HOMOGENIZATION PROBLEM WITH OSCILLATING BOUNDARY

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Abstract

In this paper, we study the asymptotic behaviour of solutions for homogenization problems of partial differential equation with a very rapidly oscillating locally periodic boundary. As a consequence, we show that \( u_\varepsilon \) converges to \( u_0 \) in the \( H^1 \)-norm and the error is \( O(\varepsilon^{2\alpha-1}) \) (if \( 1 < \alpha < 2 \)) or \( O(\varepsilon^{\alpha+1}) \) (if \( \alpha \geq 2 \)), where \( u_\varepsilon \) and \( u_0 \) are solutions of oscillating and homogenized boundary problems, respectively. Our method is based on correctors that can be used to obtain effective approximation.

1. Introduction

Homogenization of periodically oscillating boundaries, such problems arises in the context of fluid flows over a rough surface [3, 4], of reinforcement by a thin layer [11], or of electromagnetic scattering by

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an obstacle with a periodic coating [1, 15]. Indeed, boundary layers results can be applied to the homogenization of neutronic diffusion or transport equation [2, 8, 12]. Studying boundary layers is also the key for determining interface transmission conditions in various mechanical problems [7, 13, 17].

In this paper, we are interested in the following boundary-value problem for the second order elliptic equation with a very rapidly oscillating locally periodic boundary:

\[
\begin{align*}
-\Delta u_\epsilon &= f(x) \quad \text{in } \Omega_\epsilon, \\
\frac{\partial u_\epsilon}{\partial n_\epsilon} + \epsilon^{a-1} p(x', \frac{x'}{\epsilon^a}) u_\epsilon &= \epsilon^{a-1} g(x', \frac{x'}{\epsilon^a}) \quad \text{on } \Gamma_1^\epsilon, \\
u_\epsilon &= 0 \quad \text{on } \Gamma_2.
\end{align*}
\]  

(1.1)

where \( x' = (x_1, x_2, \ldots, x_{n-1}) \) and \( \nu_\epsilon \) is the outward normal.

Let \( \Omega_\epsilon \) be a bounded domain in \( \mathbb{R}^n \) and \( \partial \Omega_\epsilon = \Gamma_1^\epsilon \cup \Gamma_2 \), where a curve

\[
\Gamma_1^\epsilon : x_n = \epsilon F(x', \frac{x'}{\epsilon^a}), \quad \alpha > 1.
\]  

(1.2)

Without loss of generality, we assume that \( \Omega_\epsilon \) is a domain bounded by \( |x'| \leq (0, 1)^{n-1}, x_n = B \) and curve \( \Gamma_1^\epsilon \), where \( B \) is a positive absolutely constant.

We suppose that

\[
p(x', \zeta') > 0 \quad \text{and} \quad F(x', \zeta') \quad \text{has compactly supported on } \Gamma_1 \quad \text{uniformly in } \zeta'.
\]  

(1.3)

We also impose the smoothness condition and periodicity condition

\[
p(x', \zeta'), g(x', \zeta') \quad \text{and} \quad F(x', \zeta') \quad \text{are smooth functions in } (x', \zeta') \quad \text{and } \quad 1\text{-periodic in } \zeta',
\]  

(1.4)

where \( \zeta' = \frac{x'}{\epsilon^a} \).
As the author know, there are many papers about results of convergence rates for the elliptic homogenization problems with oscillating boundary data. In 1997, Friedman et al. [16] had studied two dimensional domain, whose boundary is oscillating according to three scales. They extended the results of Belyaev’s [9, 10] to the three-scale oscillating boundary. In 1999, Chechkin et al. [14] considered the problem (1.1). They obtained the error estimate of solutions as

\[ \|u_\varepsilon - u_0\|_{H^1(\Omega_\varepsilon)} \leq C(\varepsilon^{1/2} + \varepsilon^{\alpha-1}), \]

where \( u_0 \) is the solution of homogenized problem according to problem (1.1).

Recently, Aleksanyan et al. ([5, 6]) studied the boundary homogenization for Dirichlet problem. In particular, they proved point wise and \( L^p \) convergence results.

The main purpose of this paper is to improve the results of Chechkin et al. (see [14], Theorem 2) and obtain better error estimate up to the order of \( O(\varepsilon^{2\alpha-1}) \) (if \( 1 < \alpha < 2 \)) or \( O(\varepsilon^{\alpha+1}) \) (if \( \alpha > 2 \)) by using the correctors.

We now describe the outline of this paper as well as some key ideas used in the proof of its main results. In Section 2, we establish the corrector \( u_1 \) that plays an important role in improving the power of \( \varepsilon \). In Section 3, we show that the solution \( u_\varepsilon \) of problem (1.1) converges to the solution \( u_0 \) of the corresponding homogenized problem in the \( H^1 \)-norm with error estimate up to the order of \( O(\varepsilon^{\alpha-1}) \). This can be obtained via the corrector \( u_1 \). Following the same line of research, by using the second order expansion, we improve the error estimate up to the order of \( O(\varepsilon^{\alpha}) \) in Section 4. Finally, in Section 5, we obtain the approximation \( u_\varepsilon \) to the higher power of \( \varepsilon^{2\alpha-1} \) (if \( 1 < \alpha < 2 \)) or \( \varepsilon^{\alpha+1} \) (if \( \alpha \geq 2 \)) by using correctors \( u_1 \) and \( \phi \).
2. Preliminaries

In order to improve the power of $\varepsilon$, in this section, we shall construct the corrector $u_1$. Firstly, as a preliminary step, we compute the arc elements.

