminary step, we compute the arc elements.

Proposition 2.1. *Let* $d\sigma$ *be an element of the* $(n-1)$*-dimensional volume of* Γ_1^ε. *Then*

$$d\sigma = \varepsilon^{1-\alpha}(|\nabla_{\xi'} F(x', \xi')| + O(\varepsilon^{\alpha-1}))dx'.$$

Proof. This proposition has been proved by Chechkin et al. (see [14], Lemma 2). □

We introduce the harmonic functions $H_i(x', \xi)$, x' is treated as a parameter, as solutions of

$$\begin{cases} \Delta H_0 = 0 & \text{in } \widetilde{D}, \\ \dfrac{\partial H_0}{\partial \nu_\xi} = (g - \dfrac{p}{P} G) \dfrac{|\nabla_{\xi'} F(x', \xi')|}{(1 + |\nabla_{\xi'} F(x', \xi')|^2)^{1/2}} & \text{on } \partial\widetilde{D}, \\ |H_0| \le C \exp(-\lambda \xi_n) & \text{for some } \lambda > 0, \end{cases}$$

$$\begin{cases} \Delta H_i = 0 & \text{in } \widetilde{D}, \\ \dfrac{\partial H_i}{\partial \nu_\xi^i} = \dfrac{-1}{(1 + |\nabla_{\xi'} F(x', \xi')|^2)^{1/2}} \dfrac{\partial F(x', \xi')}{\partial \xi_i} & \text{on } \partial\widetilde{D}, \\ |H_i| \le C \exp(-\lambda \xi_n) & \text{for some } \lambda > 0, \end{cases}$$

and

$$\begin{cases} \Delta H_n = 0 & \text{in } \widetilde{D}, \\ \dfrac{\partial H_n}{\partial \nu_\xi^n} = \dfrac{1}{(1 + |\nabla_{\xi'} F(x', \xi')|^2)^{1/2}} - \dfrac{p}{P} \dfrac{|\nabla_{\xi'} F(x', \xi')|}{(1 + |\nabla_{\xi'} F(x', \xi')|^2)^{1/2}} & \text{on } \partial\widetilde{D}, \\ |H_n| \le C \exp(-\lambda \xi_n) & \text{for some } \lambda > 0, \end{cases}$$

where

$$P(x') = \int_0^1 \cdots \int_0^1 |\nabla_{\xi'} F(x', \xi')| p(x', \xi') d\xi',$$

$$G(x') = \int_0^1 \cdots \int_0^1 |\nabla_{\xi'} F(x', \xi')| g(x', \xi') d\xi',$$

$$\widetilde{D} = \{(\xi', \xi_n) : \xi' = (\xi_i),\ 0 \le \xi_i \le 1,\ F(x', \xi') < \xi_n\},$$

$$V_\xi = (\nu_\xi^1, \nu_\xi^2, \cdots, \nu_\xi^n) \text{ is the outward normal,} \tag{2.1}$$

and the curve

$$\Gamma_\xi : \xi_n = F(x', \xi'),\ 0 \le \xi_i \le 1,$$

where $\xi' = (\xi_i)$ and $i = 1 \cdots n-1$. This system was introduced in [9, 10].

To ensure these solutions H_i exist, we need to verify the compatibility condition holds true. A simple calculation then gives

$$\int_{\partial\widetilde{D}} \frac{\partial H_0}{\partial \nu_\xi} d\sigma_\xi = \int_0^1 \cdots \int_0^1 (g - \frac{p}{P} G) |\nabla_{\xi'} F(x', \xi')| d\xi' = 0,$$

$$\int_{\partial\widetilde{D}} \frac{\partial H_i}{\partial \nu_\xi^i} d\sigma_\xi = \int_0^1 \cdots \int_0^1 \frac{\partial F(x', \xi')}{\partial \xi_i} d\xi' = 0,$$

where we have used (1.3), and

$$\int_{\partial\widetilde{D}} \frac{\partial H_n}{\partial \nu_\xi^n} d\sigma_\xi = \int_0^1 \cdots \int_0^1 (1 - \frac{p}{P} |\nabla_{\xi'} F(x', \xi')|) d\xi' = 0.$$

Therefore, these harmonic functions H_i exist. Then we set corrector u_1 as follows:

$$u_1(x', \xi) = H_0(x', \xi) + \frac{\partial u_0}{\partial x_i} H_i(x', \xi). \tag{2.2}$$

Throughout this paper, the summation convention is used.

3. Error Estimate up to $O(\varepsilon^{\alpha-1})$

In this section, we will prove the error estimate up to the order of $O(\varepsilon^{\alpha-1})$. The main technique is using the corrector. Our main result is the following theorem:

Theorem 3.1. *Let* $f \in L^2(R^n)$ *and* u_ε *is a solution of problem* (1.1). *Suppose that* $p(x', \xi')$, $g(x', \xi')$, *and* $F(x', \xi')$ *satisfy* (1.3) *and* (1.4). *Then these exists a constant C does not depend on* ε, *such that*

$$\|u_\varepsilon - u_0 - \varepsilon^\alpha u_1\|_{H^1(\Omega_\varepsilon)} \le C\varepsilon^{\alpha-1}, \tag{3.1}$$

where u_0 *is a solution of homogenized problem*

$$\begin{cases} -\Delta u_0 = f(x) & \text{in } \Omega_0, \\ \dfrac{\partial u_0}{\partial \nu_0} + P(x')u_0 = G(x') & \text{on } \Gamma_1, \\ u_0 = 0 & \text{on } \Gamma_2, \end{cases} \tag{3.2}$$

and the curve $\Gamma_1 : x_n = 0$. *Functions* $u_1(x', \xi)$, $P(x')$, *and* $G(x')$ *are defined in* (2.1) *and* (2.2).

