Journal of Pure and Applied Mathematics: Advances and Applications
Volume 18, Number 2, 2017, Pages 75-86
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jpamaa_7100121874

ON COMPLETENESS AND INFERENCE ALGORITHMS OF THE LOGIC $\mathcal{CARD}$ WITH *IA*

TAHSİN ÖNER and SELÇUK TOPAL

Department of Mathematics
Ege University
Izmir 35100
Turkey
e-mail: tahsin.oner@ege.edu.tr

Department of Mathematics
Bitlis Eren University
Bitlis, 13000
Turkey
e-mail: s.topal@beu.edu.tr

Abstract

This paper extends the logic $\mathcal{CARD}$ of comparisons between the sizes of sets to the logic $\mathcal{CARD}$ with *IA* which contains intersecting adjectives. We prove completeness of the logics $\mathcal{CARD}$ and $\mathcal{CARD}$ with *IA*. We also give algorithmic analysis and some algebraic properties of the logics.

1. Introduction

First attempts of proving completeness were made for syllogistic inferences by [1] and [2]. Completeness results of some syllogistic logics containing $\mathcal{CARD}$ were given in [3] by Moss after some complexity

2010 Mathematics Subject Classification: 03B65, 68W01, 03C10.
Keywords and phrases: logic of natural languages, algorithms, completeness.
Received October 8, 2017

results of syllogistic sentences of English had been given in [4] by Pratt-Hartmann. This paper presents an extension of the logic of cardinality comparison ($\mathcal{CARD}$) by Moss et al.. Sentences in the logic $\mathcal{CARD}$ consist of *There are at least as many x as y* where *x* and *y* are plural nouns. Why we prefer the name of *intersecting adjectives* instead of *adjectives* comes from [5]. We add *intersecting adjectives* (*IA*) on top of the logic. Sentences in the logic of $\mathcal{CARD}$ *with IA* are of the form *There are at least as many red x as blue y* in which *red* and *blue* are adjectives.

On the other hand, neither language of the logic $\mathcal{CARD}$ nor language of the logic $\mathcal{CARD}$ *with IA* is expressible in FOL [6]. FOL is not decidable and $\mathcal{CARD}$ *with IA* is decidable.

For readability, we will use the notations and construction technics of the models in papers [3] and [7]. We will work on *finite* models.

2. The Logic of $\mathcal{CARD}$

We start with a *variable set* $\mathcal{P}$ of plural nouns $x, y, z \ldots \exists^{\geq}(x, y)$ is used as an abbreviation of the sentence *There are at least as many x as y*. The semantics is built on *finite* set of sentences and *finite* universe *M*. $\forall x \in \mathcal{P}, [[x]] \subseteq M$ such that $[[\ ]]$ is an *interpretation function* from $\mathcal{P}$ to subsets of *M*. Cardinality of $[[x]]$ is shown by $|[[x]]|$. A model $\mathcal{M} = (M, [[\ ]])$ has "$\mathcal{M} \models \exists^{\geq}(x, y)$ iff $|[[x]]| \geq |[[y]]|$ in $\mathcal{M}$" truth between models and sentences.

$$\frac{}{\exists^{\geq}(x, x)} \qquad \frac{\exists^{\geq}(x, y) \quad \exists^{\geq}(y, z)}{\exists^{\geq}(x, z)}$$

Figure 1. Rules for $\mathcal{CARD}$.

Lemma 2.1. *$\mathcal{CARD}$ is sound with respect to rules in Figure* 1.