**Proposition 2.1.** Let $d\sigma$ be an element of the $(n - 1)$-dimensional volume of $\Gamma^\varepsilon$. Then

$$d\sigma = \varepsilon^{1-a}(|\nabla\varphi'(x', \xi')| + O(\varepsilon^{a-1})) dx'. $$

**Proof.** This proposition has been proved by Chechkin et al. (see [14], Lemma 2).

We introduce the harmonic functions $H_i(x', \xi)$, $x'$ is treated as a parameter, as solutions of

$$\begin{cases}
\Delta H_0 = 0 & \text{in } \tilde{D}, \\
\frac{\partial H_0}{\partial v_{\xi'}} = \left( g - \frac{P}{\nu} G \right) \frac{|\nabla\varphi F(x', \xi')|}{(1 + |\nabla\varphi F(x', \xi')|^2)^{1/2}} & \text{on } \partial \tilde{D}, \\
|H_0| \leq C \exp(-\lambda \xi_n) & \text{for some } \lambda > 0,
\end{cases}$$

$$\begin{cases}
\Delta H_i = 0 & \text{in } \tilde{D}, \\
\frac{\partial H_i}{\partial v_{\xi_i}} = \left( 1 + |\nabla\varphi F(x', \xi')|^2 \right)^{1/2} \frac{\partial F(x', \xi')}{\partial \xi_i} & \text{on } \partial \tilde{D}, \\
|H_i| \leq C \exp(-\lambda \xi_n) & \text{for some } \lambda > 0,
\end{cases}$$

and

$$\begin{cases}
\Delta H_n = 0 & \text{in } \tilde{D}, \\
\frac{\partial H_n}{\partial v_{\xi_n}} = \left( 1 + |\nabla\varphi F(x', \xi')|^2 \right)^{1/2} \frac{P}{\nu} \frac{|\nabla\varphi F(x', \xi')|}{(1 + |\nabla\varphi F(x', \xi')|^2)^{1/2}} & \text{on } \partial \tilde{D}, \\
|H_n| \leq C \exp(-\lambda \xi_n) & \text{for some } \lambda > 0,
\end{cases}$$
where

\[ P(x') = \int_0^1 \cdots \int_0^1 |\nabla_{\xi'} F(x', \xi')| p(x', \xi') d\xi', \]

\[ G(x') = \int_0^1 \cdots \int_0^1 |\nabla_{\xi'} F(x', \xi')| g(x', \xi') d\xi', \]

\[ \tilde{D} = \{(\xi', \xi_n) : \xi' = (\xi_i), 0 \leq \xi_i \leq 1, F(x', \xi') < \xi_n \}, \]

\[ V_{\xi} = (v_{\xi}^1, v_{\xi}^2, \ldots, v_{\xi}^n) \] is the outward normal, \hspace{1cm} (2.1)

and the curve

\[ \Gamma_{\xi} : \xi_n = F(x', \xi'), 0 \leq \xi_i \leq 1, \]

where \( \xi' = (\xi_i) \) and \( i = 1 \cdots n - 1 \). This system was introduced in [9, 10].

To ensure these solutions \( H_i \) exist, we need to verify the compatibility condition holds true. A simple calculation then gives

\[ \int_{\partial \tilde{D}} \frac{\partial H_0}{\partial \nu_{\xi}} d\sigma_{\xi} = \int_0^1 \cdots \int_0^1 (g - \frac{p}{\tilde{P}} G) |\nabla_{\xi'} F(x', \xi')| d\xi' = 0, \]

\[ \int_{\partial \tilde{D}} \frac{\partial H_i}{\partial \nu_{\xi}} d\sigma_{\xi} = \int_0^1 \cdots \int_0^1 \frac{\partial F(x', \xi')}{\partial \xi_i} d\xi' = 0, \]

where we have used (1.3), and

\[ \int_{\partial \tilde{D}} \frac{\partial H_n}{\partial \nu_{\xi}^n} d\sigma_{\xi} = \int_0^1 \cdots \int_0^1 (1 - \frac{p}{\tilde{P}} |\nabla_{\xi'} F(x', \xi')|) d\xi' = 0. \]

Therefore, these harmonic functions \( H_i \) exist. Then we set corrector \( u_1 \) as follows:

\[ u_1(x', \xi) = H_0(x', \xi) + \frac{\partial u_0}{\partial \xi_i} H_i(x', \xi). \] (2.2)

Throughout this paper, the summation convention is used.
3. Error Estimate up to $O(\varepsilon^{a-1})$

In this section, we will prove the error estimate up to the order of $O(\varepsilon^{a-1})$. The main technique is using the corrector. Our main result is the following theorem:

**Theorem 3.1.** Let $f \in L^2(\mathbb{R}^n)$ and $u_\varepsilon$ is a solution of problem (1.1). Suppose that $p(x', \xi'), g(x', \xi')$, and $F(x', \xi')$ satisfy (1.3) and (1.4). Then these exist a constant $C$ does not depend on $\varepsilon$, such that

$$
\|u_\varepsilon - u_0 - \varepsilon^a u_1\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{a-1},
$$

where $u_0$ is a solution of homogenized problem

$$
\left\{
\begin{array}{ll}
-\Delta u_0 = f(x) & \text{in } \Omega_0, \\
\partial_\nu u_0 + P(x')u_0 = G(x') & \text{on } \Gamma_1, \\
u_0 = 0 & \text{on } \Gamma_2,
\end{array}
\right.
$$

and the curve $\Gamma_1 : x_0 = 0$. Functions $u_1(x', \xi)$, $P(x')$, and $G(x')$ are defined in (2.1) and (2.2).

