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SOME PROPERTIES OF PARABOLAS TANGENT TO TWO SIDES OF A TRIANGLE

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Abstract

In this work, we study some properties of parabolas whose vertex and focus are on bisector line and tangent to two sides of a given triangle. Also, we give the relationships among the vertex, the focus, and the intersection angle of the tangents. For the triangles with particular angles, we associate these parabolas with Euler line of the triangle ABC.

1. Introduction

A parabola is the set of points in a plane that are equidistant from a fixed point F (called the focus) and a fixed line (called the directrix). It is well known that the point called the vertex halfway between the focus and the directrix lies on the parabola and the line called the axis of the parabola through the focus perpendicular to the directrix [6].

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The parabolas which are tangent to two sides of a triangle where the tangent points are two vertices and the other side of the triangle is a chord of the corresponding parabola were first described by Artzt in 1884, [2]. When the sides of a triangle are tangents to a parabola, the triangle is said to circumscribe the parabola. But it must be clearly understood that the triangle does not enclose the parabola, for no finite triangle can enclose a parabola, which is infinite in extent. When a triangle circumscribes a parabola, the parabola is really escribed to it, that is, it touches one side of the triangle and the other two sides produced [1]. Some properties of these parabolas are mentioned in [3], [4], [5], and [6].

In this work, we establish some properties of parabolas whose vertex and focus are on bisector line and tangent to two sides of triangle. It is given the ratio of the distance between the vertex of parabola and the intersection point of tangents to the distance between the vertex and the focus. Also, in case that the vertex and the focus of parabola coincide with the special points on Euler line of the triangle, we determine the internal angle of the tangent sides of the triangle.

Theorem 1. *Let ABC be a triangle with* $m(\angle C) = \alpha$, *and a parabola with the vertex P and the focus F on the internal bisector of the angle C. If the parabola is tangent to sides AC and BC of the triangle ABC, the distance between the vertex P and the focus F of the parabola is* $\tan^2 \frac{\alpha}{2}$ *times the distance between P and the vertex C of triangle ABC.*

Proof. Consider a triangle *ABC* formed by tangents at the points *D* and *E* of a parabola. Let the focus *F* be on the internal bisector of the angle *C* of the triangle *ABC* and let the vertex *C* with angle α be the intersection point of the tangents of the parabola. Since the translations and the rotations preserve the distances in Euclidean plane, we can take the coordinates of points $F(0, 0)$, $D(x, y)$, $P(0, a)$, and $C(0, y_0)$, where $a, y_0, x, y \in \mathbb{R}$. Since the distance of the point *D* to the directrix of parabola equals to the distance of the focus F, the equation of the parabola is,

$$x^2 + 4ay - 4a^2 = 0.$$

Tangent line DC is obtained as

$$y = \cot \frac{\alpha}{2} x + y_0.$$

This will meet the parabola where,

$$x^2 + 4a \cot \frac{\alpha}{2} x + 4ay_0 - 4a^2 = 0.$$

For a tangent, the roots of the equation must be coincident and therefore;

$$y_0 = a\left(\frac{1}{\sin^2 \frac{\alpha}{2}}\right).$$

The ratio of length of PF to the length of DC;

$$\frac{PF}{PC} = \frac{a}{y_0 - a}.$$

For $y_0 = a\left(\frac{1}{\sin^2 \frac{\alpha}{2}}\right)$,

$$\frac{PF}{PC} = \tan^2 \frac{\alpha}{2}$$

is obtained. □

Corollary 2. *Let the measure of the included angle C by two adjacent sides, AC and BC, of the triangle ABC be* $\frac{\pi}{2}, \frac{\pi}{3},$ *and* $\frac{2\pi}{3},$ *respectively. If a parabola which has the vertex P and the focus F on the internal bisector of the angle C is tangent to the sides AC and BC of the triangle ABC, then the ratio* $\frac{PF}{PC}$ *is equal to* $1, \frac{1}{3},$ *and* 3*, respectively.*

Proof. In Theorem 1, if α is taken as $\frac{\pi}{2}, \frac{\pi}{3},$ and $\frac{2\pi}{3}$, then the ratio $\frac{PF}{PC}$ is obtained as 1, $\frac{1}{3}$, and 3, respectively. □

Theorem 3. *Let ABC be a triangle with* $m(\angle C) = \frac{\pi}{3}$, *O be the circumcircle center, a parabola with the vertex P and the focus F on the internal bisector of the angle C and the side AB be tangent to sides AC and BC of the triangle ABC. Then this parabola is tangent to sides OA and OB of the triangle OAB.*