Remark 3.2. Following general procedure in homogenization, we introduce the associated bilinear form

$$B(u, \nu) = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla \nu + \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} pu\nu d\sigma.$$

Then we will establish

$$B(u_\varepsilon - u_0 - \varepsilon^\alpha u_1, \nu) \le C\varepsilon^{\alpha-1} \| \nu \|_{H^1(\Omega_0)}.$$

Next, we extend $u_\varepsilon - u_0 - \varepsilon^\alpha u_1$ into Ω_0 without increasing the H^1-norm by more than a multiplicative constant. It just remains to take $\nu = u_\varepsilon - u_0 - \varepsilon^\alpha (u_1 - u_1|_{x_n = B})$. Then this theorem will be proved.

Proof. Since $\Omega_\varepsilon \subset \Omega_0$, every function $f \in H^1(\Omega_\varepsilon)$ can be easily extended by 0 in $\Omega_0 \setminus \Omega_\varepsilon$ to become a function of $H^1(\Omega_0)$. It is easy to find that

$$B(u_\varepsilon - u_0 - \varepsilon^\alpha u_1, \nu) = B(u_\varepsilon - u_0, \nu) - \int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla \nu dx + O(\varepsilon^\alpha) \| \nu \|_{H^1(\Omega_0)}. \tag{3.3}$$

Firstly, let us estimate the term $B(u_\varepsilon - u_0, \nu)$. For any $\nu \in H^1(\Omega_0)$ and $\nu = 0$ on Γ_2,

$$B(u_\varepsilon - u_0, \nu) = \int_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_0) \cdot \nabla \nu dx + \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} p(u_\varepsilon - u_0) \nu d\sigma$$

$$= \int_{\Gamma_1^\varepsilon} \left(- \varepsilon^{\alpha-1} p u_0 - \frac{\partial u_0}{\partial \nu_\varepsilon} + \varepsilon^{\alpha-1} g \right) \nu d\sigma.$$

It follows from (3.2), that

$$\frac{\partial u_0}{\partial \nu_0} + P(x') u_0 = G(x') \quad \text{on } \Gamma_1.$$

Hence

$$\left(p u_0 + \frac{p}{P} \left(\frac{\partial u_0}{\partial v_0} - G \right) \right) \Big|_{\Gamma_1^\varepsilon} \leq C\varepsilon.$$

This gives

$$B(u_\varepsilon - u_0, \nu) = \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} \left(g - \frac{p}{P} G \right) \nu d\sigma + \int_{\Gamma_1^\varepsilon} \left(\varepsilon^{\alpha-1} \frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) \nu d\sigma$$

$$+ O(\varepsilon^\alpha) \| \nu \|_{H^1(\Omega_0)}.$$

In view of (3.3), we obtain

$$B(u_\varepsilon - u_0 - \varepsilon^\alpha u_1, \nu) = \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} \left(g - \frac{p}{P} G \right) \nu d\sigma + \int_{\Gamma_1^\varepsilon} \left(\varepsilon^{\alpha-1} \frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) \nu d\sigma$$

$$- \int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla \nu dx + O(\varepsilon^\alpha) \| \nu \|_{H^1(\Omega_\varepsilon)}. \tag{3.4}$$

We shall next deal with the term $\int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla \nu dx$. Using the change of variables, we find that

$$\int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla \nu dx = \varepsilon^{(n-1)\alpha} \int_{\widetilde{D}} \nabla_\xi u_1 \cdot \nabla_\xi \nu d\xi$$

$$= \varepsilon^{(n-1)\alpha} \int_{\partial \widetilde{D}} \frac{\partial u_1}{\partial \nu_\xi} \nu d\sigma_\xi,$$

where we have used $\Delta_\xi u_1 = 0$.

It follows from (2.2), that

$$\int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla \nu dx = \varepsilon^{(n-1)\alpha} \Big(\int_{\partial \widetilde{D}} \frac{\partial H_0}{\partial \nu_\xi} \nu d\sigma_\xi + \int_{\partial \widetilde{D}} \frac{\partial H_i}{\partial \nu_\xi^i} \frac{\partial u_0}{\partial x_i} \nu d\sigma_\xi$$

$$+ \int_{\partial \widetilde{D}} \frac{\partial H_n}{\partial \nu_\xi^n} \frac{\partial u_0}{\partial x_n} \nu d\sigma_\xi \Big)$$

$$\doteq A_1 + A_2 + A_3, \tag{3.5}$$

where

$$A_1 = \varepsilon^{(n-1)\alpha} \int_{\partial \widetilde{D}} \frac{\partial H_0}{\partial \nu_\xi} \nu d\sigma_\xi$$