**Remark 3.2.** Following general procedure in homogenization, we introduce the associated bilinear form

$$
B(u, v) = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v + \varepsilon^{a-1} \int_{\Gamma_1} puv d\sigma.
$$

Then we will establish

$$
B(u_\varepsilon - u_0 - \varepsilon^a u_1, v) \leq C\varepsilon^{a-1}\|v\|_{H^1(\Omega_0)}.
$$

Next, we extend $u_\varepsilon - u_0 - \varepsilon^a u_1$ into $\Omega_0$ without increasing the $H^1$-norm by more than a multiplicative constant. It just remains to take $v = u_\varepsilon - u_0 - \varepsilon^a (u_1 - u_1|_{x_0 = 0})$. Then this theorem will be proved.
**Proof.** Since $\Omega_\varepsilon \subset \Omega_0$, every function $f \in H^1(\Omega_\varepsilon)$ can be easily extended by $0$ in $\Omega_0 \setminus \Omega_\varepsilon$ to become a function of $H^1(\Omega_0)$. It is easy to find that

$$B(u_\varepsilon - u_0 - \varepsilon^{\alpha}u_1, \nu) = B(u_\varepsilon - u_0, \nu) - \int_{\Omega_\varepsilon} \nabla \xi u_1 \cdot \nabla v dx + O(\varepsilon^\alpha) \|\nu\|_{H^1(\Omega_\varepsilon)}.$$  

(3.3)

Firstly, let us estimate the term $B(u_\varepsilon - u_0, \nu)$. For any $\nu \in H^1(\Omega_0)$ and $\nu = 0$ on $\Gamma_2$,  

$$B(u_\varepsilon - u_0, \nu) = \int_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_0) \cdot \nabla v dx + \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} p(u_\varepsilon - u_0) \nu d\sigma$$

$$= \int_{\Gamma_1^\varepsilon} (-\varepsilon^{\alpha-1} p u_0 - \frac{\partial u_0}{\partial v_0} + \varepsilon^{\alpha-1} g) \nu d\sigma.$$  

It follows from (3.2), that  

$$\frac{\partial u_0}{\partial v_0} + P(x') u_0 = G(x') \quad \text{on } \Gamma_1.$$  

Hence

$$(pu_0 + \frac{p}{P} (\frac{\partial u_0}{\partial v_0} - G))|_{\Gamma_1^\varepsilon} \leq C \varepsilon.$$  

This gives

$$B(u_\varepsilon - u_0, \nu) = \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} (g - \frac{p}{P} G) \nu d\sigma + \int_{\Gamma_1^\varepsilon} (\varepsilon^{\alpha-1} p \frac{\partial u_0}{\partial v_0} - \frac{\partial u_0}{\partial v_0} + \varepsilon^{\alpha-1} g) \nu d\sigma$$

$$+ O(\varepsilon^\alpha) \|\nu\|_{H^1(\Omega_0)}.$$  

In view of (3.3), we obtain

$$B(u_\varepsilon - u_0 - \varepsilon^{\alpha}u_1, \nu) = \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} (g - \frac{p}{P} G) \nu d\sigma + \int_{\Gamma_1^\varepsilon} (\varepsilon^{\alpha-1} p \frac{\partial u_0}{\partial v_0} - \frac{\partial u_0}{\partial v_0} + \varepsilon^{\alpha-1} g) \nu d\sigma$$

$$- \int_{\Omega_\varepsilon} \nabla \xi u_1 \cdot \nabla v dx + O(\varepsilon^\alpha) \|\nu\|_{H^1(\Omega_\varepsilon)}.$$  

(3.4)
We shall next deal with the term \( \int_{\Omega_\xi} \nabla_\xi u_1 \cdot \nabla vdx \). Using the change of variables, we find that

\[
\int_{\Omega_\xi} \nabla_\xi u_1 \cdot \nabla vdx = \varepsilon^{(n-1)\alpha} \int_D \nabla_\xi u_1 \cdot \nabla_\xi vd\xi
\]

\[
= \varepsilon^{(n-1)\alpha} \int_{\partial D \over \partial \nu_\xi} \frac{\partial u_1}{\partial \nu_\xi} vd\sigma_\xi,
\]

where we have used \( \Delta_\xi u_1 = 0 \).

It follows from (2.2), that

\[
\int_{\Omega_\xi} \nabla_\xi u_1 \cdot \nabla vdx = \varepsilon^{(n-1)\alpha} \left( \int_{\partial D \over \partial \nu_\xi} \frac{\partial H_0}{\partial \nu_\xi} vd\sigma_\xi + \int_{\partial D \over \partial \nu_\xi} \frac{\partial H_i}{\partial x_i} \frac{\partial u_0}{\partial x_i} vd\sigma_\xi \right)
\]

\[
+ \int_{\partial D \over \partial \nu_\xi} \frac{\partial H_n}{\partial x_n} \frac{\partial u_0}{\partial x_n} vd\sigma_\xi
\]

\[
= A_1 + A_2 + A_3,
\]

where

\[
A_1 = \varepsilon^{(n-1)\alpha} \int_{\partial D \over \partial \nu_\xi} \frac{\partial H_0}{\partial \nu_\xi} vd\sigma_\xi
\]

\[
= \varepsilon^{(n-1)\alpha} \int_{\Gamma_\xi} \left( g - \frac{p}{P} G \right) \nu \frac{\sqrt{\xi}}{1 + \sqrt{\xi}} d\sigma_\xi
\]

\[
= \varepsilon^{(n-1)\alpha} \int_{\Gamma_\xi} \left( g - \frac{p}{P} G \right) \nu \sqrt{\xi'} F(x', \xi') d\xi',
\]

\[
A_2 = \varepsilon^{(n-1)\alpha} \int_{\partial D \over \partial \nu_\xi} \frac{\partial H_i}{\partial x_i} \frac{\partial u_0}{\partial x_i} vd\sigma_\xi
\]

\[
= \varepsilon^{(n-1)\alpha} \int_{\Gamma_\xi} \left( -\frac{1}{\xi} \right) \frac{\partial F(x', \xi')}{\partial x_i} \frac{\partial u_0}{\partial x_i} vd\sigma_\xi
\]

\[
= \varepsilon^{(n-1)\alpha} \int_{\Gamma_\xi} \frac{\partial F(x', \xi')}{\partial x_i} \frac{\partial u_0}{\partial x_i} vd\xi',
\]
and

\[ A_3 = \varepsilon^{(n-1)\alpha} \int \frac{\partial H_n}{\partial u_0} \frac{\partial u_0}{\partial x_n} \nu d\sigma \]

\[ = \varepsilon^{(n-1)\alpha} \int \frac{1}{\varepsilon \hat{D}} \frac{1}{(1 + |\nabla_{\xi'}^2 F(x', \xi')|^2)^{1/2}} \]

\[ - \frac{p}{P} \frac{|\nabla_{\xi'}^2 F(x', \xi')|}{(1 + |\nabla_{\xi'}^2 F(x', \xi')|^2)^{1/2}} \frac{\partial u_0}{\partial x_n} \nu d\sigma \]

\[ = \varepsilon^{(n-1)\alpha} \int \Gamma_{\xi'} (1 - \frac{p}{P} |\nabla_{\xi'}^2 F(x', \xi')|) \frac{\partial u_0}{\partial x_n} \nu d\xi'. \]

