APPROXIMATION FOR THE PERCENTILES OF THE SAMPLE COEFFICIENT OF VARIATION

NOBUMICHI SHUTOH

Graduate School of Maritime Sciences
Kobe University
Japan
e-mail: shutoh@maritime.kobe-u.ac.jp

Abstract

We propose a method of approximation for the percentiles of the sample coefficient of variation based on the samples drawn from a normal population via Cornish-Fisher expansion. We derive the asymptotic expansion of the distribution of the sample coefficient of variation by using the normal approximation of the Chi-square variable and some useful expectation formulas. Finally, we observe the superiority of our result on the percentiles of the test statistic for one-sample problem by conducting Monte Carlo simulation under some selected parameters.

1. Introduction

In this paper, we consider statistical inference for the coefficient of variation R, that is, the ratio of the mean to the standard deviation which has applications in various research areas; see, for example, [7].

Under normality, one of the traditional results in this study is given by [6], who proposed an approximation to the distribution of the sample

coefficient of variation that asymptotically follows the χ^2 distribution (recently, [2] showed that approximation derived by [6] essentially follows a type-II noncentral beta distribution). Several authors concluded that approximation considered in [6] performs well under $R \leq 2/3$.

For testing the coefficients of variation, various procedures have been proposed. [1] proposed the likelihood ratio test that follows asymptotically the χ^2 distribution under the null hypothesis in k-sample problem; this was later modified by [11]. [4] derived the Wald type test and the Score test. [10] considered a divergence based test statistic. Employing the stochastic expansion used in [9], [8] proposed a simple test statistic by using the asymptotic normality of the sample coefficient of variation.

In this paper, we show the asymptotic normality of the sample coefficient of variation. It will be applicable to a test statistic for one-sample problem of R, which is asymptotically equivalent to [8]. Further, we derive the asymptotic distribution up to the higher order with respect to n^{-1} and $(n-1)^{-1}$, where n denotes the sample size, in order to obtain more accurate approximation for the percentiles of the considered test statistic via Cornish-Fisher expansion. Simulation studies observe the size of the test on the basis of our result, which is compared to the test derived by [8]. Finally, in our concluding remarks, we address the direction for further study.

2. The Distribution of the Sample Coefficient of Variation

Let $X_1, \ldots, X_n$ be the n independent samples from $N(\mu, \sigma^2)$. The sample coefficient of variation $\hat{R}$ is defined as

$$\hat{R} = \frac{S}{\overline{X}} \stackrel{d}{=} \frac{\sqrt{Y}}{Z + \frac{\sqrt{n}}{R}}, \quad \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2,$$

where the symbol $\overset{d}{=}$ denotes the equivalent in the sense of the distribution, Y is a random variable that follows the χ^2 distribution with $n-1$ degrees of freedom, Z is a random variable that follows $N(0, 1)$, Y is independent of Z, and R is the coefficient of variation σ / μ.

Employing the useful notations of the order $n = m\rho$, $n-1 = m\lambda$, where ρ and λ are some constants as in [3], and the normalizing approximation of the Chi-square variable

$$U = \frac{Y - m\lambda}{\sqrt{2m\lambda}} \overset{d}{\rightarrow} N(0, 1)\ (m \rightarrow \infty), \tag{1}$$

the distribution of the sample coefficient of variation $\hat{R}$ can be expressed as

$$\hat{R} \overset{d}{=} \frac{\sqrt{m\lambda + \sqrt{2m\lambda}U}}{Z + \frac{\sqrt{m\rho}}{R}} = R\frac{\sqrt{\lambda}}{\sqrt{\rho}}\left(1 + \frac{\sqrt{2}}{\sqrt{m\lambda}}U\right)^{\frac{1}{2}}\left(1 + \frac{R}{\sqrt{m\rho}}Z\right)^{-1}.$$