$$= \varepsilon^{(n-1)\alpha} \int_{\partial \widetilde{D}} \Big(g - \frac{p}{P} G\Big) \nu \frac{|\nabla_{\xi'} F(x', \xi')|}{(1 + |\nabla_{\xi'} F(x', \xi')|^2)^{1/2}} d\sigma_\xi$$

$$= \varepsilon^{(n-1)\alpha} \int_{\Gamma_\xi} \Big(g - \frac{p}{P} G\Big) \nu |\nabla_{\xi'} F(x', \xi')| d\xi',$$

$$A_2 = \varepsilon^{(n-1)\alpha} \int_{\partial \widetilde{D}} \frac{\partial H_i}{\partial \nu_\xi^i} \frac{\partial u_0}{\partial x_i} \nu d\sigma_\xi$$

$$= \varepsilon^{(n-1)\alpha} \int_{\partial \widetilde{D}} \frac{-1}{(1 + |\nabla_{\xi'} F(x', \xi')|^2)^{1/2}} \frac{\partial F(x', \xi')}{\partial \xi_i} \frac{\partial u_0}{\partial x_i} \nu d\sigma_\xi$$

$$= \varepsilon^{(n-1)\alpha} \int_{\Gamma_\xi} - \frac{\partial F(x', \xi')}{\partial \xi_i} \frac{\partial u_0}{\partial x_i} \nu d\xi',$$

and

$$A_3 = \varepsilon^{(n-1)\alpha} \int_{\partial \widetilde{D}} \frac{\partial H_n}{\partial \nu_\xi^n} \frac{\partial u_0}{\partial x_n} \nu d\sigma_\xi$$

$$= \varepsilon^{(n-1)\alpha} \int_{\partial \widetilde{D}} \Big(\frac{1}{(1 + |\nabla_{\xi'} F(x', \xi')|^2)^{1/2}}$$

$$- \frac{p}{P} \frac{|\nabla_{\xi'} F(x', \xi')|}{(1 + |\nabla_{\xi'} F(x', \xi')|^2)^{1/2}} \frac{\partial u_0}{\partial x_n} \nu d\sigma_\xi$$

$$= \varepsilon^{(n-1)\alpha} \int_{\Gamma_\xi} (1 - \frac{p}{P} |\nabla_{\xi'} F(x', \xi')|) \frac{\partial u_0}{\partial x_n} \nu d\xi'.$$

Note that

$$\int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} (g - \frac{p}{P} G) \nu d\sigma$$

$$= \int_{\Gamma_1} \varepsilon^{\alpha-1} (g - \frac{p}{P} G) \nu \cdot \varepsilon^{1-\alpha} |\nabla_{\xi'} F(x', \xi')| dx' + O(\varepsilon^{\alpha-1}) \| \nu \|_{H^1(\Omega_0)}$$

$$= \int_{\Gamma_1} (g - \frac{p}{P} G) \nu \cdot |\nabla_{\xi'} F(x', \xi')| dx' + O(\varepsilon^{\alpha-1}) \| \nu \|_{H^1(\Omega_0)}$$

$$= \varepsilon^{(n-1)\alpha} \int_{\Gamma_\xi} (g - \frac{p}{P} G) \nu \cdot |\nabla_{\xi'} F(x', \xi')| d\xi' + O(\varepsilon^{\alpha-1}) \| \nu \|_{H^1(\Omega_0)},$$

where we have used the change of variables $x' = \varepsilon^\alpha \xi'$. Hence

$$\int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} (g - \frac{p}{P} G) \nu d\sigma - A_1 = O(\varepsilon^{\alpha-1}) \| \nu \|_{H^1(\Omega_0)}.$$

Finally, we shall evaluate the term $\int_{\Gamma_1^\varepsilon} (\varepsilon^{\alpha-1} \frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon}) \nu d\sigma$.

Also, note that since $\nu_0 = (0, \cdots, 0, -1)$ and $\nu_\varepsilon = \varepsilon^{\alpha-1} (\frac{\partial F}{\partial \xi'}, -1)$ $\times \frac{1}{|\nabla_{\xi'} F(x', \xi')|} + O(\varepsilon^{\alpha-1})$,

$$\int_{\Gamma_1^\varepsilon}(\varepsilon^{\alpha-1}\frac{p}{P}\frac{\partial u_0}{\partial \nu_0}-\frac{\partial u_0}{\partial \nu_\varepsilon})\nu d\sigma=\int_{\Gamma_1^\varepsilon}(\varepsilon^{\alpha-1}\frac{p}{P}\nabla u_0\cdot\nu_n-\nabla u_0\cdot\nu_\varepsilon)\nu d\sigma$$

$$=\int_{\Gamma_1^\varepsilon}(\varepsilon^{\alpha-1}\frac{p}{P}\frac{\partial u_0}{\partial x_n}\nu_n-\frac{\partial u_0}{\partial x_n}\nu_\varepsilon^n)\nu d\sigma-\int_{\Gamma_1^\varepsilon}\frac{\partial u_0}{\partial x_i}\nu_\varepsilon^i\nu d\sigma$$

$$=\varepsilon^{\alpha-1}\int_{\Gamma_1^\varepsilon}\frac{\partial u_0}{\partial x_n}(\frac{1}{|\nabla_{\xi'}F(x',\xi')|}-\frac{p}{P})\nu d\sigma$$

