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ADDITIONAL PROPERTIES FOR WEAKLY *Po* AND RELATED PROPERTIES WITH AN APPLICATION

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Abstract

Within this paper, additional properties for weakly *Po* and related properties are given and the results are applied to give an internal, well-defined topological properties *WP* for which *WP* = weaker *Po* for each *Po* that exists.

1. Introduction and Preliminaries

T_0-identification spaces were introduced by Stone in 1936 [12].

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \rightarrow X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, Y))$ is the T_0-identification space of (X, T).

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Within the 1936 paper [12], T_0-identification spaces were used to further characterize metrizable spaces.

Theorem 1.1. *A space* (X, T) *is pseudometrizable iff* $(X_0, Q(X, Q(X, T)))$, *is metrizable. In the* 1975 *paper* [11], T_0*-identification spaces were used to further characterize Hausdorff spaces.*

Theorem 1.2. *A space* (X, T) *is weakly Hausdorff iff* $(X_0, Q(X, T))$ *is Hausdorff* [9].

Within the 2015 paper [1], the metrizable and Hausdorff properties were generalized to weakly *Po* properties.

Definition 1.2. Let P be topological properties such that $Po = (P$ and $T_0)$ exists. Then a space (X, T) is weakly *Po* iff its T_0-identification space $(X_0, Q(X, T))$ has property P. A topological property *Po* for which weakly *Po* exists is called a weakly *Po* property [1]. In the work below, for a topological property Q, $(Q$ and $T_0)$ will be denoted by *Qo*. In the 1936 paper [12], it was shown that for each space, its T_0-identification space has property T_0. Thus, for a topological property P for which *Po* exists, a space is weakly *Po* iff its T_0-identification space has property *Po*.

Within the paper [1], it was shown that for a weakly *Po* property *Qo*, a space is weakly *Qo* iff its T_0-identification space is weakly *Qo*, which led to the introduction and investigation of T_0-identification P properties [2].

Definition 1.3. Let S be a topological property. Then S is a T_0-identification P property iff both a space and its T_0-identification space simultaneously shares property S.

In the introductory weakly *Po* paper [1], it was shown that weakly *Po* is neither T_0 nor "not-T_0", where "not-T_0" is the negation of T_0. The need and use of "not-T_0" revealed "not-T_0" as a useful topological property and tool, motivating the inclusion of the long-neglected properties "not-P", where P is a topological property for which "not-P" exists, as important properties for investigation and use in the study of topology. As a result, within a short time period, many new, important, fundamental, foundational, never before imagined properties have been discovered, expanding and changing the study of topology.

In past studies of weakly *Po* spaces and properties, for a classical topological property *Qo*, a special topological property W was sought such that for a space with property W, its T_0-identification space has property *Qo*, which then implies the initial space has property W. If past practices continue, the study of weakly *Po* spaces and properties would continue to be tedious and never ending. Thus, the question of whether there is a shortcut for the weakly *Po* space and property search process arose, which was resolved in a recent paper [3].

Answer 1.1. Let Q be a topological property for which both *Qo* and (Q and "not-T_0") exist. Then Q is a T_0-identification P property that is weakly *Po* and Q = weakly *Qo* = (*Qo* or (Q and "not-T_0")) [3].

Answer 1.2. $\{Q \mid Q$ is a T_0-identification P property$\} = \{Qo \mid Qo$ is a weakly *Po* property$\} = \{Qo \mid Q$ is a topological property and *Qo* exists$\}$ [3].

Answer 1.3. $\{Q \mid Q$ is a T_0-identification P property $\} = \{Q \mid Q$ is weakly $Po\} = \{Q \mid Q$ is a topological property and both *Qo* and (Q and "not-T_0") exist $\}$ [3].

Thus, major progress was achieved in the study of weakly *Po* and related properties. If Q is a topological property for which both *Qo* and

(Q and "not-T_0") exist, Answer 1.1 quickly and easily gives the shortcut. If Q is a topological property for which $Q = Qo$, then $Q = Qo$ is a weakly Po property, but $Q = Qo$ is not a T_0-identification P property or weakly Po. Within the recent paper [3], a topological property W that can be both T_0 and "not-T_0" was given that is a T_0-identification P property that is weakly Po such that W = weakly Qo, again making the search process certain, and quick and easy.