Henceforth, we consider the asymptotic distribution of $\tilde{R} = \sqrt{n/(n-1)}\hat{R}$. Applying Taylor expansion to the standardized $\tilde{R}$, we have

$$T(R) = \frac{\sqrt{m}(\tilde{R} - R)}{\sigma} \overset{d}{=} t_0(R) + \frac{t_1(R)}{\sqrt{m}} + \frac{t_2(R)}{m} + O_p(m^{-\frac{3}{2}}), \tag{2}$$

where $\sigma = R\sqrt{\{1/(2\lambda)\} + (R^2/\rho)}$,

$$t_0(R) = -\frac{R^2}{\sqrt{\rho}\sigma}Z + \frac{R}{\sqrt{2}\sqrt{\lambda}\sigma}U,$$

$$t_1(R) = \frac{R^3}{\rho\sigma}Z^2 - \frac{R^2}{\sqrt{2}\sqrt{\rho\lambda}\sigma}ZU - \frac{R}{4\lambda\sigma}U^2,$$

$$t_2(R) = -\frac{R^4}{\rho\sqrt{\rho}\sigma}Z^3 + \frac{R^3}{\sqrt{2}\rho\sqrt{\lambda}\sigma}Z^2U + \frac{R^2}{4\sqrt{\rho}\lambda\sigma}ZU^2 + \frac{\sqrt{2}R}{8\lambda\sqrt{\lambda}\sigma}U^3.$$

Noting that $T(R) \overset{d}{=} t_0(R) + o_p(1)$, (2) coincides with the result shown by [8].

Theorem 1. *For a large n, $\hat{R}$ is asymptotically distributed as $N(R$,*

$\{R^2 / (n-1)\}(1/2 + R^2))$.

3. Asymptotic Expansion for the Distribution
of the Test Statistic

In this section, we show the asymptotic expansion of the distribution of $T(R)$ in a similar manner to [5]. Further, we can approximate the percentiles of $T(R)$.

The characteristic function of $T(R)$ can be expanded as

$$E\left[\exp(itT(R))\right]$$

$$= E\left[\exp(it \cdot t_0(R)) \cdot \exp\left\{it\left(\frac{t_1(R)}{\sqrt{m}} + \frac{t_2(R)}{m} + O_p(m^{-\frac{3}{2}})\right)\right\}\right]$$

$$= E\left[\exp(it \cdot t_0(R))\right] + \frac{1}{\sqrt{m}} E\left[-\theta \exp(it \cdot t_0(R)) t_1(R)\right]$$

$$+ \frac{1}{m} E\left[-\theta \exp(it \cdot t_0(R)) t_2(R) + \frac{\theta^2}{2} \exp(it \cdot t_0(R)) \{t_1(R)\}^2\right] + O(m^{-\frac{3}{2}}),$$

(3)

where $\theta = -it$. In order to obtain the expectations, we have a lemma.

Lemma 2. *The density function of u can be obtained as*

$$f_u(u) = \phi(u)\left\{1 + \frac{1}{\sqrt{m}}\left(\frac{\sqrt{2}}{3\sqrt{\lambda}} u^3 - \frac{\sqrt{2}}{\sqrt{\lambda}} u\right)\right.$$

$$\left. + \frac{1}{m}\left(\frac{1}{9\lambda} u^6 - \frac{7}{6\lambda} u^4 + \frac{2}{\lambda} u^2 - \frac{1}{6\lambda}\right)\right\} + O(m^{-\frac{3}{2}}),$$

where $\phi(z)$ is the density function of Z.

Further, let $I_i(a)$ be the following integration:

$$\int_{-\infty}^{\infty} x^i \, \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x - (-\theta a))^2\right] dx.$$

Then we have

$$\int_{-\infty}^{\infty} x^i \, \exp(it \cdot ax)\phi(x)dx = \exp\left(-\frac{1}{2}a^2t^2\right)I_i(a). \tag{4}$$

To calculate $I_i(a)$, we also have the following corollary:

Corollary 3. *For $i = 0, \ldots, 6$, the integration $I_i(a)$ can be obtained as*

$$I_0(a) = 1, \quad I_1(a) = -(\theta a), \quad I_2(a) = 1 + (\theta a)^2,$$

$$I_3(a) = -3(\theta a) - (\theta a)^3, \quad I_4(a) = 3 + 6(\theta a)^2 + (\theta a)^4,$$

$$I_5(a) = -15(\theta a) - 10(\theta a)^3 - (\theta a)^5,$$

$$I_6(a) = 15 + 45(\theta a)^2 + 15(\theta a)^4 + (\theta a)^6.$$

By using Lemma 2 and Corollary 3, we have the characteristic function stated in (3) and the distribution function of $T(R)$.