$$-\varepsilon^{\alpha-1}\int_{\Gamma_1^\varepsilon}\frac{\partial u_0}{\partial x_i}\frac{1}{|\nabla_{\xi'}F(x',\xi')|}\frac{\partial F}{\partial \xi_i}\nu d\sigma+O(\varepsilon^\alpha)\|\nu\|_{H^1(\Omega_0)}$$

$$=\int_{\Gamma_1}\frac{\partial u_0}{\partial x_n}(1-\frac{p}{P}|\nabla_{\xi'}F(x',\xi')|)\nu dx'$$

$$-\int_{\Gamma_1}\frac{\partial u_0}{\partial x_i}\frac{\partial F}{\partial \xi_i}\nu dx'+O(\varepsilon^{\alpha-1})\|\nu\|_{H^1(\Omega_0)}$$

$$=\varepsilon^{(n-1)\alpha}\int_{\Gamma_\xi}\frac{\partial u_0}{\partial x_n}(1-\frac{p}{P}|\nabla_{\xi'}F(x',\xi')|)\nu d\xi'$$

$$-\varepsilon^{(n-1)\alpha}\int_{\Gamma_\xi}\frac{\partial u_0}{\partial x_i}\frac{\partial F}{\partial \xi_i}\nu d\xi'+O(\varepsilon^{\alpha-1})\|\nu\|_{H^1(\Omega_0)}.$$

By the definition of A_2 and A_3 in (3.5), this implies that

$$\int_{\Gamma_1^\varepsilon}(\varepsilon^{\alpha-1}\frac{p}{P}\frac{\partial u_0}{\partial \nu_0}-\frac{\partial u_0}{\partial \nu_\varepsilon})\nu d\sigma-A_2-A_3=O(\varepsilon^{\alpha-1})\|\nu\|_{H^1(\Omega_0)}.$$

This together with (3.4), gives

$$B(u_\varepsilon-u_0-\varepsilon^\alpha u_1,\nu)\le C\varepsilon^{\alpha-1}\|\nu\|_{H^1(\Omega_0)}.$$

Choosing $\nu=u_\varepsilon-u_0-\varepsilon^\alpha(u_1-u_1|_{x_n=B})$, we obtain the desired result.

□

4. Error Estimate up to $O(\varepsilon^{\alpha})$

In this section, we will improve the error estimate up to the order of $O(\varepsilon^{\alpha})$. The main technique is using the second order Taylor's expansion. Our main result is the following theorem:

Theorem 4.1. *Let* $f \in L^2(R^n)$ *and* u_ε *is a solution of problem* (1.1). *Suppose that* $p(x', \xi')$, $g(x', \xi')$, *and* $F(x', \xi')$ *satisfy* (1.3) *and* (1.4). *Then these exists a constant C does not depend on* ε, *such that*

$$\|u_\varepsilon - u_0 - \varepsilon^{\alpha} u_1\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{\alpha}, \tag{4.1}$$

where u_0 *is a solution of homogenized problem* (3.2) *and* u_1 *is defined in* (2.2).

Proof. Following the same line of research, we consider

$$B(u_\varepsilon - u_0, \nu) = \int_{\Gamma_1^\varepsilon} \left(-\varepsilon^{\alpha-1} p u_0 - \frac{\partial u_0}{\partial \nu_\varepsilon} + \varepsilon^{\alpha-1} g \right) \nu d\sigma.$$

This together with (3.2) and the Taylor's expansion, gives

$$\left(\frac{\partial u_0}{\partial \nu_0} + P(x')u_0 - G(x')\right)\Big|_{\Gamma_1^\varepsilon} = \varepsilon F(x', \xi') \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial \nu_0} + P(x')u_0 - G(x')\right)\Big|_{\Gamma_1} + O(\varepsilon^2).$$

It follows that

$$\begin{aligned} B(u_\varepsilon - u_0, \nu) = & \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} \left(g - \frac{p}{P} G\right) \nu d\sigma + \int_{\Gamma_1^\varepsilon} \left(\varepsilon^{\alpha-1} \frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon}\right) \nu d\sigma \\ & - \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha} \frac{p}{P} F(x', \xi') \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial \nu_0} + P(x')u_0\right)\Big|_{\Gamma_1} \nu d\sigma \\ & + O(\varepsilon^{\alpha+1}) \|\nu\|_{H^1(\Omega_0)}. \end{aligned}$$

Note that

$$B(-\varepsilon^{\alpha}u_1, \nu) = \int_{\Omega_\varepsilon} \varepsilon^{\alpha}\Delta u_1 \nu dx - \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha} \frac{\partial u_1}{\partial \nu_\varepsilon} \nu d\sigma - \int_{\Gamma_1^\varepsilon} \varepsilon^{2\alpha-1} p u_1 \nu d\sigma$$

$$= \int_{\Omega_\varepsilon} \varepsilon^{\alpha}\Delta_x u_1 \nu dx + \int_{\Omega_\varepsilon} 2\nabla_x \nabla_\xi u_1 \nu dx$$

$$- \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha} \frac{\partial u_1}{\partial \nu_\varepsilon} \nu d\sigma - \int_{\Gamma_1^\varepsilon} \varepsilon^{2\alpha-1} p u_1 \nu d\sigma,$$

where we have used $\nabla_\xi u_1 = 0$.