Proof. We use *complete induction* to prove soundness of the logic of $\mathcal{CARD}$ on number of nodes of proof tree over Γ. Let Γ be a set of sentences and φ be a sentence in $\mathcal{CARD}$. We must show that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. Suppose that all proof trees over Γ with less than $\mathcal{CARD}$ nodes. Suppose the root and its parents are labelled as follow:

$$\frac{\exists^{\geq}(x,\ y) \quad \exists^{\geq}(y,\ z)}{\exists^{\geq}(x,\ z)}.$$

Let Υ_1 and Υ_2 be sub trees at ending $\exists^{\geq}(y, z)$ and $\exists^{\geq}(x, y)$. For some y, the root of Υ_1 is $\exists^{\geq}(y, z)$ and the root of Υ_2 is $\exists^{\geq}(x, y)$. Also, lengths of Υ_1 and Υ_2 are less than n. $\Gamma \vDash \exists^{\geq}(y, z)$ and $\Gamma \vDash \exists^{\geq}(x, y)$ by the induction hypothesis. Now we argue that $\Gamma \vDash \exists^{\geq}(x, z)$. To prove this, we must take an arbitrary model $\mathcal{M}$ in which all sentences in Γ are true. So, $\mathcal{M} \vDash \exists^{\geq}(x, y)$ and $\mathcal{M} \vDash \exists^{\geq}(y, z)$. Then $|[[x]]| \geq |[[y]]|$ and $|[[y]]| \geq |[[z]]|$. We obtain $|[[x]]| \geq |[[z]]|$ by using *transitivity of greater than or equal relation*. Because $\mathcal{M}$ is arbitrary, we conclude $\Gamma \vDash \exists^{\geq}(x, z)$. On the other hand, if we consider the proof tree with one node, then φ is in Γ or form of $\exists^{\geq}(x, x)$. If φ is in Γ, every model that satisfies sentences in Γ, since φ in Γ, satisfies φ as well. If φ is form of $\exists^{\geq}(x, x)$, then every model satisfies φ.

□

Definition 2.2. Let Γ be a subset of $\mathcal{CARD}$. $x \geq_c y$ means that $\Gamma \vdash \exists^{\geq}(x,\ y)$.

Proposition 2.3. $\geq_c$ *is a preorder relation on set of variables of* Γ.

Proof. We obtain reflexivity of $\geq_c$ from rule Figure 1. To show transitivity of $\geq_c$, assume that $x \geq_c y \geq_c z$. Then we have proof trees at ending $x \geq_c y$ and $y \geq_c z$. Finally, by using second rule of Figure 1, we conclude $x \geq_c z$. □

Theorem 2.4. *$\mathcal{CARD}$ is complete with respect to rules in Figure* 1.

Proof. We define a model $\mathcal{M}$ with one element universe $M = \{*\}$ by using [3] to prove completeness. Note that variable y is the same everywhere in the proof.

$$[[z]] = \begin{cases} \{*\}, & \text{if } \Gamma \vdash \exists^{\geq}(z, y), \\ \varnothing, & \text{otherwise.} \end{cases}$$

Suppose that $\Gamma \models \exists^{\geq}(x, y)$. Firstly, we argue that if $\exists^{\geq}(w, u)$ in Γ, then $|[[w]]| \geq |[[u]]|$. Assume that $|[[u]]| \neq \varnothing$. Then $|[[u]]| = \{*\}$. This means $\Gamma \vdash \exists^{\geq}(u, y)$.

$$\frac{\exists^{\geq}(w, u) \quad \overset{\vdots}{\exists^{\geq}(u, y)}}{\exists^{\geq}(w, u)}.$$

As it can be seen the above, we have $\exists^{\geq}(w, u)$ from proof tree. Then $|[[w]]| = \{*\}$. We conclude $|[[w]]| \geq |[[u]]|$. If $\mathcal{M}$ makes all sentences in Γ true, it also must the conclusion true. Thus $|[[x]]| \geq |[[y]]|$. We have $\Gamma \vdash \exists^{\geq}(y, y)$. Then $|[[y]]| = \{*\}$. Therefore $|[[x]]| = \{*\}$. We conclude $\Gamma \vdash \exists^{\geq}(x, y)$.

□

3. The Logic $\mathcal{CARD}$ *with IA*

The purpose of this section is to obtain the following kind of inferences:

There are at least as many red y as x

There are at least as many red x as h

There are at least as many blue h as z

———————————————————— $(\star)$

$\therefore$ There are at least as many y as z.