Theorem 4. *The distribution function of $T(R)$ is obtained as*

$$\Phi(x) + [a_{11}(R) + a_{12}(R)H_2(x)$$

$$+ a_{21}(R)H_1(x) + a_{22}(R)H_3(x) + a_{23}(R)H_5(x)]\phi(x) + O_{\frac{3}{2}},$$

where $O_{\frac{3}{2}}$ denotes the remainder terms of the $3/2$-th order with respect to n^{-1} and $(n-1)^{-1}$,

$$c^{\frac{1}{2}}a_{11}(R) = \frac{1}{4(n-1)} - \frac{R^2}{n},$$

$$c^{\frac{3}{2}}a_{12}(R) = -\frac{1}{24(n-1)^2} - \frac{R^2}{2n(n-1)} - \frac{R^4}{n^2},$$

$$ca_{21}(R) = \frac{1}{32(n-1)^2} - \frac{R^2}{4n(n-1)} - \frac{9R^4}{2n^2},$$

$$c^2 a_{22}(R) = \frac{1}{96(n-1)^3} - \frac{R^2}{6n(n-1)^2} - \frac{2R^4}{n^2(n-1)} - \frac{4R^6}{n^3},$$

$$c^3 a_{23}(R) = -\frac{1}{1152(n-1)^4} - \frac{R^2}{48n(n-1)^3} - \frac{R^4}{6n^2(n-1)^2}$$

$$- \frac{R^6}{2n^3(n-1)} - \frac{R^8}{2n^4},$$

$$c = \frac{1}{2(n-1)} + \frac{R^2}{n},$$

and $H_r(x)$ denotes the r-th Hermite polynomial:

$$H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x,$$

$$H_5(x) = x^5 - 10x^3 + 15x.$$

Proof. Using Lemma 2 and (4), we obtain the following expressions:

$$\mathbb{I}_z(j) = E\left[Z^j \exp\left(it \cdot -\frac{R^2}{\sqrt{\rho\sigma}} Z \right) \right]$$

$$= \int_{-\infty}^{\infty} z^j \exp\left(it \cdot -\frac{R^2}{\sqrt{\rho\sigma}} z \right) f_z(z) dz$$

$$= \exp\left(-\frac{1}{2}\frac{R^4}{\rho\sigma^2} t^2 \right) I_j\left(-\frac{R^2}{\sqrt{\rho\sigma}} \right), \tag{5}$$

$$\mathbb{I}_u(j) = E\left[U^j \exp\left(it \cdot \frac{R}{\sqrt{2}\sqrt{\lambda}\sigma} U \right) \right]$$

$$= \int_{-\infty}^{\infty} u^j \exp\left(it \cdot \frac{R}{\sqrt{2}\sqrt{\lambda}\sigma} u \right) f_u(u)\, du$$

$$= \exp\left(-\frac{1}{2}\frac{R^2}{2\lambda\sigma^2} t^2 \right)\left[I_j\left(\frac{R}{\sqrt{2}\sqrt{\lambda}\sigma} \right) \right.$$

$$+ \frac{1}{\sqrt{m}}\left\{ \frac{\sqrt{2}}{3\sqrt{\lambda}} I_{3+j}\left(\frac{R}{\sqrt{2}\sqrt{\lambda}\sigma} \right) - \frac{\sqrt{2}}{\sqrt{\lambda}} I_{1+j}\left(\frac{R}{\sqrt{2}\sqrt{\lambda}\sigma} \right) \right\}$$

$$+ \frac{1}{m}\left\{ \frac{1}{9\lambda} I_{6+j}\left(\frac{R}{\sqrt{2}\sqrt{\lambda}\sigma} \right) - \frac{7}{6\lambda} I_{4+j}\left(\frac{R}{\sqrt{2}\sqrt{\lambda}\sigma} \right) \right.$$

$$\left. + \frac{2}{\lambda} I_{2+j}\left(\frac{R}{\sqrt{2}\sqrt{\lambda}\sigma} \right) - \frac{1}{6\lambda} I_j\left(\frac{R}{\sqrt{2}\sqrt{\lambda}\sigma} \right) \right\} + O(m^{-\frac{3}{2}}). \qquad (6)$$

By using the above expressions, we derive some moments in order to obtain the main result.