Note that

\[ \int_{\Gamma_1} \varepsilon^{\alpha-1} (g - \frac{p}{P} G) \nu d\sigma \]

\[ = \int_{\Gamma_1} \varepsilon^{\alpha-1} (g - \frac{p}{P} G) \nu \cdot \varepsilon^{1-\alpha} |\nabla_{\xi'}^2 F(x', \xi')| dx' + O(\varepsilon^{\alpha-1}) \| \nu \|_{H^1(\Omega_0)} \]

\[ = \int_{\Gamma_1} (g - \frac{p}{P} G) \nu \cdot |\nabla_{\xi'}^2 F(x', \xi')| dx' + O(\varepsilon^{\alpha-1}) \| \nu \|_{H^1(\Omega_0)} \]

\[ = \varepsilon^{(n-1)\alpha} \int_{\Gamma_{\xi'}} (g - \frac{p}{P} G) \nu \cdot |\nabla_{\xi'}^2 F(x', \xi')| d\xi' + O(\varepsilon^{\alpha-1}) \| \nu \|_{H^1(\Omega_0)}, \]

where we have used the change of variables \( x' = \varepsilon^{\alpha} \xi' \). Hence

\[ \int_{\Gamma_1} \varepsilon^{\alpha-1} (g - \frac{p}{P} G) \nu d\sigma - A_1 = O(\varepsilon^{\alpha-1}) \| \nu \|_{H^1(\Omega_0)}. \]

Finally, we shall evaluate the term \( \int_{\Gamma_1} (\varepsilon^{\alpha-1} \frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_{\xi'}}) \nu d\sigma \).

Also, note that since \( \nu_0 = (0, \ldots, 0, -1) \) and \( \nu_{\xi'} = \varepsilon^{\alpha-1} (\frac{\partial F}{\partial \xi'}, -1) \)

\[ \frac{1}{|\nabla_{\xi'}^2 F(x', \xi')|} + O(\varepsilon^{\alpha-1}), \]
\[
\int_{\Gamma_1^e} (\varepsilon^{-1} \frac{p}{P} \frac{\partial u_0}{\partial v_0} - \frac{\partial u_0}{\partial v_\varepsilon}) v d\sigma = \int_{\Gamma_1^e} (\varepsilon^{-1} \frac{p}{P} \nabla u_0 \cdot \nu_\varepsilon - \nabla u_0 \cdot \nu_\varepsilon) v d\sigma
\]
\[
= \int_{\Gamma_1^e} (\varepsilon^{-1} \frac{p}{P} \frac{\partial u_0}{\partial \tilde{x}_n} \nu_\varepsilon - \frac{\partial u_0}{\partial \tilde{x}_n} \nu_\varepsilon^e) v d\sigma - \int_{\Gamma_1^e} \frac{\partial u_0}{\partial \tilde{x}_i} v^i \nu_\varepsilon v d\sigma
\]
\[
= \varepsilon^{-1} \int_{\Gamma_1^e} \frac{\partial u_0}{\partial \tilde{x}_n} \left( \frac{1}{|\nabla F(x', \xi')|} - \frac{p}{P} \right) v d\sigma
\]
\[
- \varepsilon^{-1} \int_{\Gamma_1^e} \frac{\partial u_0}{\partial \tilde{x}_i} \frac{1}{|\nabla F(x', \xi')|} \frac{\partial F}{\partial \xi_i} v d\sigma + O(\varepsilon^a) \| v \|_{H^1(\Omega_0)}
\]
\[
= \int_{\Gamma_1} \frac{\partial u_0}{\partial \tilde{x}_n} (1 - \frac{p}{P} |\nabla F(x', \xi')|) v dx'
\]
\[
- \int_{\Gamma_1} \frac{\partial u_0}{\partial \tilde{x}_i} \frac{\partial F}{\partial \xi_i} v dx' + O(\varepsilon^{a-1}) \| v \|_{H^1(\Omega_0)}
\]
\[
= \varepsilon^{(n-1)a} \int_{\Gamma_1} \frac{\partial u_0}{\partial \tilde{x}_n} (1 - \frac{p}{P} |\nabla F(x', \xi')|) v d\xi'
\]
\[
- \varepsilon^{(n-1)a} \int_{\Gamma_1} \frac{\partial u_0}{\partial \tilde{x}_i} \frac{\partial F}{\partial \xi_i} v d\xi' + O(\varepsilon^{a-1}) \| v \|_{H^1(\Omega_0)}
\]

By the definition of \( A_2 \) and \( A_3 \) in (3.5), this implies that
\[
\int_{\Gamma_1^e} (\varepsilon^{-1} \frac{p}{P} \frac{\partial u_0}{\partial v_0} - \frac{\partial u_0}{\partial v_\varepsilon}) v d\sigma - A_2 - A_3 = O(\varepsilon^{a-1}) \| v \|_{H^1(\Omega_0)}.
\]

This together with (3.4), gives
\[
B(u_\varepsilon - u_0 - \varepsilon^a u_1, v) \leq C \varepsilon^{a-1} \| v \|_{H^1(\Omega_0)}.
\]

Choosing \( v = u_\varepsilon - u_0 - \varepsilon^a (u_1 - u_1 \mid_{\xi_n = B}) \), we obtain the desired result.

\[\square\]
4. Error Estimate up to $O(\varepsilon^\alpha)$

In this section, we will improve the error estimate up to the order of $O(\varepsilon^\alpha)$. The main technique is using the second order Taylor's expansion. Our main result is the following theorem:

**Theorem 4.1.** Let $f \in L^2(R^n)$ and $u_\varepsilon$ is a solution of problem (1.1). Suppose that $p(x', \xi'), g(x', \xi')$, and $F(x', \xi')$ satisfy (1.3) and (1.4). Then these exist a constant $C$ does not depend on $\varepsilon$, such that

$$
\|u_\varepsilon - u_0 - \varepsilon^\alpha u_1\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^\alpha,
$$

(4.1)

where $u_0$ is a solution of homogenized problem (3.2) and $u_1$ is defined in (2.2).

