PERFORMANCE RATING OF TRANSMUTED NADARAJAH AND HAGHIGHI EXPONENTIAL DISTRIBUTION: AN ANALYTICAL APPROACH

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Abstract

This work investigates the potential usefulness of the transmuted Nadarajah Haghighi exponential distribution for modelling lifetime data. This distribution can be obtained by using the quadratic rank transmutation map scheme. Various structural properties of the transmuted Nadarajah Haghighi exponential model were investigated including moment generating function, order statistics, moment, mean which represent the average life span of a system, variance, estimation of the parameters using maximum likelihood and the potential usefulness of the transmuted Nadarajah Haghighi exponential model was shown by means of Kevlar 373/epoxy data.

1. Introduction

In this paper, we introduce the transmuted Nadarajah Haghighi exponential distribution which stems from the following idea; Shaw & Buckley [20] considered composite maps of the following two forms: sample transmutation maps (STMs), $y = G^{-1}(F(x))$ and rank transmutation maps (RTMs), $\nu = G(F^{-1}(w))$, where F and G are cumulative distribution functions (CDFs). Gilchrist [9] refers to STMs and RTMs as Q-transformation and P-transformation, respectively. Shaw & Buckley [21] focused on the RTM, which uses as a tool for the construction of new families of non-Gaussian distributions. They used it to modulate a given base distribution for the purposes of modifying the moments, in particular the skew and kurtosis. An attraction of the approach is that if the CDF and inverse CDF (or quantile function (QF)) are tractable for the base distribution, there is a good chance for the transmuted distribution to be so. According to this approach, a random variable X is said to have a transmuted distribution if its cumulative distribution function (cdf) satisfies the following relationship:

$$G(x) = (1 + \lambda)F(x) - \lambda F(x)^2, \qquad |\lambda| \le 1; \tag{1}$$

$$g(x) = f(x)[(1 + \lambda) - 2\lambda F(x)], \tag{2}$$

where $F(x)$ is the cdf of the baseline model, $g(x)$ and $f(x)$ are the corresponding probability density functions (pdf) associated with $G(x)$ and $F(x)$, respectively. This paper investigates the statistical properties of the transmuted Nadarajah Haghighi exponential distribution. Aryal and Tsokos [3] studied the transmuted Weibull distribution to analyze two lifetime data sets. Transmuted Lomax distribution (Ashour and Eltehiwy [2]), transmuted exponentiated Gamma distribution (Hussian [10]), transmuted inverse Rayleigh distribution (Ahmad et al. [2]), transmuted Pareto distribution (Merovci and Puka [18]), transmuted

modified inverse Weibull distribution (Elbatal [6]), transmuted additive Weibull distribution (Elbatal and Aryal [7]), transmuted complementary Weibull geometric distribution (Afify et al. [1]), transmuted inverse exponential (Oguntunde and Adejumo [19]). Khan and King [13, 14] proposed the transmuted modified Weibull and the transmuted generalized inverse Weibull distributions and discussed structural properties with application to reliability data. Khan et al. [12] proposed the transmuted inverse Weibull distribution and discussed various structural properties with application to reliability data. More recently, Khan et al. [15, 16, 17] studied the transmuted generalized exponential, transmuted Weibull, and transmuted generalized Gompertz distributions also, Owoloko et al. [8] examined the performance rating of the transmuted exponential distribution by using QRTM technique which extends the baseline models for modelling lifetime data. Abdul-Moniem and Seham [4, 5] examined the statistical properties of the transmuted Gompertz distribution and also the exponentiated Nadarajah Haghighi exponential distribution.