For the leading terms, we need to obtain the expectation $E[\exp(it \cdot t_0(R))] = \mathbb{I}_z(0)\mathbb{I}_u(0)$ up to the terms of m^{-1}. Substituting $j = 0$ into (5) and (6) and making use of Corollary 3, we have

$$\mathbb{I}_z(0) = \exp\left(-\frac{1}{2}\frac{R^4}{\rho\sigma^2} t^2 \right),$$

$$\mathbb{I}_u(0) = \exp\left(-\frac{1}{2}\frac{R^2}{2\lambda\sigma^2} t^2 \right)\left\{ 1 + \frac{1}{\sqrt{m}}\left(-\frac{\theta^3 R^3}{6\lambda^2\sigma^3} \right) \right.$$

$$\left. + \frac{1}{m}\left(\frac{\theta^4 R^4}{8\lambda^3\sigma^4} + \frac{\theta^6 R^6}{72\lambda^4\sigma^6} \right) \right\} + O(m^{-\frac{3}{2}}).$$

Because it holds that

$$\exp\left(-\frac{1}{2} t^2 \right) = \exp\left(-\frac{1}{2}\frac{R^4}{\rho\sigma^2} t^2 \right)\exp\left(-\frac{1}{2}\frac{R^2}{2\lambda\sigma^2} t^2 \right),$$

we can obtain

$$E[\exp(it \cdot t_0(R))] = \exp\left(-\frac{1}{2}t^2\right)\left\{1 + \frac{1}{\sqrt{m}}\left(-\frac{\theta^3 R^3}{6\lambda^2 \sigma^3}\right)\right.$$

$$\left. + \frac{1}{m}\left(\frac{\theta^4 R^4}{8\lambda^3 \sigma^4} + \frac{\theta^6 R^6}{72\lambda^4 \sigma^6}\right)\right\} + O(m^{-\frac{3}{2}}). \qquad (7)$$

We also need to obtain the expectation

$$E\left[-\theta \exp(it \cdot t_0(R)) t_1(R)\right]$$

$$= -\frac{\theta R^3}{\rho\sigma}\mathbb{I}_z(2)\mathbb{I}_u(0) + \frac{\theta R^2}{\sqrt{2}\sqrt{\rho\lambda}\sigma}\mathbb{I}_z(1)\mathbb{I}_u(1) + \frac{\theta R}{4\lambda\sigma}\mathbb{I}_z(0)\mathbb{I}_u(2).$$

Substituting $j = 1, 2$ into (5) and (6) and making use of Corollary 3, we have

$$\mathbb{I}_z(1) = \exp\left(-\frac{1}{2}\frac{R^4}{\rho\sigma^2}t^2\right)\left(\frac{\theta R^2}{\sqrt{\rho\sigma}}\right),$$

$$\mathbb{I}_z(2) = \exp\left(-\frac{1}{2}\frac{R^4}{\rho\sigma^2}t^2\right)\left(1 + \frac{\theta^2 R^4}{\rho\sigma^2}\right),$$

$$\mathbb{I}_u(1) = \exp\left(-\frac{1}{2}\frac{R^2}{2\lambda\sigma^2}t^2\right)\left\{-\frac{\theta R}{\sqrt{2}\sqrt{\lambda}\sigma}\right.$$

$$\left. + \frac{1}{\sqrt{m}}\left(\frac{\theta^2 R^2}{\sqrt{2}\lambda\sqrt{\lambda}\sigma^2} + \frac{\theta^4 R^4}{6\sqrt{2}\lambda^2\sqrt{\lambda}\sigma^4}\right)\right\} + O(m^{-1}),$$

$$\mathbb{I}_u(2) = \exp\left(-\frac{1}{2}\frac{R^2}{2\lambda\sigma^2}t^2\right)\left\{1 + \frac{\theta^2 R^2}{2\lambda\sigma^2}\right.$$

$$\left. + \frac{1}{\sqrt{m}}\left(-\frac{2\theta R}{\lambda\sigma} - \frac{7\theta^3 R^3}{6\lambda^2\sigma^3} - \frac{\theta^5 R^5}{12\lambda^3\sigma^5}\right)\right\} + O(m^{-1}).$$

Recalling the results of $\mathbb{I}_z(0)$ and $\mathbb{I}_u(0)$, we can obtain

$$E\left[-\theta\exp(it\cdot t_0(R))\,t_1(R)\right]$$

$$= \exp\left(-\frac{1}{2}t^2\right)\left\{\frac{\theta R}{4\lambda\sigma} - \frac{\theta R^3}{\rho\sigma} + \frac{\theta^3 R^3}{8\lambda^2\sigma^3} - \frac{\theta^3 R^5}{2\rho\lambda\sigma^3} - \frac{\theta^3 R^7}{\rho^2\sigma^3}\right.$$

$$+ \frac{1}{\sqrt{m}}\left(-\frac{\theta^2 R^2}{2\lambda^2\sigma^2} - \frac{7\theta^4 R^4}{24\lambda^3\sigma^4} + \frac{2\theta^4 R^6}{3\rho\lambda^2\sigma^4}\right.$$

$$\left.\left. - \frac{\theta^6 R^6}{48\lambda^4\sigma^6} + \frac{\theta^6 R^8}{12\rho\lambda^3\sigma^6} + \frac{\theta^6 R^{10}}{6\rho^2\lambda^2\sigma^6}\right)\right\} + O(m^{-1}). \qquad (8)$$