**Proof.** Following the same line of research, we consider

$$
B(u_\varepsilon - u_0, v) = \int_{\Gamma_1^\varepsilon} \left( - \varepsilon^{\alpha-1} p u_0 - \frac{\partial u_0}{\partial n_\varepsilon} + \varepsilon^{\alpha-1} g \right) v d\sigma.
$$

This together with (3.2) and the Taylor's expansion, gives

$$
\left( \frac{\partial u_0}{\partial n_0} + P(x')u_0 - G(x') \right) \big|_{\Gamma_1^\varepsilon} = \varepsilon F(x', \xi') \frac{\partial}{\partial x_n} \left( \frac{\partial u_0}{\partial n_0} + P(x')u_0 - G(x') \right) \big|_{\Gamma_1} + O(\varepsilon^2).
$$

It follows that

$$
B(u_\varepsilon - u_0, v) = \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} (g - \frac{P}{P} G) v d\sigma + \int_{\Gamma_1^\varepsilon} \left( \varepsilon^{\alpha-1} \frac{P}{P} \frac{\partial u_0}{\partial n_0} - \frac{\partial u_0}{\partial n_\varepsilon} \right) v d\sigma
$$

$$
- \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha \frac{P}{P} F(x', \xi') \frac{\partial}{\partial x_n} \left( \frac{\partial u_0}{\partial n_0} + P(x')u_0 \right) \big|_{\Gamma_1} v d\sigma
$$

$$
+ O(\varepsilon^{\alpha+1}) \|v\|_{H^1(\Omega_0)}.
$$
Note that

\[ B(-\varepsilon^a u_1, \nu) = \int_{\Omega_\varepsilon} \varepsilon^a \Delta u_1 v d\sigma - \int_{\Gamma_1^\varepsilon} \varepsilon^a \frac{\partial u_1}{\partial \nu_\varepsilon} v d\sigma - \int_{\Gamma_1^\varepsilon} \varepsilon^{2a-1} p u_1 v d\sigma \]

\[ = \int_{\Omega_\varepsilon} \varepsilon^a \Delta u_1 v d\sigma + \int_{\Omega_\varepsilon} 2\nabla_x \nabla_{\xi_\varepsilon} u_1 v d\sigma \]

\[ - \int_{\Gamma_1^\varepsilon} \varepsilon^a \frac{\partial u_1}{\partial \nu_\varepsilon} v d\sigma - \int_{\Gamma_1^\varepsilon} \varepsilon^{2a-1} p u_1 v d\sigma, \]

where we have used \( \nabla_{\xi_\varepsilon} u_1 = 0. \)

Thus we obtain

\[ B(u_\varepsilon - u_0 - \varepsilon^a u_1, \nu) = \int_{\Gamma_1^\varepsilon} \varepsilon^{a-1} (g - \frac{p}{P} G) v d\sigma \]

\[ + \int_{\Gamma_1^\varepsilon} \left( \varepsilon^{a-1} \frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma \]

\[ - \int_{\Gamma_1^\varepsilon} \varepsilon^a F(x', \xi') \frac{\partial}{\partial x_n} \left( \frac{\partial u_0}{\partial v_0} + P(x') u_0 \right) |_{\Gamma_1^\varepsilon} v d\sigma \]

\[ + \int_{\Omega_\varepsilon} \varepsilon^a \Delta x u_1 v + 2\nabla_x \nabla_{\xi_\varepsilon} u_1 v d\sigma - \int_{\Gamma_1^\varepsilon} \varepsilon^a \frac{\partial u_1}{\partial \nu_\varepsilon} v d\sigma \]

\[ - \int_{\Gamma_1^\varepsilon} \varepsilon^{2a-1} p u_1 v d\sigma + O(\varepsilon^{a+1}) \| \nu \|_{H^1(\Omega_\varepsilon)}, \quad (4.2) \]

where \( \nu_0 = (0, \ldots, 0, -1) \), and

\[ \nu_\varepsilon = \varepsilon^{a-1} \left( \frac{\partial F}{\partial \xi_\varepsilon}, -1 \right) \frac{1}{|\nabla_{\xi_\varepsilon} F(x', \xi')|} + O(\varepsilon^a). \]

In view of change of variables and (2.2), we have

\[ 2\int_{\Omega_\varepsilon} \nabla_x \nabla_{\xi_\varepsilon} u_1 v dx \]

\[ = 2\varepsilon^{na} \int_{\bar{D}} \nabla_x \nabla_{\xi_\varepsilon} u_1 v d\xi \]
\[ \begin{align*}
&\leq 2\varepsilon^2\alpha^{-1} \int_{\Gamma_1^\varepsilon} \nabla_x u_1 \cdot \nu_{\varepsilon} \nu d\sigma + C\varepsilon^2 \alpha^{-1} \int_{\Omega_\varepsilon^\varepsilon} \exp(-\lambda \xi_n) |\nabla_x \nu| dx \\
&\leq 2\varepsilon^2\alpha^{-1} \int_{\Gamma_1^\varepsilon} \nabla_x u_1 \left( \frac{\partial F}{\partial \xi'}, -1 \right) \frac{1}{[\nabla_\xi F(x', \xi')]^2} \nu d\sigma + C\varepsilon^3 \alpha^{2/3} \| \nu \|_{H^1(\Omega_0)}.
\end{align*} \]