We want to add some intersecting adjectives in English to sentences of language of the logic $\mathcal{CARD}$. We can give some examples of intersecting adjectives and nouns for the purpose of illustrating objects in natural language (*IA*) such as *red*, *blue*, *yellow*, ... etc and plural nouns are cats, cars, ... etc. In this language, syntax begins with *basic nouns* $x, y, z, \ldots$ and *intersecting adjectives* $a_1, a_2, \ldots$. A *complex noun ax* is a noun such that x is a *basic noun* and a is an *intersecting adjective*. We shall use x, y, z for basic nouns, p, q, r for nouns and (there are too many intersecting adjectives in English) *red*, *blue*, *green* for *intersecting adjectives* to avoid abstruseness and to provide more readability. This language is called $\mathcal{CARD}$ *with IA* containing of combinations of sentences $\exists^{\geq}(p, q)$, where p and q nouns. The semantics is built on *finite* set of sentences and *finite* set M. For all basic noun x, $[[x]] \subseteq M$. A complex noun ax is defined by $[[ax]] = [[a]] \cap [[x]]$. A model $\mathcal{M} = (M, [[\ ]])$ has the relation of truth between models and sentences: "$\mathcal{M} \models \exists^{\geq}(p, q)$ iff $|[[p]]| \geq |[[q]]|$ in $\mathcal{M}$".

$$\frac{}{\exists^{\geq}(p, p)}\ (ax1) \qquad \frac{}{\exists^{\geq}(x, red\ x)}\ (ax2)$$

$$\frac{\exists^{\geq}(p, q) \quad \exists^{\geq}(q, r)}{\exists^{\geq}(p, r)}\ (T)$$

Figure 2. The logic for $\mathcal{CARD}$ with *IA*. x is a basic noun and p, q, and r nouns that are complex or basic nouns.

Lemma 3.1. *$\mathcal{CARD}$ with IA is sound with respect to rules in Figure* 2.

Remark 3.2. We will skip to prove soundness of the logic because the logic has only one new axiom $(ax2)$ and not a new rule.

Theorem 3.3. *$\mathcal{CARD}$ with IA is complete with respect to rules in Figure* 2.

Proof. We use a similar model we used for the proof of completeness of $\mathcal{CARD}$. We could take two semantics *red* and q. Instead of this, as we shall see, we do not need to do this. We use x, y, z are basic nouns and p, q, m, n, and r nouns that are complex or basic nouns. Now we have to show that "if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$" to prove completeness of the logic. We construct model $\mathcal{M}$ having the universe M with singleton and by given

$$[[q]] = \begin{cases} \{*\}, & \text{if } \Gamma \vdash \exists^{\geq}(q, n), \\ \varnothing, & \text{otherwise.} \end{cases}$$

First, we show that $\mathcal{M} \models \Gamma$. Note that the atom n is the same one as in the sentence $\exists^{\geq}(q, n)$ throughout this proof. Now, take an arbitrary sentence $\exists^{\geq}(\alpha_1, \alpha_2) \in \Gamma$. We must show that $\|[[\alpha_2]]\| \leq \|[[\alpha_1]]\|$. This is trivially true if $[[\alpha_2]] = \varnothing$. So, assume without loss of generality that $[[\alpha_2]] \neq \varnothing$. By the definition of $\mathcal{M}$, it follows that $[[\alpha_2]] = \{\star\}$, i.e., $\Gamma \vdash \exists^{\geq}(\alpha_2, n)$. Since $\exists^{\geq}(\alpha_1, \alpha_2) \in \Gamma$, we have $\Gamma \vdash \exists^{\geq}(\alpha_1, \alpha_2)$. By ($T$), we obtain $\Gamma \vdash \exists^{\geq}(\alpha_1, n)$. Thus, $[[\alpha_1]] = \{\star\}$, and therefore $|[[\alpha_2]]| \leq |[[\alpha_1]]|$ as desired.