Finally, we need to obtain the expectation

$$E\left[-\theta\exp(it\cdot t_0(R))\,t_2(R) + \frac{\theta^2}{2}\exp\left(it\cdot t_0(R)\right)\{t_1(R)\}^2\right]$$

$$= \frac{\theta R^4}{\rho\sqrt{\rho\sigma}}\,\mathbb{I}_z(3)\mathbb{I}_u(0) - \frac{\theta R^3}{\sqrt{2}\rho\sqrt{\lambda\sigma}}\,\mathbb{I}_z(2)\mathbb{I}_u(1) - \frac{\theta R^2}{4\sqrt{\rho\lambda\sigma}}\,\mathbb{I}_z(1)\mathbb{I}_u(2)$$

$$- \frac{\sqrt{2}\theta R}{8\lambda\sqrt{\lambda\sigma}}\,\mathbb{I}_z(0)\mathbb{I}_u(3) + \frac{\theta^2 R^6}{2\rho^2\sigma^2}\,\mathbb{I}_z(4)\mathbb{I}_u(0) - \frac{\theta^2 R^5}{\sqrt{2}\rho\sqrt{\rho\lambda}\sigma^2}\,\mathbb{I}_z(3)\mathbb{I}_u(1)$$

$$+ \frac{\theta^2 R^3}{4\sqrt{2}\sqrt{\rho\lambda}\lambda\sigma^2}\,\mathbb{I}_z(1)\mathbb{I}_u(3) + \frac{\theta^2 R^2}{32\lambda^2\sigma^2}\,\mathbb{I}_z(0)\mathbb{I}_u(4).$$

Substituting $j = 3, 4$ into (5) and (6) and making use of Corollary 3, we have

$$\mathbb{I}_z(3) = \exp\left(-\frac{1}{2}\frac{R^4}{\rho\sigma^2}t^2\right)\left(\frac{3\theta R^2}{\sqrt{\rho\sigma}} + \frac{\theta^3 R^6}{\rho\sqrt{\rho\sigma^3}}\right),$$

$$\mathbb{I}_z(4) = \exp\left(-\frac{1}{2}\frac{R^4}{\rho\sigma^2}t^2\right)\left(3 + \frac{6\theta^2 R^4}{\rho\sigma^2} + \frac{\theta^4 R^8}{\rho^2\sigma^4}\right),$$

$$\mathbb{I}_u(3) = \exp\left(-\frac{1}{2}\frac{R^2}{2\lambda\sigma^2}t^2\right)\left(-\frac{3\theta R}{\sqrt{2}\sqrt{\lambda\sigma}} - \frac{\theta^3 R^3}{2\sqrt{2}\lambda\sqrt{\lambda\sigma^3}}\right) + O(m^{-\frac{1}{2}}),$$

$$\mathbb{I}_u(4) = \exp\left(-\frac{1}{2}\frac{R^2}{2\lambda\sigma^2}t^2\right)\left(3 + \frac{3\theta^2 R^2}{\lambda\sigma^2} + \frac{\theta^4 R^4}{4\lambda^2\sigma^4}\right) + O(m^{-\frac{1}{2}}).$$