(4.3)

A direct computation (using (2.2)) shows that

\[ \varepsilon^\alpha \frac{\partial u_1}{\partial \nu_{\varepsilon}} \big|_{\Gamma_1^\varepsilon} = (\nabla_{\nu_{\varepsilon}} u_1 + \varepsilon^\alpha \nabla_x u_1 \cdot \nu_{\varepsilon}) \big|_{\Gamma_1^\varepsilon} = P_1 + P_2 + O(\varepsilon^{\alpha}), \]

(4.4)

where

\[ P_1 = \nabla_{\nu_{\varepsilon}} u_1 \frac{\varepsilon^{-1} \left( \frac{\partial F}{\partial \xi'}, -1 \right)}{[\nabla_\xi F(x', \xi')]^2} \]

\[ = \varepsilon^{-1} \frac{\partial u_1}{\partial \nu_{\varepsilon}} \left( 1 + \frac{[\nabla_\xi F(x', \xi')]^2}{[\nabla_\xi F(x', \xi')]^2} \right)^{1/2} \]

\[ = \varepsilon^{-1} \left( \frac{P}{\frac{\partial F}{\partial \xi'}} \right) - \varepsilon^{-1} \frac{\partial u_0}{\partial \xi_i} \frac{\partial F(x', \xi')}{[\nabla_\xi F(x', \xi')]^2} \frac{1}{[\nabla_\xi F(x', \xi')]^2} \]

\[ + \varepsilon^{-1} \frac{\partial u_0}{\partial x_n} \left( \frac{1}{[\nabla_\xi F(x', \xi')]^2} \frac{\partial F}{\partial \xi'} - \frac{P}{\frac{\partial F}{\partial \xi'}} \right), \]

and

\[ P_2 = \varepsilon^{2\alpha-1} \nabla_x u_1 \left( \frac{\partial F}{\partial \xi'}, -1 \right) \frac{1}{[\nabla_\xi F(x', \xi')]^2}. \]

It follows from (4.4), that

\[ \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha \frac{\partial u_1}{\partial \nu_{\varepsilon}} \nu d\sigma = \int_{\Gamma_1^\varepsilon} P_1 \nu d\sigma + \int_{\Gamma_1^\varepsilon} P_2 \nu d\sigma + O(\varepsilon^{2\alpha}) \| \nu \|_{H^1(\Omega_0)}. \]
A similar computation yields the following result:

\[
\int_{\Gamma_1^e} (\varepsilon^{a-1} \frac{p}{\varepsilon} \frac{\partial u_0}{\partial v} - \frac{\partial v_0}{\partial v}) \, v \, d\sigma = \int_{\Gamma_1^e} (1 - \varepsilon^{a-1} \frac{p}{\varepsilon} \frac{\partial u_0}{\partial x_n} - Q + O(\varepsilon^{a+1})) \| v \|_{H^1(\Omega_0)}^2.
\] (4.5)

where

\[
\tilde{Q} = \int_{\Gamma_1^e} \varepsilon^{a-1} \frac{1}{|V.x|} \left( \frac{\partial u_0}{\partial x_1} \frac{\partial F}{\partial x_1} - \frac{\partial u_0}{\partial x_n} \right) v \, d\sigma.
\]

As in Section 3, we extend \( u_\varepsilon - u_0 - \varepsilon^{a} u_1 \) into \( \Omega_0 \) without increasing the \( H^1 \)-norm. It remains to substitute \( v = u_\varepsilon - u_0 - \varepsilon^{a} (u_1 - u_1|_{x_n=B}) \). This, together with (4.2)-(4.5), yields the desired estimate. \( \Box \)

### 5. Error Estimate up to \( O(\varepsilon^{2a-1}) \) or \( O(\varepsilon^{a+1}) \)

The goal of this section is to improve the error estimate up to the order from \( O(\varepsilon^{a}) \) to \( O(\varepsilon^{2a-1}) \) or \( O(\varepsilon^{a+1}) \). This can be obtained via correctors. The main result is the following theorem:

**Theorem 5.1.** Let \( f \in L^2(R^n) \) and \( u_\varepsilon \) is a solution of problem (1.1). Suppose that \( p(x', \xi'), g(x', \xi'), \) and \( F(x', \xi') \) satisfy (1.3) and (1.4). Then these exist a constant \( C \) does not depend on \( \varepsilon \), such that

\[
\left\{ \begin{array}{ll}
\| u_\varepsilon - u_0 - \varepsilon^{a} u_1 - \varepsilon \phi \|_{H^1(\Omega_\varepsilon)} & \leq C \varepsilon^{2a-1}, \text{ if } 1 < a < 2, \\
\| u_\varepsilon - u_0 - \varepsilon^{a} u_1 - \varepsilon \phi \|_{H^1(\Omega_\varepsilon)} & \leq C \varepsilon^{a+1}, \text{ if } a \geq 2,
\end{array} \right.
\] (5.1)

where \( u_0 \) is a solution of homogenized problem (3.2) and \( u_1, \phi \) are defined in (2.2) and (5.7).
Proof. Analogously to Sections 3 and 4, we consider again

\[ B(u_\varepsilon - u_0 - \varepsilon^a u_1, \nu) = \int_{\Gamma_1^\varepsilon} \varepsilon^{a-1}(g - \frac{P}{G})v d\sigma + \int_{\Gamma_1^\varepsilon} (\varepsilon^{a-1} \frac{p}{P} \frac{\partial u_0}{\partial \nu} + \frac{\partial u_0}{\partial \nu})v d\sigma \]

\[ - \int_{\Gamma_1^\varepsilon} \varepsilon^a \frac{p}{P} F(x', \xi') \frac{\partial}{\partial x_n} \left( \frac{\partial u_0}{\partial \nu} + P(x')u_0 \right) |_{\Gamma_1^\varepsilon}v d\sigma \]