Now, we argue that if $\exists^{\geq}(red\ x, q) \in \Gamma$ then $|[[x]]| \geq |[[q]]|$. Taking $[[q]] = \{\star\}$, $\Gamma \vdash \exists^{\geq}(q, n)$. Then we have proof tree as the following:

$$\frac{\exists^{\geq}(red\ x,\ q) \quad \overset{\vdots}{\exists^{\geq}(q,\ n)}}{\exists^{\geq}(red\ x,\ n)}.$$

Then we have $\Gamma \vdash \exists^{\geq}(red\ x, n)$ and $\Gamma \vdash \exists^{\geq}(x, red\ x)$ from ($ax2$). We obtain $\Gamma \vdash \exists^{\geq}(x, n)$ by using (T). This means that $[[x]] = \{\star\}$ and we conclude $|[[x]]| \geq |[[q]]|$.

We obtain $\mathcal{M} \models \exists^{\geq}(p, n)$, i.e., $|[[n]]| \leq |[[p]]|$ because $\exists^{\geq}(p, n) \in \Gamma$ and we have shown that $\mathcal{M} \models \Gamma$. $[[n]] = \{\star\}$ by ($ax1$). It then follows that $\Gamma \vdash \exists^{\geq}(p, n)$ by the definition of $\mathcal{M}$. Now, we argue that if $\mathcal{M} \models \exists^{\geq}(red\ x, n)$ then $\Gamma \vdash \exists^{\geq}(x, n)$. $[[n]] = \{\star\}$ by ($ax1$) and $|[[red\ x]]| \geq |[[n]]|$. Then we have $[[red\ x]] = \{\star\}$ and $\Gamma \vdash \exists^{\geq}(red\ x, n)$. $\Gamma \vdash \exists^{\geq}(x, n)$ by using ($ax2$) and (T).

□

4. Algorithms of $\mathcal{CARD}$ and $\mathcal{CARD}$ *with IA*

We showed that relation $\geq_c$ is a preorder on set of sentences Γ in $\mathcal{CARD}$ in Proposition 2.3. We will use the relation and graph theoretic methods to produce $\mathcal{CARD}$ *with IA* proofs for any provable sequent. To do this, we will give some definitions and constructions.

Definition 4.1. A graph $\mathcal{G}$ is a pair $\mathcal{G} = (G, E)$ such that G is set of nodes and edge is said to be each elements of E which is a subset of $G \times G$.

Definition 4.2. Let $\mathcal{G} = (G, E)$ be any graph. $\forall a, b \in G,$ if there is a path from a to b in $\mathcal{G}^{\star}$, $\mathcal{G}^{\star}$ is called reflexive transitive closure of $\mathcal{G}$ and it's relation is called reachability [8], [9].

Definition 4.3. For all sentences and all nouns in $\mathcal{CARD}$, relation $\hookrightarrow$ is defined by

$$x \underset{\Gamma}{\hookrightarrow} y \quad \text{iff} \quad \exists^{\geq}(y, x) \in \Gamma.$$

This defines a graph on Γ that $\mathcal{G}_{\Gamma} = (P, \hookrightarrow)$ such that P is set of all nouns of Γ.

Definition 4.4. Let Γ be a set of sentences and P is set of nouns in $\mathcal{CARD}$. $\forall x, y \in P,$ relation $\overset{\star}{\hookrightarrow}$ is defined by

$$x \underset{\Gamma}{\overset{\star}{\hookrightarrow}} y \quad \text{iff} \quad \Gamma \vdash \exists^{\geq}(y, x).$$

This defines the relation reachability on $\mathcal{G}_{\Gamma}$. [1]

Proposition 4.5. *Let Γ be a set of sentences in $\mathcal{CARD}$. Let P be set of nouns in Γ. Let $\mathcal{G}_{\Gamma}$ be a graph of Γ on P. Let $\underset{\Gamma}{\overset{\star}{\hookrightarrow}}$ be reachability relation of $\mathcal{G}_{\Gamma}$ and $x, y \in \mathcal{G}_{\Gamma}$. Then the following are equivalent*:

(1) $x \underset{\Gamma}{\overset{\star}{\hookrightarrow}} y$.