Recalling the results of $\mathbb{I}_z(j)$ and $\mathbb{I}_u(j)$ for $j = 0, 1, 2$, we can obtain

$$E\left[-\theta\exp(it\cdot t_0(R))t_2(R) + \frac{\theta^2}{2}\exp\left(it\cdot t_0(R)\right)\{t_1(R)\}^2\right]$$

$$= \exp\left(-\frac{1}{2}t^2\right)\left(\frac{15\theta^2 R^2}{32\lambda^2\sigma^2} + \frac{\theta^2 R^4}{4\rho\lambda\sigma^2} + \frac{9\theta^2 R^6}{2\rho^2\sigma^2} + \frac{5\theta^4 R^4}{32\lambda^3\sigma^4} - \frac{\theta^4 R^6}{2\rho\lambda^2\sigma^4}\right.$$

$$\left. + \frac{2\theta^4 R^8}{\rho^2\lambda\sigma^4} + \frac{4\theta^4 R^{10}}{\rho^3\sigma^4} + \frac{\theta^6 R^6}{128\lambda^4\sigma^6} - \frac{\theta^6 R^8}{16\rho\lambda^3\sigma^6} + \frac{\theta^6 R^{12}}{2\rho^3\lambda\sigma^6} + \frac{\theta^6 R^{14}}{2\rho^4\sigma^6}\right)$$

$$+ O(m^{-\frac{1}{2}}). \tag{9}$$

Combining (7)-(9), we have

$$\exp\left(-\frac{1}{2}t^2\right)\left\{1 + \frac{1}{\sqrt{m}}\left(\frac{\theta R}{4\lambda\sigma} - \frac{\theta R^3}{\rho\sigma} - \frac{\theta^3 R^3}{24\lambda^2\sigma^3} - \frac{\theta^3 R^5}{2\rho\lambda\sigma^3} - \frac{\theta^3 R^7}{\rho^2\sigma^3}\right)\right.$$

$$+ \frac{1}{m}\left(-\frac{\theta^2 R^2}{32\lambda^2\sigma^2} + \frac{\theta^2 R^4}{4\rho\lambda\sigma^2} + \frac{9\theta^2 R^6}{2\rho^2\sigma^2} - \frac{\theta^4 R^4}{96\lambda^3\sigma^4} + \frac{\theta^4 R^6}{6\rho\lambda^2\sigma^4} + \frac{2\theta^4 R^8}{\rho^2\lambda\sigma^4} + \frac{4\theta^4 R^{10}}{\rho^3\sigma^4}\right.$$

$$\left.\left. + \frac{\theta^6 R^6}{1152\lambda^4\sigma^6} + \frac{\theta^6 R^8}{48\rho\lambda^3\sigma^6} + \frac{\theta^6 R^{10}}{6\rho^2\lambda^2\sigma^6} + \frac{\theta^6 R^{12}}{2\rho^3\lambda\sigma^6} + \frac{\theta^6 R^{14}}{2\rho^4\sigma^6}\right)\right\} + O(m^{-\frac{3}{2}}).$$

Inverting the characteristic function, the distribution function can be obtained as

$$\Phi(x) + \frac{1}{\sqrt{m}}\left\{\frac{R}{4\lambda\sigma} - \frac{R^3}{\rho\sigma} - \left(\frac{R^3}{24\lambda^2\sigma^3} + \frac{R^5}{2\rho\lambda\sigma^3} + \frac{R^7}{\rho^2\sigma^3}\right)H_2(x)\right\}\phi(x)$$

$$+ \frac{1}{m}\left\{\left(\frac{R^2}{32\lambda^2\sigma^2} - \frac{R^4}{4\rho\lambda\sigma^2} - \frac{9R^6}{2\rho^2\sigma^2}\right)H_1(x)\right.$$

$$+\left(\frac{R^4}{96\lambda^3\sigma^4} - \frac{R^6}{6\rho\lambda^2\sigma^4} - \frac{2R^8}{\rho^2\lambda\sigma^4} - \frac{4R^{10}}{\rho^3\sigma^4}\right)H_3(x)$$

$$-\left.\left(\frac{R^6}{1152\lambda^4\sigma^6} + \frac{R^8}{48\rho\lambda^3\sigma^6} + \frac{R^{10}}{6\rho^2\lambda^2\sigma^6} + \frac{R^{12}}{2\rho^3\lambda\sigma^6} + \frac{R^{14}}{2\rho^4\sigma^6}\right)H_5(x)\right\}\phi(x)$$

$$+ O(m^{-\frac{3}{2}}),$$

which completes the proof. $\square$

By the last theorem, we have approximation to the upper 100α percentile of $T(R)$, i.e., $t(R, \alpha)$ that satisfies $\Pr[T(R) > t(R, \alpha)] \simeq \alpha$ can be obtained as