Definition 1.4. Let Q be a topological property for which Qo exists. A space (X, T) has property QNO iff (X, T) is "not-T_0" and $(X_0, Q(X, T))$ has property Qo [3].

In the recent paper [3], it was shown that QNO exists and is a topological property, and $W = (Qo$ or $QNO)$ is a T_0-identification P property that is weakly Po with W = weakly Qo. Thus, as stated above, for a topological property Q for which Qo exists and $Q = Qo$, there is a certain, quick, and easy answer.

In the weakly Po paper [4], the use of T_0 and "not-T_0" revealed that $L = (T_0$ or "not-T_0") is the least of all topological properties. Since the existence of the least topological property L had not even been considered in past studies, its existence required a change in the definition of subspace properties with the removal of L as a subspace property [5]. Prior to the study of weakly Po spaces and properties, it was unknown whether for subspace properties P and Q, if (P and Q) exists. Within the 2016 paper [5], it was shown that for subspace properties P and Q, (P and Q) exists and, thus, is a subspace property. In the paper [5], it was shown that $\mathcal{P} = \{P \mid P$ is a subspace property$\}$ has no least element and in the paper [6], it was shown that $\mathcal{P}$ has strongest element the singleton set property. Since, for the most part, "not-P", where P is a topological property for which "not-P" exists, had been ignored in the study of topology, the introduction and investigation of "not-(subspace property P)" in the paper [4] revealed that for a subspace property P, "not-(subspace property P)" is not a subspace property, which instantaneously gave many previously unknown examples of topological properties that are not subspace properties. Also, in the paper [4], it was shown that for subspace properties P and Q for which (P and "not-Q") exist, (P and "not-Q") is not a subspace property, giving many additional new topological properties that are now known to be not subspace properties. Within the 2017 paper [7], similar results were obtained for product properties.

Thus, the study of weakly Po and related properties has exposed a previously unknown, very fruitful territory within topology with properties and tools needed to successfully address previously unasked and/or unanswered questions. Below, the exploration of the recently discovered new territory within topology continues.

2. Additional Properties of Weakly *Po* and Related Properties

Within the paper [8], it was shown that L is weakly Po and $Lo = T_0$ is a weakly Po property. In the paper [9], it was shown that for a "not-T_0" space (X, T), there exists a proper subspace (XTO, T_{XTO}) that is homeomorphic to $(X_0, Q(X, T))$.

Definition 2.1. Let (X, T) be a "not-T_0" space and let C_x be the T_0-identification space equivalence class containing x. Then XTO is a subset of X that contains exactly one element from each equivalence class C_x [9].

Theorem 2.1. *Let* (X, T) *be "not-*T_0*" and let XTO be as defined above. Then* (XTO, T_{XTO}) *is homeomorphic to* $(X_0, Q(X, T))$ [9].

Below, the results above are used to further investigate weakly *Po* and related properties.

Theorem 2.2. *Let* (X, T) *be a "not-*T_0*" space and let XTO be as defined above. Then* (XTO, T_{XTO}) *is* T_0.

Proof. Since $(X_0, Q(X, T))$ is T_0 and T_0 is a topological property, then, by the results above, (XTO, T_{XTO}) is T_0.

Theorem 2.3. *For each space* (X, T) *that is "not-*T_0*",* (XTO, T_{XTO}) *is a proper, dense subspace of* (X, T) *that is* T_0.

Proof. By the results above, *XTO* is a proper subset of *X* and (XTO, T_{XTO}) is T_0. Let $x \in X \setminus XTO$. Let $O \in T$ such that $x \in O$. Let $y \in XTO$ such that $y \in C_x$. Since $Cl(\{x\}) = Cl(\{y\})$, then $y \in O$. Hence, *XTO* is dense in (X, T).

Theorem 2.4. *Let Q be a topological property for which* $(Q$ *and "not-*T_0*")* *exists. Then the following are equivalent*: (a) *Q is a* T_0*-identification P property*, (b) *Q is a weakly Po property*, (c) *for each space* (X, T) *with property* $(Q$ *and "not-*T_0*")*, (XTO, T_{XTO}) *has property Q, and* (d) *for each space* (X, T) *with property* $(Q$ *and "not-*T_0*")*, (XTO, T_{XTO}) *has property Qo.*

Proof. By the results above, (a) and (b) are equivalent.