A random variable X is said to have a Nadarajah and Haghighi exponential distribution (NHED) with shape parameter α and the scale parameter β, if its probability density function (pdf) is in the form:

$$f(x; \alpha, \beta) = \alpha\beta(1 + \alpha x)^{\beta-1} e^{1-(1+\alpha x)^{\beta}}, \qquad x > 0, (\alpha, \beta > 0). \qquad (3)$$

The cumulative distribution function (CDF) and survival function (SF) are:

$$F(x) = 1 - e^{1-(1+\alpha x)^{\beta}}, \qquad x > 0, (\alpha, \beta > 0), \qquad (4)$$

and

$$S(x) = e^{1-(1+\alpha x)^{\beta}}, \ x > 0, (\alpha, \beta > 0). \qquad (5)$$

2. Transmuted Nadarajah and Haghighi
Exponential Distribution

Using (1) and (4), we can define the CDF of TNHED as follows:

$$G(x;\, \alpha,\, \beta,\, \lambda) = \left\{1 - e^{1-(1+\alpha x)^{\beta}}\right\}\left\{1 + \lambda e^{1-(1+\alpha x)^{\beta}}\right\}, \quad x > 0,\, (\alpha,\, \beta \text{ and } \lambda > 0).$$

$$(6)$$

The pdf of TNHED is given as:

$$g(x;\, \alpha,\, \beta,\, \lambda) = \alpha\beta(1 + \alpha x)^{\beta-1} e^{1-(1+\alpha x)^{\beta}}\left[1 - \lambda + 2\lambda e^{1-(1+\alpha x)^{\beta}}\right], \, x > 0,$$

$$(\alpha,\, \beta \text{ and } \lambda > 0). \qquad (7)$$

We can get the pdf for the transmuted exponential (TED), Nadarajah and Haghighi exponential (NHED), and exponential (ED) distributions by taking $\beta = 1$, $\lambda = 1$, and $\alpha = \beta = 1$, respectively. The survival (reliability) function $S(x)$ and the hazard rate function $h(x)$ for TNHED were obtained as follows:

$$S(x;\, \alpha,\, \beta,\, \lambda) = 1 - \left\{1 - e^{1-(1+\alpha x)^{\beta}}\right\}\left\{1 + \lambda e^{1-(1+\alpha x)^{\beta}}\right\}, \, x > 0,$$

$$(\alpha,\, \beta \text{ and } \lambda > 0), \qquad (8)$$

and

$$h(x) = \frac{\alpha\beta(1 + \alpha x)^{\beta-1} e^{1-(1+\alpha x)^{\beta}}\left[1 - \lambda + 2\lambda e^{1-(1+\alpha x)^{\beta}}\right]}{1 - \left\{1 - e^{1-(1+\alpha x)^{\beta}}\right\}\left\{1 + \lambda e^{1-(1+\alpha x)^{\beta}}\right\}}, \, x > 0,$$

$$(\alpha,\, \beta \text{ and } \lambda > 0), \qquad (9)$$

3. Quantile Function, Median and Mode of the
Transmuted Nadarajah Haghighi Exponential Distribution

The quantile function x_q of the TNHED distribution can be obtained as the inverse of Equation (6)

$$x_q = \frac{1}{\alpha}\left[\left\{1 + \ln\left(\frac{(1-\lambda) - \sqrt{(1-\lambda)^2 + 4\lambda}}{2\lambda}\right)\right\}^{\frac{1}{\beta}} - 1\right], \tag{10}$$

and the median can be obtained as

$$x_{0.5} = \frac{1}{\alpha}\left[\left\{1 + \ln\left(\frac{(1-\lambda) - \sqrt{(1-\lambda)^2 + 4\lambda}}{2\lambda}\right)\right\}^{\frac{1}{\beta}} - 1\right]. \tag{11}$$

The lower quartile and upper quartile can also be derived from Equation (11) when $q = 0.25$ and $q = 0.75$, respectively.

The mode of TNHED is the root of the following equation:

$$\alpha(\beta - 1)\left\{1 - (1 + \alpha x)^\beta\right\} = \alpha\beta(1 + \alpha x)^\beta e^{1-(1+\alpha x)^\beta}. \tag{12}$$

