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WEAKLY Po, T_0-IDENTIFICATION P, THEIR NEGATIONS, AND APPLICATIONS FOR R_0 AND R_1

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Abstract

Within this paper weakly Po properties, T_0-identification P properties, and their negations are further investigated, with the focus on the negations with the results applied to the R_0 and R_1 properties.

1. Introduction and Preliminaries

T_0-Identification spaces were introduced in 1936 [13].

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let

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$Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, Y))$ is the T_0- identification space of (X, T).

Within the 1936 paper [13], T_0-identification spaces were used to further characterize metrizable spaces.

Theorem 1.1. *A space* (X, T) *is pseudometrizable iff* $(X_0, Q(X,$ $Q(X, T))$ *is metrizable. In the* 1975 *paper* [11], T_0*-identification spaces were used to further characterize Hausdorff spaces.*

Theorem 1.2. *A space* (X, T) *is weakly Hausdorff iff* $(X_0, Q(X, T))$ *is Hausdorff* [11]. *Within the* 2015 *paper* [2], *metrizable and Hausdorff were generalized to weakly Po properties.*

Definition 1.2. Let P be topological properties such that $Po = (P \text{ and } T_0)$ exists. Then a space (X, T) is weakly Po iff its T_0- identification space $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property [2].

In the 1936 paper [13], it was shown that for each space, its T_0- identification space has property T_0. Thus, for a topological property P for which Po exists, a space is weakly Po iff its T_0-identification space has property Po.

Within the 2015 paper [2], it was shown that for a weakly Po property Qo, a space is weakly Qo iff its T_0-identification space is weakly Qo, which motivated the introduction and investigation of T_0-identification P properties [3].

Definition 1.3. Let S be a topological property. Then S is a T_0-identification P property iff both a space and its T_0-identification space simultaneously shares property S.

In the introductory weakly Po property paper [2], it was shown that weakly Po is neither T_0 nor "not-T_0", where "not-T_0" is the negation of

T_0. The need and use of "not-T_0" revealed "not-T_0" as a useful topological tool, motivating the investigation of "not-P", where P is a topological property for which "not-P" exists. For the most part, "not-P" had been neglected in past studies in topology. The inclusion of "not-P" in the study of topology has proved to be a productive, fruitful addition providing tools needed to reveal many fundamental, foundational, never before imagined properties, changing the study of topology forever. As an example, the existence of the least of all topological properties was never before even imagined in the study of topology, but in the paper [4], the use of T_0 and "not-T_0" revealed that $L = (T_0$ or "not-T_0"$)$ is the least of all topological properties. As another example, in the paper [5], topological properties P and "not-P", where both P and "not-P" exist, were used to quickly and easily prove there is no strongest topological property revealing yet another foundational, never before known property within the study of topology.

As is often the case, the existence of something never before imagined can create problems as was the case in topology. The knowledge and investigation of the least of all topological properties L revealed needed changes in the definitions of product properties [6] and subspace properties [7] in order to preserve continuity in the study of each of those properties, while, at the same time, revealing many new properties and examples for each of those properties.

In past studies of weakly Po spaces and properties, for a classical topological property Qo, a special topological property W was sought such that if a space (X, T) has property W, then its T_0-identification space $(X_0, Q(X, T))$ has property Qo, which then implies the initial space (X, T) has property W. If past practices were continued, work in weakly Po spaces and properties would be tedious and never ending. Thus, the question of whether there is a shortcut for the weakly Po space and property search process arose, which was resolved in a paper [8].

Answer 1.1. Let Q be a topological property for which both Qo and (Q and "not-T_0") exist. Then Q is a T_0-identification P property that is weakly Po and Q = weakly Qo = (Qo or (Q and "not-T_0")) [8].

Answer 1.2. $\{Q \mid Q$ is a T_0-identification P property$\}$ = $\{Qo \mid Qo$ is *a weakly Po* property$\}$ = $\{Qo \mid Q$ is a topological property and Qo exists$\}$ [8].