Thus we obtain

$$B(u_\varepsilon - u_0 - \varepsilon^{\alpha}u_1, \nu) = \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1}(g - \frac{p}{P}G)\nu d\sigma$$

$$+ \int_{\Gamma_1^\varepsilon} (\varepsilon^{\alpha-1} \frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon})\nu d\sigma$$

$$- \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha} \frac{p}{P} F(x', \xi') \frac{\partial}{\partial x_n} (\frac{\partial u_0}{\partial \nu_0} + P(x')u_0)|_{\Gamma_1} \nu d\sigma$$

$$+ \int_{\Omega_\varepsilon} \varepsilon^{\alpha}\Delta_x u_1 \nu + 2\nabla_x \nabla_\xi u_1 \nu dx - \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha} \frac{\partial u_1}{\partial \nu_\varepsilon} \nu d\sigma$$

$$- \int_{\Gamma_1^\varepsilon} \varepsilon^{2\alpha-1} p u_1 \nu d\sigma + O(\varepsilon^{\alpha+1}) \| \nu \|_{H^1(\Omega_0)}, \tag{4.2}$$

where $\nu_0 = (0, \cdots, 0, -1)$, and

$$\nu_\varepsilon = \varepsilon^{\alpha-1}(\frac{\partial F}{\partial \xi'}, -1) \frac{1}{|\nabla_{\xi'} F(x', \xi')|} + O(\varepsilon^{\alpha}).$$

In view of change of variables and (2.2), we have

$$2\int_{\Omega_\varepsilon} \nabla_x \nabla_\xi u_1 \nu dx$$

$$= 2\varepsilon^{n\alpha} \int_{\tilde{D}} \nabla_x \nabla_\xi u_1 \nu d\xi$$

$$= 2\varepsilon^{n\alpha}\int_{\partial\widetilde{D}}\nabla_x u_1\cdot\nu_\xi\nu d\sigma_\xi - 2\varepsilon^{n\alpha}\int_{\widetilde{D}}\nabla_x u_1\cdot\nabla_\xi\nu d\xi$$

$$\leq 2\varepsilon^{\alpha}\int_{\Gamma_1^\varepsilon}\nabla_x u_1\cdot\nu_\xi\nu d\sigma + C\varepsilon^{\alpha}\int_{\Omega_\varepsilon}\exp(-\lambda\xi_n)|\nabla_x\nu|\,dx$$

$$\leq 2\varepsilon^{2\alpha-1}\int_{\Gamma_1^\varepsilon}\nabla_x u_1\Big(\frac{\partial F}{\partial\xi'},-1\Big)\frac{1}{|\nabla_{\xi'}F(x',\xi')|}\nu d\sigma + C\varepsilon^{3\alpha/2}\|\nu\|_{H^1(\Omega_0)}. \tag{4.3}$$

A direct computation (using (2.2)) shows that

$$\varepsilon^{\alpha}\frac{\partial u_1}{\partial\nu_\varepsilon}\Big|_{\Gamma_1^\varepsilon} = (\nabla_\xi u_1\cdot\nu_\varepsilon + \varepsilon^{\alpha}\nabla_x u_1\cdot\nu_\varepsilon)\big|_{\Gamma_1^\varepsilon} = P_1 + P_2 + O(\varepsilon^{2\alpha}), \tag{4.4}$$

where

$$P_1 = \nabla_\xi u_1\frac{\varepsilon^{\alpha-1}\Big(\frac{\partial F}{\partial\xi'},-1\Big)}{|\nabla_{\xi'}F(x',\xi')|}$$

$$= \varepsilon^{\alpha-1}\frac{\partial u_1}{\partial\nu_\xi}\frac{(1+|\nabla_{\xi'}F(x',\xi')|^2)^{1/2}}{|\nabla_{\xi'}F(x',\xi')|}$$

$$= \varepsilon^{\alpha-1}\Big(g-\frac{p}{P}G\Big) - \varepsilon^{\alpha-1}\frac{\partial u_0}{\partial x_i}\frac{\partial F(x',\xi')}{\partial\xi_i}\frac{1}{|\nabla_{\xi'}F(x',\xi')|}$$

$$+\varepsilon^{\alpha-1}\frac{\partial u_0}{\partial x_n}\Big(\frac{1}{|\nabla_{\xi'}F(x',\xi')|}-\frac{p}{P}\Big),$$

and

$$P_2 = \varepsilon^{2\alpha-1}\nabla_x u_1\frac{\Big(\frac{\partial F}{\partial\xi'},-1\Big)}{|\nabla_{\xi'}F(x',\xi')|}.$$

It follows from (4.4), that

$$\int_{\Gamma_1^\varepsilon}\varepsilon^{\alpha}\frac{\partial u_1}{\partial\nu_\varepsilon}\nu d\sigma = \int_{\Gamma_1^\varepsilon}P_1\nu d\sigma + \int_{\Gamma_1^\varepsilon}P_2\nu d\sigma + O(\varepsilon^{2\alpha})\|\nu\|_{H^1(\Omega_0)}.$$