(b) implies (c): Let (X, T) be a space with property $(Q$ and "not-T_0"). Since Q is weakly Qo, then $(X_0, Q(X, T))$ has property Q. Since (X, T) has property "not-T_0", then (XTO, T_{XTO}) is homeomorphic to $(X_0, Q(X, T))$ and since Q is a topological property, (XTO, T_{XTO}) has property Q.

By Theorem 2.2, (c) implies (d).

(d) implies (b): Let (X, T) be a space with property (Q and not-T_0”). Since (XTO, T_{XTO}) has property Qo, then both Qo and (Q and not-T_0”) exist and, by Answer 1.3 above, Q is weakly Po.

In the 2015 paper [2], it was proven that for topological properties Q and W, which are both weakly Po, weakly Qo = weakly Wo iff $Qo = Wo$. Could $Q = W$ be added as an equivalent statement?

Theorem 2.5. *Let Q be a topological property that is weakly Po. Then Q is a T_0-identification P property and Q = weakly Qo.*

Proof. By Answer 1.3 above, both Qo and (Q and “not-T_0”) exist. Then by Answer 1.1 above, Q is a T_0-identification P property and Q = weakly Qo.

Theorem 2.6. *Let Q and W be topological properties, both of which are weakly Po. Then the following are equivalent*: (a) *weakly Qo = weakly Wo*, (b) *$Qo = Wo$*, *and* (c) *$Q = W$*.

Proof. By the results above, (a) and (b) are equivalent.

(a) implies (c): By Theorem 2.5, Q = weakly Qo and W = weakly Wo and, since weakly Qo = weakly Wo, then $Q = W$.

Clearly, (c) implies (a).

Theorem 2.7. *Let Q and W be topological properties, both of which are weakly Po. Then the following are equivalent*: (a) *weakly Qo implies weakly Wo*, (b) *Q implies W*, *and* (c) *Qo implies Wo*.

Proof. (a) implies (b): Since Q = weakly Qo, W = weakly Wo, and weakly Qo implies weakly Wo, then Q implies W.

Clearly (b) implies (c).

(c) implies (a): Let (X, T) be weakly *Qo*. Then $(X_0, Q(X, T))$ is *Qo*, which implies $(X_0, Q(X, T))$ is *Wo*, which implies (X, T) is weakly *Wo*. Thus, weakly *Qo* implies weakly *Wo*.

3. Weakly *Po* for Each Existent *Qo*

Definition 3.1. A space (X, T) has property *SM* iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exists an open set U containing only one of x and y.

Theorem 3.1. *Every space has property SM.*

Proof. Let (X, T) be a space. Let x and y be elements in X such that $Cl(\{x\}) \neq Cl(\{y\})$. Then $x \notin Cl(\{y\})$ or $y \notin Cl(\{x\})$, say $x \notin Cl(\{y\})$. Then $x \in U = X \setminus Cl(\{y\})$ is open and $y \notin U$. Thus (X, T) has property *SM*.

Corollary 3.1. *SM is a topological property.*

Theorem 3.2. *Let (X, T) be a space. Then (X, T) has property L iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exists an open set U containing only one of x and y.*

Proof. By Theorem 3.1, if (X, T) has property L, then (X, T) has property *SM*. Thus L implies *SM* and, since L is the least of all topological properties, then $L = SM$.

Definition 3.2. Let (X, T) be a space and for each $x \in X$, let C_x be the T_0-identification class containing x. Then *OXTO* is a subset of X containing exactly one element from each equivalence class C_x.

Note that if (X, T) is "not-T_0", then a subset *OXTO* of X is an *XTO* subset of X.

Theorem 3.3. *Let* (X, T) *be a space. Then* $(OXTO, T_{OXTO})$ *is homeomorphic to* $(X_0, Q(X, T))$.

Proof. By Theorem 3.2, (X, T) is (T_0 or "not-T_0"). Consider the case that (X, T) is T_0. Then $X = OXTO$ and $(X, T) = (OXTO, T_{OXTO})$ is T_0. Since a space (Y, S) is T_0 iff the natural map $N : (Y, S) \rightarrow (Y_0, Q(Y, S))$ is a homeomorphism [10], then $(OXTO, T_{OXTO})$ is homeomorphic to $(X_0, Q(X, T))$. Thus, consider the case that (X, T) is "not-T_0". Then $OXTO = XTO$ and, by Theorem 2.1, $(OXTO, T_{OXTO})$ is homeomorphic to $(X_0, Q(X, T))$.