$$t(R, \alpha) = z_\alpha + F_1(R, z_\alpha) + F_2(R, z_\alpha),$$

where z_α is the upper 100α percentile of the standard normal distribution,

$$F_1(R, x) = -a_{11}(R) - a_{12}(R)H_2(x),$$

$$F_2(R, x) = -\frac{1}{2}a_{11}(R)\left[a_{11}(R) - 4a_{12}(R)\right]x$$

$$-a_{12}(R)\left[a_{11}(R) - 2a_{12}(R)\right]xH_2(x)$$

$$-\frac{1}{2}\{a_{12}(R)\}^2 x\{H_2(x)\}^2 - a_{21}(R)H_1(x)$$

$$-a_{22}(R)H_3(x) - a_{23}(R)H_5(x).$$

4. Simulation Studies

In order to investigate the performance of the testing procedure based on $T(R)$ and $t(R, \alpha)$ for one-sample problem, we conduct a Monte Carlo simulation with 1,000,000 replications under the selected parameters: $\alpha = 0.10, 0.05,$ $n = 10, 15, 20, 40, R = 0.3, 0.6.$ We primarily evaluate

the empirical size in the following one-sample testing for the coefficient of variation $H_1 : R = R_0$, $H_2 : R \leq R_0$, and $H_3 : R \geq R_0$. All of the simulation results under $R = R_0$ are listed in Tables 4.1-4.3. In this paper, we compare the attained type I error and the nominal type I error denoted by α for the following tests:

- Test 1 ([8]; cf. Theorem 1): The test statistic is

$$T_M(R_0) = \frac{\hat{R} - R_0}{R_0 \sqrt{\dfrac{1}{2(n-1)} + \dfrac{R_0^2}{n-1}}};$$

H_1 is rejected if $|T_M(R_0)| > z_{\frac{\alpha}{2}}$, H_2 is rejected if $T_M(R_0) > z_\alpha$, and H_3 is rejected if $T_M(R_0) < -z_\alpha$.

- Test 2 (Test based on $T(R)$): The test statistic is $T(R_0)$, H_1 is rejected if $|T(R_0)| > z_{\frac{\alpha}{2}}$, H_2 is rejected if $T(R_0) > z_\alpha$, and H_3 is rejected if $T(R_0) < -z_\alpha$

- Test 3 (Test based on $T(R)$ with the critical value obtained by Cornish-Fisher expansion up to the terms of $1/2$-th order): The test statistic is $T(R_0)$, H_1 is rejected if $|T(R_0)| > t^*(R_0, \alpha/2)$, H_2 is rejected if $T(R_0) > t^*(R_0, \alpha)$, and H_3 is rejected if $T(R_0) < t^*(R_0, 1 - \alpha)$, where $t^*(R_0, \alpha) = z_\alpha + F_1(R_0, z_\alpha)$.

- Test 4 (Test based on $T(R)$ with the critical value obtained by Cornish-Fisher expansion up to the terms of 1st order): The test statistic is $T(R_0)$, H_1 is rejected if $|T(R_0)| > t(R_0, \alpha/2)$, H_2 is rejected if $T(R_0) > t(R_0, \alpha)$, and H_3 is rejected if $T(R_0) < t(R_0, 1 - \alpha)$.

For testing H_1, there are several cases such that Tests 1 and 2 are recommended. In particular, Test 2 is a nearly exact test in all the cases for $R_0 = 0.3$. This implies that the approximated percentiles we proposed are not very useful in these cases. On the other hand, for testing H_2 and H_3 the tests based on the modified percentiles derived in this paper, i.e., applying Tests 3 and 4 are obviously recommended. In particular, we observe that Test 4 can almost exactly control type I error.

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Table 4.1. Empirical size of the test for H_1

n	$R_0 = 0.3$ and $\alpha = 0.10$			
	Test 1	Test 2	Test 3	Test 4
10	0.088799	0.096023	0.087964	0.089321
15	0.092960	0.097974	0.090986	0.091797
20	0.094837	0.098067	0.091989	0.092481
40	0.097217	0.099045	0.094441	0.094689
n	$R_0 = 0.6$ and $\alpha = 0.10$			
	Test 1	Test 2	Test 3	Test 4
10	0.083176	0.101864	0.064327	0.057172
15	0.088983	0.101481	0.067334	0.061362
20	0.091760	0.101129	0.069853	0.064910
40	0.095809	0.100495	0.076505	0.073543
n	$R_0 = 0.3$ and $\alpha = 0.05$			
	Test 1	Test 2	Test 3	Test 4
10	0.035804	0.046823	0.035433	0.035755
15	0.042853	0.048058	0.038147	0.038298
20	0.044876	0.048638	0.039828	0.039911
40	0.047581	0.049608	0.043036	0.043054
n	$R_0 = 0.6$ and $\alpha = 0.05$			
	Test 1	Test 2	Test 3	Test 4
10	0.042427	0.058993	0.032642	0.027590
15	0.044780	0.056288	0.031283	0.027172
20	0.045895	0.054776	0.031242	0.027636
40	0.048238	0.052736	0.033449	0.031012