A similar computation yields the following result:

$$\int_{\Gamma_1^\varepsilon} (\varepsilon^{\alpha-1} \frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon}) \nu d\sigma = \int_{\Gamma_1^\varepsilon} - \varepsilon^{\alpha-1} \frac{p}{P} \frac{\partial u_0}{\partial x_n} \nu d\sigma - \widetilde{Q} + O(\varepsilon^{\alpha+1}) \| \nu \|_{H^1(\Omega_0)}, \tag{4.5}$$

where

$$\widetilde{Q} = \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} \frac{1}{|\nabla_{\xi'} F|} (\frac{\partial u_0}{\partial x_i} \frac{\partial F}{\partial \xi_i} - \frac{\partial u_0}{\partial x_n}) \nu d\sigma.$$

As in Section 3, we extend $u_\varepsilon - u_0 - \varepsilon^\alpha u_1$ into Ω_0 without increasing the H^1-norm. It remains to substitute $\nu = u_\varepsilon - u_0 - \varepsilon^\alpha (u_1 - u_1|_{x_n=B})$. This, together with (4.2)-(4.5), yields the desired estimate. □

5. Error Estimate up to $O(\varepsilon^{2\alpha-1})$ or $O(\varepsilon^{\alpha+1})$

The goal of this section is to improve the error estimate up to the order from $O(\varepsilon^\alpha)$ to $O(\varepsilon^{2\alpha-1})$ or $O(\varepsilon^{\alpha+1})$. This can be obtained via correctors. The main result is the following theorem:

Theorem 5.1. *Let* $f \in L^2(R^n)$ *and* u_ε *is a solution of problem* (1.1). *Suppose that* $p(x', \xi')$, $g(x', \xi')$, *and* $F(x', \xi')$ *satisfy* (1.3) *and* (1.4). *Then these exists a constant C does not depend on* ε, *such that*

$$\begin{cases} \|u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon\phi\|_{H^1(\Omega_\varepsilon)} \le C\varepsilon^{2\alpha-1}, \text{ if } 1 < \alpha < 2, \\ \|u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon\phi\|_{H^1(\Omega_\varepsilon)} \le C\varepsilon^{\alpha+1}, \text{ if } \quad \alpha \ge 2, \end{cases} \tag{5.1}$$

where u_0 *is a solution of homogenized problem* (3.2) *and* u_1, ϕ *are defined in* (2.2) *and* (5.7).

Proof. Analogously to Sections 3 and 4, we consider again

$$B(u_\varepsilon - u_0 - \varepsilon^\alpha u_1, \nu) = \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1}(g - \frac{p}{P}G)\nu d\sigma + \int_{\Gamma_1^\varepsilon}(\varepsilon^{\alpha-1}\frac{p}{P}\frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon})\nu d\sigma$$

$$- \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha \frac{p}{P} F(x', \xi')\frac{\partial}{\partial x_n}(\frac{\partial u_0}{\partial \nu_0} + P(x')u_0)|_{\Gamma_1} \nu d\sigma$$

$$+ \int_{\Omega_\varepsilon} \varepsilon^\alpha \Delta_\varepsilon u_1 \nu + 2\nabla_x \nabla_\xi u_1 \nu dx - \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha \frac{\partial u_1}{\partial \nu_\varepsilon}\nu d\sigma$$

$$- \int_{\Gamma_1^\varepsilon} \varepsilon^{2\alpha-1} p u_1 \nu d\sigma + O(\varepsilon^{\alpha+1})\| \nu \|_{H^1(\Omega_\varepsilon)}, \tag{5.2}$$

where $\nu_0 = (0, \cdots, 0, -1)$. Using the second order expansion, we have

$$\nu_\varepsilon = \frac{\varepsilon^{\alpha-1}(\frac{\partial F}{\partial \xi'}, -1)}{|\nabla_{\xi'} F(x', \xi')|} + \varepsilon^\alpha \frac{\partial F}{\partial x'}(\frac{1}{|\nabla_{\xi'} F(x', \xi')|} - \frac{(\frac{\partial F}{\partial \xi'})^2}{|\nabla_{\xi'} F(x', \xi')|^3},$$

$$\frac{\frac{\partial F}{\partial \xi'}}{|\nabla_{\xi'} F(x', \xi')|^3}) + O(\varepsilon^{\alpha+1}). \tag{5.3}$$

Note that

$$|\int_{\Omega_\varepsilon} \varepsilon^\alpha \Delta_x u_1 \nu dx| \le C\varepsilon^\alpha \int_{\Omega_\varepsilon} \exp(-\lambda x_n / \varepsilon^\alpha)|\nu| dx$$

$$\le C\varepsilon^{3\alpha/2}\| \nu \|_{H^1(\Omega_0)}, \tag{5.4}$$

and

$$2\int_{\Omega_\varepsilon} \nabla_x \nabla_\xi u_1 \nu dx$$

$$\le 2\varepsilon^{n\alpha}\int_{\partial \widetilde{D}} \nabla_x u_1 \cdot \nu_\xi \nu d\xi + C\varepsilon^\alpha \int_{\Omega_\varepsilon} \exp(-\lambda \varepsilon_n)|\nabla_x \nu| dx$$

$$\le 2\varepsilon^{2\alpha-1}\int_{\Gamma_1^\varepsilon} \nabla_x u_1 (\frac{\partial F}{\partial \xi'}, -1)\frac{1}{|\nabla_{\xi'} F(x', \xi')|}\nu d\sigma$$