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[1] The form of *the logic* All of definitions in Definition 4.3 and Definition 4.4 can be found in [3].

(2) $\Gamma \vdash \exists^{\geq}(y, x)$.

(3) $\Gamma \models \exists^{\geq}(y, x)$.

Proof. Proofs of the equivalences can be showed easily from Definition 4.4, soundness and completeness. □

Definition 4.6. Let Γ be a set of sentences in $\mathcal{CARD}$ with *IA*. Let *P* be set of nouns in Γ. Let $\mathcal{G}_\Gamma$ be a graph of Γ on *P*. Let $\overset{\star}{\underset{\Gamma}{\hookrightarrow}}$ be reachability relation of $\mathcal{G}_\Gamma$ and $x, y \in \mathcal{G}_\Gamma$. The conclusion can not be derived from the premises without $(ax2)$ in $(\star)$. Then we need to construct larger graph and relation. Now, for all $x \in P$, we define six new sets:

(a) $I = \{ a$: if x is a complex noun then take the part of adjective $a\}$.

(b) $B = \{ x$: if x is not basic noun then take the part of basic, otherwise take $x\}$.

(c) $IB = \bigcup\{ax : a \in I \text{ and } \in B\}$.

(d) $P^{\star\star} = P \cup IB$.

(e) $\Theta = \overset{\star}{\underset{\Gamma}{\hookrightarrow}} \cup \{(x, ax) : a \in I \text{ and } x \in IB\}$.

(f) $\Gamma^{\star\star} = \Gamma \cup \{\exists^{\geq}(x, ax) : \forall (x, ax) \in \Theta\}$.

Let $\mathcal{G}_\Gamma^{\star\star}$ be a graph of $\Gamma^{\star\star}$ on $P^{\star\star}$. Then $\overset{\star\star}{\underset{\Gamma}{\hookrightarrow}}$ is called reachability relation on $\mathcal{G}_\Gamma^{\star\star}$.

Proposition 4.7. *Let* $\overset{\star\star}{\underset{\Gamma}{\hookrightarrow}}$ *be reachability relation of* $\mathcal{G}_\Gamma^{\star\star}$ *as in Definition* 4.6 *and* $p, q \in \mathcal{G}_\Gamma^{\star\star}$. *Then the following are equivalent*:

(1) $p \overset{\star\star}{\underset{\Gamma}{\hookrightarrow}} q$.

(2) $\Gamma \vdash \exists^{\geq}(q,\ p)$.

(3) $\Gamma \models \exists^{\geq}(q,\ p)$.

Proof. (1) $\Rightarrow$ (2) We suppose $p \overset{\star\star}{\underset{\Gamma}{\hookrightarrow}} q$ and we show that $\Gamma \vdash \exists^{\geq}(q,\ p)$. We use *induction* on a path of length n from p to q. If $n = 0$, then $p = q$ and it is clear that $\Gamma \vdash \exists^{\geq}(p,\ p)$ by $(ax1)$. Now suppose that our assumption is *true* for n. This means there is a path of length $n+1$ that p is reachable from q in $\mathcal{G}_{\Gamma}^{\star\star}$. Since our assumption, there is a path of length n and a r in $\mathcal{G}_{\Gamma}^{\star\star}$ such that p is reachable from r. Then by *induction* hypothesis, we have $\Gamma \vdash \exists^{\geq}(p,\ r)$ and since the existence of a edge between q and r, $\exists^{\geq}(q,\ r) \in \Gamma$. Hence we have a proof tree as the following:

$$\frac{\exists^{\geq}(q,\ r) \qquad \overset{\vdots}{\exists^{\geq}(p,\ y)}}{\exists^{\geq}(p,\ q)}\,(T).$$

Therefore $\Gamma \vdash \exists^{\geq}(p,\ q)$.