Answer 1.3. $\{Q \mid Q$ is a T_0-identification P property$\}$ = $\{Q \mid Q$ is weakly $Po\}$ = $\{Q \mid Q$ is a topological property and both Qo and (Q and "not-T_0") exist$\}$ [8].

Thus, major progress was achieved in the study of weakly Po and related properties. If Q is a topological property for which both Qo and (Q and "not-T_0") exist, Answer 1.1 quickly and easily gives the shortcut. If Q is a topological property for which $Q = Qo$, then $Q = Qo$ is a weakly Po property, but $Q = Qo$ is not a T_0-identification P property or weakly Po. Within the 2017 paper [8], a topological property W that can be both T_0 and "not-T_0" was given that is a T_0-identification P property that is weakly Po such that W = weakly Qo, again making the search process certain, and quick and easy.

Definition 1.4. Let Q be a topological property for which Qo exists. A space (X, T) has property QNO iff (X, T) is "not-T_0" and $(X_0, Q(X, T))$ has property Qo [8].

In the 2017 paper [8], it was shown that QNO exists and is a topological property, and W = (Qo or QNO) is a T_0-identification P property that is weakly Po with W = weakly Qo. Thus, as stated above, for a topological property Q for which Qo exists and $Q = Qo$, there is a certain, quick, and easy answer.

Hence, there is now a certain, quick, and easy solution to the weakly Po space and property questions. However, for topological properties $Q = Qo$, the topological property QNO is utilized in the solution and, even though *QNO* is well-defined, precisely determining it using the definition is challenging, at best, and thus, the question of whether the weakly *Po* problem could somehow be internalized to more easy determine the needed *QNO* property arose leading to an answer in the paper [9].

Definition 1.5. Let (X, T) be a space and for each $x \in X$, let C_x be the T_0-identification class containing x. Then *OXTO* is a subset of X containing exactly one element from each equivalence class C_x.

Within the paper [9], it was shown that for a space (X, T), (*OXTO*, T_{OXTO}) is homeomorphic to $(X_0, Q(X, T))$ leading to the definition of the property *WQ*.

Definition 1.6. Let Q be a topological property for which *Qo* exists. Then a space (X, T) has property *WQ* iff $(OXTO, T_{OXTO})$ has property *Qo*.

Within the 2017 paper [8], it was shown that for a topological property Q such that *Qo* exists, (*Qo* or *QNO*) is a T_0-identification P property, (*Qo* or *QNO*) = weakly (*Qo* or *QNO*)*o* = weakly *Qo*, and (*Qo* or *QNO*)) is a weakly *Po* property, which was used in the paper [9] to show that for a topological property Q for which *Qo* exists and a space (X, T), (X, T) has property *WQ* iff (X, T) has property (*Qo* or *QNO*).

Since both *Qo* and *QNO* are topological properties, then *WQ* is a topological property and *WQ* is the special property for which a space has property *WQ* iff its T_0-identification space has property *Qo* [9].

In the paper [9], the results above were applied to establish that a space (X, T) has property $W(T_1)$ iff $\{Cl(\{x\}) \mid x \in X\}$ is a decomposition of *X*. Below the results above are used to determine $W(T_2)$ and to further investigate weakly *Po* and "not-(weakly *Po*) spaces and properties, with applications to R_0 and R_1.

2. Weakly $(T_2)o$ Spaces and Additional Properties

Let (X, T) have property $W(T_2)$. Then $(X_0, Q(X, T))$ has property T_2, which implies $(OXTO, T_{OXTO})$ has property T_2. Let x and y be elements of X such that $Cl(\{x\}) \neq Cl(\{y\})$. Let u and v be elements of $OXTO$ such that $Cl(\{x\}) = Cl(\{u\})$ and $Cl(\{y\}) = Cl(\{v\})$. Let U and V be disjoint open sets in $OXTO$ such that $u \in U$ and $v \in V$. Let O and W be open sets in X such that $U = O \cap OXTO$ and $V = W \cap OXTO$. Then O and W are disjoint, for suppose not. Let $z \in (O \cap W)$. Let $s \in OXTO$ such that $Cl(\{s\}) = Cl(\{z\})$. Then $s \in (U \cap V)$, which is a contradiction. Thus for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets O and W in X such that $x \in O$ and $y \in W$.