Definition 3.3. Let Q be a topological property for which Qo exists. Then a space (X, T) has property WQ iff $(OXTO, T_{OXTO})$ has property Qo.

Within the paper [3], it was shown that for a topological property Q such that Qo exists, (Qo or QNO) is a T_0-identification P property, (Qo or QNO) = weakly (Qo or QNO)o = weakly Qo, and (Qo or QNO) is a weakly Po property, which is used below.

Theorem 3.4. *Let Q be a topological property for which Qo exists and let* (X, T) *be a space. Then* (X, T) *has property WQ iff* (X, T) *has property* (Qo *or* QNO).

Proof. Suppose (X, T) has property WQ. Then $(OXTO, T_{OXTO})$ has property Qo and, since $(OXTO, T_{OXTO})$ is homeomorphic to $(X_0, Q(X, T))$, then $(X_0, Q(X, T))$ has property Qo. Since Qo exists, then (Qo or QNO) is a T_0-identification P property and (Qo or QNO) = weakly Qo. Thus, a space (Y, S) is (Qo or QNO) iff $(Y_0, Q(Y, S))$ is Qo, and, since $(X_0, Q(X, T))$ is Qo, then (X, T) is (Qo or QNO).

Conversely, suppose (X, T) has property (Qo or QNO). Then $(X_0, Q(X, T))$ has property Qo and, since $(OXTO, T_{OXTO})$ is homeomorphic to $(X_0, Q(X, T))$, then $(OXTO, T_{OXTO})$ has property Qo. Hence (X, T) has property WQ.

Therefore, $WQ = (Qo$ or $QNO)$.

Since both Qo and QNO are topological properties, then WQ is a topological property.

Corollary 3.2. *Let Q be a topological property for which Qo exists. Then WQ is the special property for which a space has property WQ iff its T_0-identification space has property Qo.*

Below the results above are applied to determine $W(T_1)$.

Let (X, T) have property $W(T_1)$.

Then $(OXTO, T_{OXTO})$ has property $(T_1)o = T_1$. Suppose there exists a $x \in X$ such that $C_x \neq Cl(\{x\})$. Since $C_x \subseteq Cl(\{x\})$, let $y \in Cl(\{x\})$ that is not in C_x. Let $u, v \in OXTO$ such that $x \in C_u$ and $y \in C_v$. Then u and v are distinct elements in $OXTO$ and since $(OXTO, T_{OXTO})$ is T_1, there exists and open set V containing v and not u. Let $O \in T$ such that $V = O \cap OXTO$. Then $y \in O$ and $x \notin O$, which contradicts $y \in Cl(\{x\})$. Thus, for each $x \in X$, $C_x = Cl(\{x\})$ and $\{Cl(\{x\}) \mid x \in X\}$ is a decomposition of X.

Conversely, suppose $\{Cl(\{x\}) \mid x \in X\}$ is a decomposition of X. Let u and v be distinct elements of $OXTO$. Then $\{v\} = Cl(\{v\}) \cap OXTO$ is closed in $OXTO$. Hence singleton sets are closed in $OXTO$ and $(OXTO, T_{OXTO})$ is T_1.

Hence, a space (X, T) has property $W(T_1)$ iff $\{Cl(\{x\}) | x \in X\}$ is a decomposition of *X*.

Past work in T_0-identification spaces and weakly *Po* spaces verify the result above. The application above is not required, but it does add comfort to use of the work above.

In the paper [3], it was shown that if *Q* is a topological property for which both *Qo* and (Q and not-T_0") exist, then *Q* is a T_0-identification *P* property, $QNO = (Q$ or "not-T_0"), and *Q* = weakly *Qo* = (*Qo* or *QNO*) = (*Qo* or (Q and "not-T_0")), which greatly simplifies the above process.

Corollary 3.3. *Let Q be a topological property for which both Qo and* (Q *and "not-*T_0") *exist. Then WQ = Q.*

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