The graph of cumulative density, probability density, and the hazard functions were plotted below for various values of $\alpha = a$, $\beta = b$, and $\lambda = b_1$.
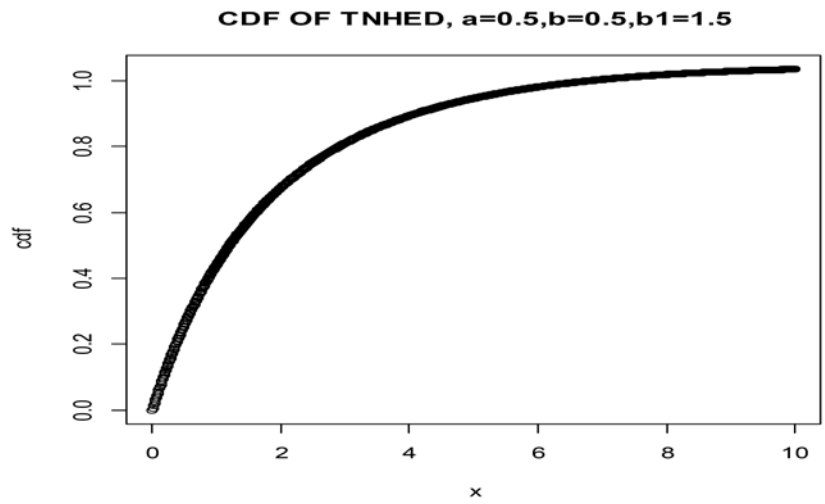

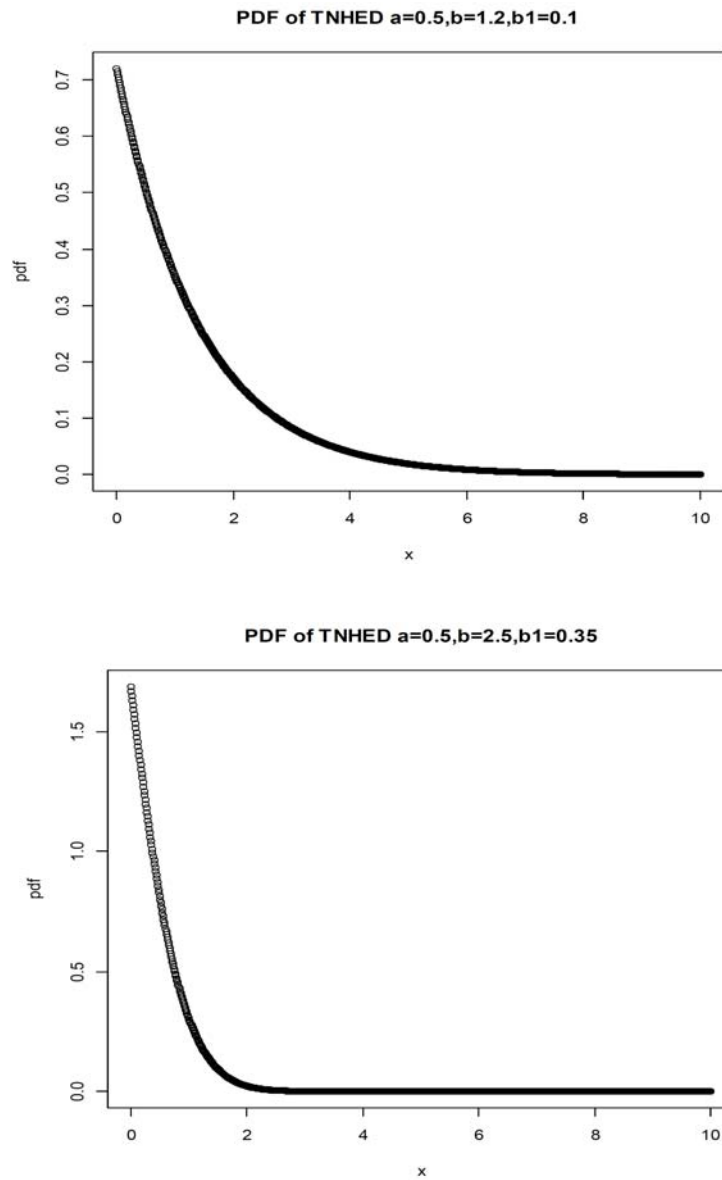**Figure 1.** Plot of the cumulative density function for TNHED.

PDF of TNHED a=0.5,b=1.2,b1=0.1

PDF of TNHED a=0.5,b=2.5,b1=0.35

Figure 2. Plot of the density function for TNHED for various values of the parameters.