Let (X, T) be a space such that for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets O and W in X such that $x \in O$ and $y \in W$. Let u and v be distinct elements in $OXTO$. Then $Cl(\{u\}) \neq Cl(\{v\})$. Let O and W be disjoint open sets in X such that $u \in O$ and $v \in W$. Then $(O \cap OXTO)$ and $(W \cap OXTO)$ are disjoint open sets in $OXTO$ containing u and v respectively. Hence, $(OXTO, T_{OXTO})$ has property T_2, which implies $(X_0, Q(X, T))$ has property T_2, and (X, T) has property $W(T_2)$.

Therefore, a space (X, T) has property $W(T_2)$ iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets O and W in X such that $x \in O$ and $y \in W$.

In the study of mathematics, for a particular problem, there are often multiple solution techniques that can be used to solve the problem. One of the many beauties of mathematics is regardless which correct solution technique is used, if used correctly, all solutions are equivalent. In the study of weakly *Po* spaces and properties, two solution techniques are given above: (1) the initial seek and search technique and (2) the solution above using the *WQ* property process. It is not required to use multiple solution techniques to solve a problem, but, when multiple solutions are used, it is both required and comforting to obtain equivalent solutions.

In the 1975 paper [11], it was shown that the weakly Hausdorff property is equivalent to the R_1 property, which was introduced in 1961 [1].

Definition 2.1. A space (X, T) is R_1 iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Thus, by the 1975 paper [11], a space is R_1 iff its T_0-identification space is Hausdorff, denoted by T_2, and, as determined above, $R_1 =$ weakly T_2.

In the 1943 paper [12], the R_0 property was introduced.

Definition 2.2. A space is R_0 iff for each closed set C and each $x \notin C$, $(C \cap Cl(\{x\})) = \phi$.

In the 1961 paper [1], it was shown that a space is R_0 iff $\{Cl(\{x\}) \mid x \in X\}$ is a decomposition of X and in the 2015 paper [2], it was shown that R_0 = weakly T_1 agreeing with the result in the earlier paper [9].

Initially the study of weakly *Po* spaces and properties was challenging having to move from a space to its T_0-identification space and then back to the initial space. However, the discovery of the subspace $(OXTO, T_{OXTO})$ for a space (X, T) and the introduction and use of *WQ* as given above reduced the process to only the initial space, while, at the same time providing clarity and understanding, greatly simplifying the process. Thus, additional, significant progress continued to be made in the study of weakly *Po* spaces and properties.

Also, in the 1961 paper [1], it was shown that R_1 implies R_0 and that a space is T_i iff it is R_{i-1} and T_{i-1}; $i = 1, 2$, raising the question of whether R_i, $i = 0, 1$, is the least topological property, which together with T_0, equals T_{i+1}. The investigation of those questions revealed that for a topological property Q which is weakly *Po*, weakly *Qo* is the least topological property that is weakly *Po*, which together with T_0, equals *Qo* and the least of all topological properties, which together with T_0, equals *Qo* is ((weakly *Qo*) or "not-T_0"). Below, these results are combined with the new results given above to give quick and easy answers to the question immediately above.

Corollary 2.1. *Let Q be a topological property for which both Qo and (Q and "not-T_0") exist. Then Q is the least topological property that is weakly Po, which together with T_0, equals Qo and (Q or "not-T_0") is the least of all topological properties, which together with T_0, equals Qo.*

Since for $Q = R_0$ or $Q = R_1$, both *Qo* and (Q and "not-T_0") exist, then in the result above, Q can be replaced by each of R_0 and R_1.