\[ + \int_{\Omega_\varepsilon} \varepsilon^a \Delta_\varepsilon u_1 \nu + 2\nu_x \nu_{\xi} u_1 \nu dx - \int_{\Gamma_1^\varepsilon} \varepsilon^a \frac{\partial u_1}{\partial \nu} v d\sigma \]

\[ - \int_{\Gamma_1^\varepsilon} \varepsilon^{2a-1} p \nu_1 v d\sigma + O(\varepsilon^{a+1}) \| \nu \|_{H^1(\Omega_\varepsilon)}, \quad (5.2) \]

where \( \nu_0 = (0, \cdots, 0, -1) \). Using the second order expansion, we have

\[ \nu_\varepsilon = \varepsilon^{a-1} \left( \frac{\partial F}{\partial \xi'}, -1 \right) + \varepsilon^a \frac{\partial F}{\partial x'} \left( \frac{1}{|\nabla_x F(x', \xi')|} - \frac{(\frac{\partial F}{\partial \xi'})^2}{|\nabla_x F(x', \xi')|^3} \right) + O(\varepsilon^{a+1}). \quad (5.3) \]

Note that

\[ \int_{\Omega_\varepsilon} \varepsilon^a \Delta_\varepsilon u_1 \nu dx \leq C_0^a \int_{\Omega_\varepsilon} \exp(-\lambda x_n / \varepsilon^a) |\nu| dx \]

\[ \leq C_0^{3a/2} \| \nu \|_{H^1(\Omega_\varepsilon)}, \quad (5.4) \]

and

\[ 2 \int_{\Omega_\varepsilon} \nu_x \nabla_{\xi} u_1 \nu dx \]

\[ \leq 2\varepsilon^{na} \int_{\partial D} \nabla_x u_1 \cdot \nu_{\xi} d\xi + C_0 \int_{\Omega_\varepsilon} \exp(-\lambda x_n) |\nabla_x \nu| dx \]

\[ \leq 2\varepsilon^{2a-1} \int_{\Gamma_1^\varepsilon} \nabla_x u_1 \left( \frac{\partial F}{\partial \xi'}, -1 \right) \frac{1}{|\nabla_x F(x', \xi')|} v d\sigma \]
\[ + 2\varepsilon^{2a} \int_{\Gamma_1^e} \nabla \! x u_1 \frac{\partial F}{\partial x'} \left( \frac{1}{|\nabla \! z F(x', \xi')|^3} \right) - \left( \frac{\partial F}{\partial \xi'} \right)^2 \left( \frac{\partial F}{\partial \xi''} \right)^2 \right) \nu d\sigma \\
+ C \varepsilon^{3a/2} \| \nu \|_{H^1(\Omega_0)} \quad (5.5) \]

It follows from (2.2) and (5.3), that

\[ \varepsilon^a \frac{\partial u_1}{\partial \nu_\varepsilon} \bigg|_{\Gamma_1^e} = (\nabla \! z u_1 \cdot \nu_\varepsilon + \varepsilon^a \nabla \! x u_1 \cdot \nu_\varepsilon) \bigg|_{\Gamma_1^e} = P_1 + P_2 + O(\varepsilon^{2a}), \quad (5.6) \]

where

\[ P_1 = \varepsilon^{a-1} \frac{\partial u_1}{\partial \nu_\varepsilon} \frac{(1 + |\nabla \! z F(x', \xi')|^2)^{1/2}}{|\nabla \! z F(x', \xi')|} \\
= \varepsilon^{a-1} \left( g - \frac{P}{P} G \right) - \varepsilon^{a-1} \frac{\partial u_0}{\partial x_i} \frac{\partial F(x', \xi')}{\partial \xi_i} \frac{1}{|\nabla \! z F(x', \xi')|} \\
+ \varepsilon^{a-1} \frac{\partial u_0}{\partial x_n} \left( \frac{1}{|\nabla \! z F(x', \xi')|} - \frac{P}{P} \right), \]

and

\[ P_2 = \nabla \! z u_1 \varepsilon^a \frac{\partial F}{\partial x'} \left( \frac{1}{|\nabla \! z F(x', \xi')|^3} \right) - \left( \frac{\partial F}{\partial \xi'} \right)^2 \left( \frac{\partial F}{\partial \xi''} \right)^2 \left( \frac{\partial F}{\partial \xi''} \right)^3 \right) \\
+ \varepsilon^{2a-1} \nabla \! x u_1 \frac{(\partial F)}{|\nabla \! z F(x', \xi')|}. \]

Combining all these terms in (5.6), we have proved that

\[ \int_{\Gamma_1^e} \varepsilon^a \frac{\partial u_1}{\partial \nu_\varepsilon} v d\sigma = \int_{\Gamma_1^e} P_1 v d\sigma + \int_{\Gamma_1^p} P_2 v d\sigma + O(\varepsilon^{2a}) \| \nu \|_{H^1(\Omega_0)}. \]

A similar calculation yields the following result:

\[ \int_{\Gamma_1^p} (\varepsilon^{a-1} \frac{P}{P} \frac{\partial u_0}{\partial \nu_\varepsilon} - \frac{\partial u_0}{\partial \nu_\varepsilon}) v d\sigma = \int_{\Gamma_1^p} -\varepsilon^{a-1} \frac{P}{P} \frac{\partial u_0}{\partial x_n} v d\sigma - \tilde{Q} + O(\varepsilon^{a-1}) \| \nu \|_{H^1(\Omega_0)} , \]
where
\begin{equation*}
\widetilde{Q} = \int_{\Gamma_1} \varepsilon^{\alpha-1} \left( \frac{1}{|\nabla \varepsilon F|} \left( \frac{\partial u_0}{\partial x_i} \frac{\partial F}{\partial x_i} - \frac{\partial u_0}{\partial x_n} \right) + \varepsilon^{\alpha} \nabla u_0 \frac{\partial F}{\partial x} \frac{1}{\left| \nabla \varepsilon F(x', \xi') \right|^2} \left( \frac{\partial F}{\partial \xi'} \right)^2, \frac{\partial F}{\partial \xi''} \right) \nu d\sigma.
\end{equation*}

Then, we introduced a corrector so as to cancel lower power of $\varepsilon^\alpha$.