(2) $\Rightarrow$ (3) We saw the proof of soundness in Lemma 3.1.

(3) $\Rightarrow$ (1) Let's take a model H with an interpretation $[[q]] = \{p \in \mathcal{G}_{\Gamma}^{\star\star} : p \overset{\star\star}{\underset{\Gamma}{\hookrightarrow}} q\}$ and suppose that $p \overset{\star\star}{\underset{\Gamma}{\not\hookrightarrow}} q$. It is clear that if $\exists^{\geq}(q,\ p)$, then $H \models \exists^{\geq}(q,\ p)$. Hence $[[p]] \subseteq [[q]]$. By the hypothesis, $p \in \mathcal{G}_{\Gamma}^{\star\star}$. $p \in [[p]]$, since $p \overset{\star\star}{\underset{\Gamma}{\hookrightarrow}} p$. Since $p \overset{\star\star}{\underset{\Gamma}{\not\hookrightarrow}} q$, $p \notin [[q]]$. Hence, $H \not\models \exists^{\geq}(p,\ q)$. This is a contradiction. Then $p \overset{**}{\underset{\Gamma}{\hookrightarrow}} q$.

□

Algorithm 1: Given a finite set Γ and a sentence $\exists^{\geq}(q,\ p)$ in $\mathcal{CARD}$ *with IA*, tell whether or not $\Gamma \vdash \exists^{\geq}(q,\ p)$.

1: **If** $q = p$ **Then Print** $(ax1)$ **End If**

2: **If** $\exists^{\geq}(q,\ p)$ is the form of $\exists^{\geq}(x,\ ax)$ **Then Print** $(ax2)$ **End If**

3: Take $\mathcal{G}_{\Gamma}^{\star\star} \leftarrow P^{\star\star}$ with p and q ▷ see Definition 4.6

4: **If** $p \overset{\star\star}{\underset{\Gamma}{\hookrightarrow}} q$ **Then Print** the path in $\mathcal{G}_{\Gamma}^{\star\star}$ ▷ the path from p to q is the proof tree of $p \overset{\star\star}{\underset{\Gamma}{\hookrightarrow}} q$

5: **Else Print** $p \overset{\star\star}{\underset{\Gamma}{\not\hookrightarrow}} q$

6: **End If**

5. Conclusion

In this paper, we have given a soundness and completeness theorem of the logic $\mathcal{CARD}$ and $\mathcal{CARD}$ *with IA*. Although languages of the logics are beyond FOL [6] (FOL is not decidable), the logics are decidable and also they have efficient proof search and model construction.

References

[1] J. Lukasiewicz, Aristotle's Syllogistic from the Standpoint of Modern Formal Logic, Clarendon Press, Oxford, 1951.

[2] J. Corcoran, Completeness of an ancient logic, Journal of Symbolic Logic 37 (1972), 696-702.

[3] L. S. Moss, Completeness theorems for syllogistic fragments, eds. Logics for linguistic structures, Vol. 201, Walter de Gruyter (2008), 143-175.

[4] I. Pratt-Hartmann, Fragments of language, Journal of Logic Language and Information 13(2) (2004), 207-223.

[5] E. L. Keenan and L. M. Faltz, Boolean Semantics for Natural Language, Syntheses Language Library, Vol. 23, D. Reidel Publishing Co., Dordrecht, 1985.

[6] A. Baltag and S. Smets, Three Etudes on Logical Dynamics and the Program of Natural Logic, Johan van Benthem on Logic and Information Dynamics (2014), 705-728.

[7] L. S. Moss, Intersecting adjectives in syllogistic logic, The Mathematics of Language, Springer Berlin Heidelberg (2010), 223-237.

[8] Phokion G. Kolaitis, On the expressive power of logics on finite models, Finite Model Theory and its Applications, Springer Berlin Heidelberg (2007), 27-123.

[9] S. S. Skiena, Transitive closure and reduction, The algorithm design manual (2nd ed.), Springer (2011), 495-497.

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