$$+ 2\varepsilon^{2\alpha}\int_{\Gamma_1^\varepsilon}\nabla_x u_1 \frac{\partial F}{\partial x'}\Big(\frac{1}{|\nabla_{\xi'}F(x',\xi')|} - \frac{(\frac{\partial F}{\partial \xi'})^2}{|\nabla_{\xi'}F(x',\xi')|^3}, \frac{\frac{\partial F}{\partial \xi'}}{|\nabla_{\xi'}F(x',\xi')|^3}\Big)\nu d\sigma$$

$$+ C\varepsilon^{3\alpha/2}\| \nu \|_{H^1(\Omega_0)}. \tag{5.5}$$

It follows from (2.2) and (5.3), that

$$\varepsilon^\alpha \frac{\partial u_1}{\partial \nu_\varepsilon}\Big|_{\Gamma_1^\varepsilon} = (\nabla_\xi u_1 \cdot \nu_\varepsilon + \varepsilon^\alpha \nabla_x u_1 \cdot \nu_\varepsilon)\big|_{\Gamma_1^\varepsilon} = P_1 + P_2 + O(\varepsilon^{2\alpha}), \tag{5.6}$$

where

$$\begin{aligned} P_1 &= \varepsilon^{\alpha-1}\frac{\partial u_1}{\partial \nu_\xi}\frac{(1+|\nabla_{\xi'}F(x',\xi')|^2)^{1/2}}{|\nabla_{\xi'}F(x',\xi')|} \\ &= \varepsilon^{\alpha-1}(g - \frac{p}{P}G) - \varepsilon^{\alpha-1}\frac{\partial u_0}{\partial x_i}\frac{\partial F(x',\xi')}{\partial \xi_i}\frac{1}{|\nabla_{\xi'}F(x',\xi')|} \\ &\quad + \varepsilon^{\alpha-1}\frac{\partial u_0}{\partial x_n}\Big(\frac{1}{|\nabla_{\xi'}F(x',\xi')|} - \frac{p}{P}\Big), \end{aligned}$$

and

$$\begin{aligned} P_2 &= \nabla_\xi u_1 \varepsilon^\alpha \frac{\partial F}{\partial x'}\Big(\frac{1}{|\nabla_{\xi'}F(x',\xi')|} - \frac{(\frac{\partial F}{\partial \xi'})^2}{|\nabla_{\xi'}F(x',\xi')|^3}, \frac{\frac{\partial F}{\partial \xi'}}{|\nabla_{\xi'}F(x',\xi')|^3}\Big) \\ &\quad + \varepsilon^{2\alpha-1}\nabla_x u_1 \frac{(\frac{\partial F}{\partial \xi'}, -1)}{|\nabla_{\xi'}F(x',\xi')|}. \end{aligned}$$

Combining all these terms in (5.6), we have proved that

$$\int_{\Gamma_1^\varepsilon}\varepsilon^\alpha \frac{\partial u_1}{\partial \nu_\varepsilon}\nu d\sigma = \int_{\Gamma_1^\varepsilon} P_1 \nu d\sigma + \int_{\Gamma_1^\varepsilon} P_2 \nu d\sigma + O(\varepsilon^{2\alpha})\| \nu \|_{H^1(\Omega_0)}.$$

A similar calculation yields the following result:

$$\int_{\Gamma_1^\varepsilon}(\varepsilon^{\alpha-1}\frac{p}{P}\frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon})\nu d\sigma = \int_{\Gamma_1^\varepsilon} -\varepsilon^{\alpha-1}\frac{p}{P}\frac{\partial u_0}{\partial x_n}\nu d\sigma - \widetilde{Q} + O(\varepsilon^{\alpha+1})\| \nu \|_{H^1(\Omega_0)},$$

where

$$\widetilde{Q} = \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} \frac{1}{|\nabla_{\xi'} F|} \left(\frac{\partial u_0}{\partial x_i} \frac{\partial F}{\partial \xi_i} - \frac{\partial u_0}{\partial x_n} \right)$$

$$+ \varepsilon^\alpha \nabla u_0 \frac{\partial F}{\partial x'} \left(\frac{1}{|\nabla_{\xi'} F(x', \xi')|} - \frac{\left(\frac{\partial F}{\partial \xi'}\right)^2}{|\nabla_{\xi'} F(x', \xi')|^3}, \frac{\frac{\partial F}{\partial \xi'}}{|\nabla_{\xi'} F(x', \xi')|^3} \right) \nu d\sigma.$$

Then, we introduced a corrector so as to cancel lower power of ε^α.