Table 4.2. Empirical size of the test for H_2

	$R_0 = 0.3$ and $\alpha = 0.10$			
n	Test 1	Test 2	Test 3	Test 4
10	0.059291	0.094065	0.098529	0.099800
15	0.066672	0.095718	0.099156	0.099948
20	0.070467	0.095653	0.098684	0.099268
40	0.079379	0.097370	0.099361	0.099629
	$R_0 = 0.6$ and $\alpha = 0.10$			
n	Test 1	Test 2	Test 3	Test 4
10	0.086272	0.119602	0.104625	0.101295
15	0.089378	0.116410	0.102808	0.100319
20	0.090970	0.114373	0.102117	0.100141
40	0.094155	0.110297	0.100897	0.099811
	$R_0 = 0.3$ and $\alpha = 0.05$			
n	Test 1	Test 2	Test 3	Test 4
10	0.030765	0.052833	0.049489	0.050066
15	0.034387	0.052849	0.049719	0.050092
20	0.036585	0.052142	0.049425	0.049647
40	0.040605	0.051766	0.049609	0.049733
	$R_0 = 0.6$ and $\alpha = 0.05$			
n	Test 1	Test 2	Test 3	Test 4
10	0.054028	0.077903	0.056929	0.052126
15	0.054084	0.073017	0.054641	0.051053
20	0.053825	0.069762	0.053378	0.050604
40	0.053519	0.064110	0.051818	0.050254

Table 4.3. Empirical size of the test for H_3

n	$R_0 = 0.3$ and $\alpha = 0.10$			
	Test 1	Test 2	Test 3	Test 4
10	0.136235	0.104466	0.097801	0.099521
15	0.130466	0.103482	0.098891	0.099904
20	0.126678	0.102754	0.099014	0.099718
40	0.119087	0.101731	0.099316	0.099631
n	$R_0 = 0.6$ and $\alpha = 0.10$			
	Test 1	Test 2	Test 3	Test 4
10	0.096378	0.078200	0.104922	0.098339
15	0.099715	0.082293	0.103866	0.099533
20	0.100487	0.084312	0.102722	0.099488
40	0.101434	0.088586	0.101260	0.099630
n	$R_0 = 0.3$ and $\alpha = 0.05$			
	Test 1	Test 2	Test 3	Test 4
10	0.058034	0.043190	0.048243	0.049116
15	0.058573	0.045125	0.049268	0.049734
20	0.058252	0.045925	0.049546	0.049921
40	0.056612	0.047279	0.049794	0.049933
n	$R_0 = 0.6$ and $\alpha = 0.05$			
	Test 1	Test 2	Test 3	Test 4
10	0.029148	0.023961	0.058551	0.046692
15	0.034889	0.028464	0.055862	0.048403
20	0.037935	0.031367	0.054452	0.049027
40	0.042290	0.036385	0.051962	0.049422

5. Conclusion and Discussion

We can approximate the percentiles of the sample coefficient of variation via the normal approximation of the Chi-square variable and Cornish-Fisher expansion. We also present the direction to derive the expectation in order to obtain the main result in this paper.

By the simulation results, in particular, the tests based on the approximate percentiles are recommended for the use of the one-sided test in many cases: it is obviously implied that our approximation performs well when the critical value of the test statistic is not so large. For one-sided test, when the accuracy of the critical value of the test statistic is strongly required, we particularly recommend applying Test 4 to the dataset. On the other hand, when the simplicity of the critical value is favorable, we recommend applying Test 2 or Test 3.

Finally, we address future problems. A similar theoretical result may also be derived on the basis of Wilson and Hilferty's transformation instead of (1). In this case, its effect appears in the terms of the higher order and it may improve our results. In the case that Test 3 is not conservative and Test 4 is conservative, the correction of $t_1(R_0, \alpha)$ and $t_0(R_0, \alpha)$ by the other tuning parameter will be considered in order to obtain more accurate percentiles in a view of the practical application.

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