For any given function of the form $\tilde{g}$, suppose that $\tilde{u}_\varepsilon$ is the solution of the following problem:

\begin{align*}
- \Delta \tilde{u}_\varepsilon & = 0, \quad \text{in } \Omega_\varepsilon, \\
\frac{\partial \tilde{u}_\varepsilon}{\partial \nu} + \varepsilon^{\alpha-1} p(x', \frac{x'}{\varepsilon^\alpha}) \tilde{u}_\varepsilon & = \varepsilon^{\alpha-1} \tilde{g}(x', \frac{x'}{\varepsilon^\alpha}) \quad \text{on } \Gamma_1^\varepsilon, \\
\tilde{u}_\varepsilon & = 0 \quad \text{on } \Gamma_2.
\end{align*}

Meanwhile, assume that $\phi(x)$ is the solution of the homogenized problem:

\begin{align*}
- \Delta \phi & = 0, \quad \text{in } \Omega_0, \\
\frac{\partial \phi}{\partial \nu_0} + \varepsilon^{\alpha-1} P(x') \phi & = \varepsilon^{\alpha-1} \tilde{G}(x') \quad \text{on } \Gamma_1, \\
\phi & = 0 \quad \text{on } \Gamma_2,
\end{align*}

where $P(x')$ is defined in (2.1), and

\begin{equation*}
\tilde{G}(x') = \int_0^1 \cdots \int_0^1 |\nabla \varepsilon F(x', \xi')| \tilde{g}(x', \xi') d\xi'.
\end{equation*}

We may invoke Theorem 4.1 to conclude that

\begin{equation*}
\|\tilde{u}_\varepsilon - \phi - \varepsilon^\alpha \tilde{u}_1\|_{H^1(\Omega_\varepsilon)} \leq C \varepsilon^\alpha,
\end{equation*}

where $\tilde{u}_1$ follows the essentially the same steps as Section 2, with the obvious modifications.
It follows that
\[
B(\varepsilon \phi, \nu) = B(\varepsilon (\phi - \tilde{u}_\varepsilon + \varepsilon^a \tilde{u}_1), \nu) + B(\varepsilon \tilde{u}_\varepsilon - \varepsilon^{a+1} \tilde{u}_1, \nu) \\
= B(\varepsilon \tilde{u}_\varepsilon, \nu) + O(\varepsilon^{a+1}) \| \nu \|_{H^1(\Omega_0)}.
\]

We now write
\[
B(u_\varepsilon - u_0 - \varepsilon^a u_1 + \varepsilon \phi, \nu) = \varepsilon^\alpha \int_{\Gamma_1^\varepsilon} \tilde{g} v d\sigma - \varepsilon^\alpha \sum_i I_i + O(\varepsilon^{a+1}) \| \nu \|_{H^1(\Omega_0)},
\]
where
\[
I_1 = \int_{\Gamma_1^\varepsilon} \nabla u_0 \frac{\partial F}{\partial x^i} \left( \frac{1}{|\nabla_{\xi'} F(x', \xi')|^3} \right) - \left( \frac{\partial F}{\partial \xi^i} \right)^2 \frac{\partial F}{|\nabla_{\xi'} F(x', \xi')|^3}, \frac{\partial F}{|\nabla_{\xi'} F(x', \xi')|^3} v d\sigma,
\]
\[
I_2 = \int_{\Gamma_1^\varepsilon} \frac{P}{P} F(x', \xi') \frac{\partial}{\partial x_n} \left( \frac{\partial u_0}{\partial v_0} + P(x') u_0 \right) v d\sigma,
\]
\[
I_3 = \int_{\Gamma_1^\varepsilon} \nabla \tilde{u}_1 \frac{\partial F}{\partial x^i} \left( \frac{1}{|\nabla_{\xi'} F(x', \xi')|^3} \right) - \left( \frac{\partial F}{\partial \xi^i} \right)^2 \frac{\partial F}{|\nabla_{\xi'} F(x', \xi')|^3}, \frac{\partial F}{|\nabla_{\xi'} F(x', \xi')|^3} v d\sigma.
\]

Then we just need to take that
\[
\tilde{g} = \sum_i \tilde{I}_i,
\]
where
\[
\tilde{I}_1 = \nabla u_0 \frac{\partial F}{\partial x^i} \left( \frac{1}{|\nabla_{\xi'} F(x', \xi')|^3} \right) - \left( \frac{\partial F}{\partial \xi^i} \right)^2 \frac{\partial F}{|\nabla_{\xi'} F(x', \xi')|^3}, \frac{\partial F}{|\nabla_{\xi'} F(x', \xi')|^3} v d\sigma,
\]
\[
\tilde{I}_2 = \frac{P}{P} F(x', \xi') \frac{\partial}{\partial x_n} \left( \frac{\partial u_0}{\partial v_0} + P(x') u_0 \right),
\]
\[
\tilde{I}_3 = \nabla \xi^1 \frac{\partial F}{\partial x^i} \left( \frac{1}{|\nabla_{\xi'} F(x', \xi')|^3} \right) - \left( \frac{\partial F}{\partial \xi^i} \right)^2 \frac{\partial F}{|\nabla_{\xi'} F(x', \xi')|^3}, \frac{\partial F}{|\nabla_{\xi'} F(x', \xi')|^3} v d\sigma.
\]
Combining all these terms and choosing \( \nu = u_\varepsilon - u_0 - \varepsilon^\alpha (u_1 - u_1 |_{x_n=B}) + \varepsilon\phi \), we conclude that
\[
\| u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon\phi \|_{H^1(\Omega_\varepsilon)} \leq C(\varepsilon^{2\alpha-1} + \varepsilon^{\alpha+1}).
\]

This completes the proof after taking into account the different case when \( 1 < \alpha < 2 \) and \( \alpha \geq 2 \). □

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