Proof. Let the measure of the included angle C by two adjacent sides, AC and BC, of the triangle ABC be $\frac{\pi}{3}$. Since the translations and the rotations preserve the distances in Euclidean plane, we can take the coordinates of points $F(0, 0)$, $P(0, a)$, and $C(0, 4a)$, $a \in \mathbb{R}$. Then the equation of the parabola tangent to sides AC and BC of the triangle ABC is $x^2 + 4ay - 4a^2 = 0$. If it is taken the coordinate of the point A as $(x_1, \sqrt{3}x_1 + 4a)$, the coordinates of the vertex B and the circumcircle center O are calculated as $\left(\frac{2ax_1}{\sqrt{3}x_1 + 2a}, \frac{2a(\sqrt{3}x_1 + 4a)}{\sqrt{3}x_1 + 2a}\right)$ and $\left(\frac{\sqrt{3}x_1^2 + 4ax_1}{\sqrt{3}x_1 + 2a}, \frac{x_1^2 + 4\sqrt{3}ax_1 + 8a^2}{\sqrt{3}x_1 + 2a}\right)$. It is easily seen that this parabola is tangent to the sides OA and OB of the triangle OAB. □

Corollary 4. *Let ABC be an equilateral triangle and a parabola with the vertex P and the focus F on the internal bisector of the angle C and the side AB be tangent to sides AC and BC of the triangle ABC. Then, this parabola is tangent to the internal bisectors of the angles A and B.*

Proof. Let ABC be an equilateral triangle. Since the circumcircle coincides with the orthocenter in an equilateral triangle ABC, the corollary is obtained from the previous theorem. □

The most famous line in the subject of triangle geometry is the Euler line, named in honor of Leonhard Euler (pronounced Oiler), who penned more pages of original mathematics than any other human being. If G centroid, O circumcenter, H the orthocenter, N ninepoint center of any triangle ABC, these well-known points lie on the Euler line and when you vary the shape of triangle ABC, the relative distances between the points G, O, H, N remain the same: G always lies $1/3$ of the way from O to H; N always lies $1/2$ of the way from O to H.

In the following theorems deal with the tangent parabolas and Euler line.

Theorem 5. *Let H, G, O, N be the orthocenter, the centroid, the circumcircle center, and the ninepoint center in any scalene triangle ABC, respectively. If the measure of the included angle at the vertex C is $\frac{\pi}{3}$, the parabola tangent to the altitudes from the vertices A and B has the vertex N and the focus G.*

Proof. Suppose the measure of the included angle at the vertex C is $\frac{\pi}{3}$. Then the measure of angle H, the angle between the altitudes from the vertices A and B is $\frac{\pi}{3}$ and Euler line is the internal bisector of the angle H. It is well-known that the distance from the ninepoint center N to the orthocenter H is always three times the distance of the ninepoint center N to the centroid G on the Euler line. From Corollary 2, the parabola tangent to the altitudes from the vertices A and B has the vertex N and the focus G. □

Theorem 6. *Let H, G, N be the orthocenter, the centroid, and the ninepoint center in any isoscales triangle ABC, $CA = CB$, respectively. There is the parabola with the vertex N and the focus H tangent to the medians from the vertices A and B. Also the included angle between these medians is $\frac{\pi}{3}$.*

Proof. Let any isoscales triangle be ABC with $CA = CB$. Then Euler line coincides with the internal bisector of the angles C and G. It is well known that the distance from the ninepoint center N to the orthocenter H is always three times of the distance from the ninepoint center N to the centroid G on the Euler line. From Theorem 1, the parabola with the vertex N and the focus H is tangent to the medians from the vertices A and B. And the measure of angle G, the angle between the medians from the vertices A and B is $\frac{\pi}{3}$. □

Theorem 7. *Let H, O, N be the orthocenter, the circumcircle center and the ninepoint center in any isoscales triangle ABC, $CA = CB$, respectively. The parabola with the vertex N and the focus H is tangent to both the sides AC and BC and the radii from the vertices A and B of the triangle ABC.*

Proof. Let any isoscales triangle be ABC with $CA = CB$. So, Euler line is the internal bisector of the angle C. The distance from the ninepoint center N to the orthocenter H is always equal to the distance from the ninepoint center N to the centroid O. From Theorem 1, the parabola with the vertex N and the focus H is tangent to the radii from the vertices A and B. And the measure of angle O, the angle between the radii from the vertices A and B is $\frac{\pi}{2}$. Since the measure of the included angle at the vertex C is $\frac{\pi}{4}$ and $\frac{NH}{NC} = 3 - 2\sqrt{2}$ for the parabola with the vertex N and the focus H, this parabola is tangent to both sides AC and BC of the triangle ABC. □

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