For any given function of the form $\widetilde{g}$, suppose that $\widetilde{u}_\varepsilon$ is the solution of the following problem:

$$\begin{cases} -\Delta \widetilde{u}_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial \widetilde{u}_\varepsilon}{\partial \nu_\varepsilon} + \varepsilon^{\alpha-1} p\left(x', \dfrac{x'}{\varepsilon^\alpha}\right) \widetilde{u}_\varepsilon = \varepsilon^{\alpha-1} \widetilde{g}\left(x', \dfrac{x'}{\varepsilon^\alpha}\right) & \text{on } \Gamma_1^\varepsilon, \\ \widetilde{u}_\varepsilon = 0 & \text{on } \Gamma_2. \end{cases}$$

Meanwhile, assume that $\phi(x)$ is the solution of the homogenized problem:

$$\begin{cases} -\Delta \phi = 0 & \text{in } \Omega_0, \\ \dfrac{\partial \phi}{\partial \nu_0} + \varepsilon^{\alpha-1} P(x') \phi = \varepsilon^{\alpha-1} \widetilde{G}(x') & \text{on } \Gamma_1, \\ \phi = 0 & \text{on } \Gamma_2, \end{cases} \tag{5.7}$$

where $P(x')$ is defined in (2.1), and

$$\widetilde{G}(x') = \int_0^1 \cdots \int_0^1 |\nabla_{\xi'} F(x', \xi')| \widetilde{g}(x', \xi') d\xi'.$$

We may invoke Theorem 4.1 to conclude that

$$\|\widetilde{u}_\varepsilon - \phi - \varepsilon^\alpha \widetilde{u}_1\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^\alpha,$$

where $\widetilde{u}_1$ follows the essentially the same steps as Section 2, with the obvious modifications.

It follows that

$$B(\varepsilon\phi, \nu) = B(\varepsilon(\phi - \widetilde{u}_\varepsilon + \varepsilon^\alpha \widetilde{u}_1), \nu) + B(\varepsilon\widetilde{u}_\varepsilon - \varepsilon^{\alpha+1}\widetilde{u}_1, \nu)$$

$$= B(\varepsilon\widetilde{u}_\varepsilon, \nu) + O(\varepsilon^{\alpha+1}) \| \nu \|_{H^1(\Omega_0)}.$$

We now write

$$B(u_\varepsilon - u_0 - \varepsilon^\alpha u_1 + \varepsilon\phi, \nu) = \varepsilon^\alpha \int_{\Gamma_1^\varepsilon} \widetilde{g}\nu d\sigma - \varepsilon^\alpha \sum_i I_i + O(\varepsilon^{\alpha+1}) \| \nu \|_{H^1(\Omega_0)},$$

where

$$\begin{cases} I_1 = \int_{\Gamma_1^\varepsilon} \nabla u_0 \dfrac{\partial F}{\partial x'} \left(\dfrac{1}{|\nabla_{\xi'} F(x', \xi')|} - \dfrac{(\frac{\partial F}{\partial \xi'})^2}{|\nabla_{\xi'} F(x', \xi')|^3}, \dfrac{\frac{\partial F}{\partial \xi'}}{|\nabla_{\xi'} F(x', \xi')|^3} \right) \nu d\sigma, \\ I_2 = \int_{\Gamma_1^\varepsilon} \dfrac{p}{P} F(x', \xi') \dfrac{\partial}{\partial x_n} \left(\dfrac{\partial u_0}{\partial \nu_0} + P(x') u_0 \right) \nu d\sigma, \\ I_3 = \int_{\Gamma_1^\varepsilon} \nabla_\xi u_1 \dfrac{\partial F}{\partial x'} \left(\dfrac{1}{|\nabla_{\xi'} F(x', \xi')|} - \dfrac{(\frac{\partial F}{\partial \xi'})^2}{|\nabla_{\xi'} F(x', \xi')|^3}, \dfrac{\frac{\partial F}{\partial \xi'}}{|\nabla_{\xi'} F(x', \xi')|^3} \right) \nu d\sigma. \end{cases}$$

Then we just need to take that

$$\widetilde{g} = \sum_i \widetilde{I}_i,$$

where

$$\begin{cases} \widetilde{I}_1 = \nabla u_0 \dfrac{\partial F}{\partial x'} \left(\dfrac{1}{|\nabla_{\xi'} F(x', \xi')|} - \dfrac{(\frac{\partial F}{\partial \xi'})^2}{|\nabla_{\xi'} F(x', \xi')|^3}, \dfrac{\frac{\partial F}{\partial \xi'}}{|\nabla_{\xi'} F(x', \xi')|^3} \right), \\ \widetilde{I}_2 = \dfrac{p}{P} F(x', \xi') \dfrac{\partial}{\partial x_n} \left(\dfrac{\partial u_0}{\partial \nu_0} + P(x') u_0 \right), \\ \widetilde{I}_3 = \nabla_\xi u_1 \dfrac{\partial F}{\partial x'} \left(\dfrac{1}{|\nabla_{\xi'} F(x', \xi')|} - \dfrac{(\frac{\partial F}{\partial \xi'})^2}{|\nabla_{\xi'} F(x', \xi')|^3}, \dfrac{\frac{\partial F}{\partial \xi'}}{|\nabla_{\xi'} F(x', \xi')|^3} \right). \end{cases}$$

Combining all these terms and choosing $\nu = u_\varepsilon - u_0 - \varepsilon^\alpha (u_1 - u_1|_{x_n=B}) + \varepsilon\phi$, we conclude that

$$\|u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon\phi\|_{H^1(\Omega_\varepsilon)} \leq C(\varepsilon^{2\alpha-1} + \varepsilon^{\alpha+1}).$$

This completes the proof after taking into account the different case when $1 < \alpha < 2$ and $\alpha \geq 2$. □

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