Corollary 2.2. *Let Q be a topological property for which $Q = Qo$. Then W = (Qo or QNO) is the least topological property that is weakly Po, which together with T_0, equals Qo and ((Qo or QNO) or "not-T_0") is the least of all topological properties, which together with T_0, equals Qo.*

With the clarity gained by the use of *WQ* for a topological property Q for which *Qo* exists given above, Corollary 2.2 can be simplified by replacing (*Qo* or *QNO*) by *WQ*, as given below.

Corollary 2.3. *Let Q be a topological property for which $Q = Qo$. Then WQ is the least topological property that is weakly Po, which together with T_0, equals Qo and (WQ or "not-T_0") is the least of all topological properties, which together with T_0, equals Qo.*

Below the results above are used to continue the study of "not-(weakly *Po*)" spaces and properties.

3. "Not-(Weakly *Po*)" with Applications to R_0 and R_1 Properties

Within the paper [9], it was shown that the least topological property L is weakly *Po* and $Lo = T_0$ is a weakly *Po* property. Since the negation of L does not exist, then "not-(weakly T_0)" does not exist. However, if Q is weakly *Po* and not L, then in the paper [10], it was shown that "not-(weakly *Qo*)" exists and is a topological property with "not-(weakly *Qo*)" = weakly ("not-Q")o = "not-Q". Thus, for a topological property Q different from L, Q is a T_0-identification P property and weakly *Po* iff "not-Q" is a T_0-identification P property and weakly *Po*. Combining these results with those above give the following results.

Corollary 3.1. *"not-R_0" = weakly ("not-R_0)o = weakly ("not-T_1")o = "not-(weakly (R_0)o)".*

Corollary 3.2. *"not-R_1" = weakly ("not-R_1)o = weakly ("not-T_2")o = "not-(weakly (R_1)o)".*

Corollary 3.3. *Let Q be a topological property different from L such that both Qo and (Q and "not-T_0") exist and let (X, T) be a space. Then the following are equivalent*: (a) *(X, T) has property "not-Q"*, (b) *$(OXTO, T_{OXTO})$ has property "not-Q"*, (c) *$(OXTO, T_{OXTO})$ has property ("not-Q)o*, (d) *(X, T) is weakly ("not-Q")o*, (e) *$(X_0, Q(X, T))$ has property ("not-Q")*, *and* (f) *$(X_0, Q(X, T))$ has property ("not-Q")o.*

Corollary 3.4. *Let Q be a topological property different from L such that both Qo and (Q and "not-T_0") exist. Then "not-Q" is the least topological property that is weakly Po, which together with T_0, equals ("not-Q")o and (("not-Q") or "not-T_0") is the least of all topological properties, which together with T_0, equal ("not-Q")o.*

As above, for $Q = R_0$ or $Q = R_1$, both Qo and (Q and "not-T_0") exist and in the results above, Q can be replaced by each of R_0 and R_1.

The contrapositive of the 1961 characterizations of T_1 and T_2 [1] give the next results.

Corollary 3.5. *A space is "not-T_1" iff it is ("not-R_0" or "not-T_0").*

Corollary 3.6. *"not-T_1" is the least topological property, which together with T_0, equals ("not-R_0")o.*

Corollary 3.7. *Let (X, T) be a space. Then the following are equivalent*: (a) (X, T) *is "not-T_2"*, (b) (X, T) *is ("not-R_1" or "not-T_1"), and* (c) (X, T) *is ("not-R_1" or "not-T_0").*

Corollary 3.8. *"not-T_2" is the least topological property, which together with T_0, equals ("not-R_1")o.*

Corollary 3.9. *Let Q be a topological property such that $Q = Qo$. Then "not-(Qo or QNO)" is a T_0-identification P property that is weakly Po and "not-(Qo or QNO)" = weakly ("not-Q")o.*

Corollary 3.10. *Let Q be a topological property such that $Q = Qo$. Then "not-WQ" is a T_0-identification P property that is weakly Po and "not-WQ" = weakly ("not-Q")o.*

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