Hazard curve for TNHED,a=0.25,b=0.5,b1=0.75

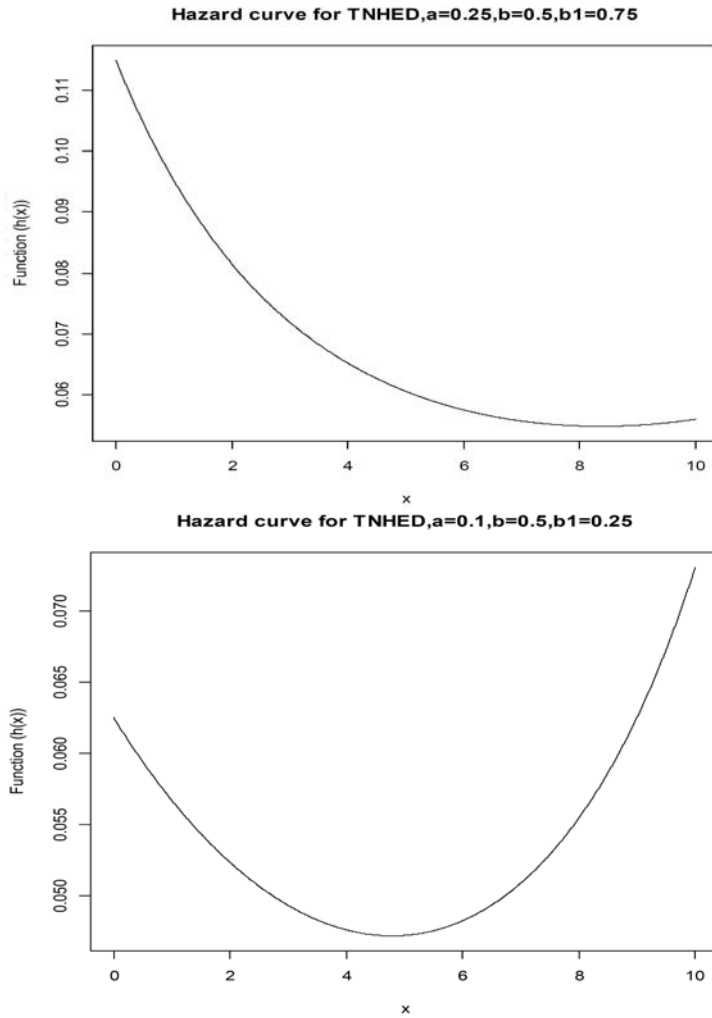**Hazard curve for TNHED,a=0.1,b=0.5,b1=0.25**

Figure 3. Plot of the hazard function for TNHED for various value of the parameters.

From the diagram above Figure 1 depict that the cdf of TNHED converges to one as $n \to \infty$ and that shows that the pdf of TNHED is a proper pdf. Figure 2 demonstrates the flexibility of the distribution in modelling lifetime data. Figure 3 gives a clearer picture of the fact that the distribution is reversible both to the right and to the left and it can be used to model lifetime data even those that possess the bathtub-shape failure rate.

4. Traditional Moments for TNHED

Here we obtain $E(1 + \alpha x)^r$, hence we can obtain the mean and the variance for TNHED as follows:

$$E(1 + \alpha x)^r = \alpha\beta \int_{-\infty}^{\infty} (1 + \alpha x)^r (1 + \alpha x)^{\beta-1} e^{1-(1+\alpha x)^\beta} \left[1 - \lambda + 2\lambda e^{1-(1+\alpha x)^\beta}\right] dx.$$

(13)

Using the transformation, $1 - (1 + \alpha x)^\beta = -y$, this implies that $1 + y = (1 + \alpha x)^\beta$ and $dx = \dfrac{1}{\alpha\beta}(1 + y)^{\frac{1}{\beta}-1} dy$, then Equation (13) we transform to

$$E(1 + \alpha x)^r = \int_{(1+\alpha x)^\beta}^{\infty} (1 + y)^{\frac{r}{\beta}} e^{-y}(1 - \lambda + 2\lambda e^{-y}) dy.$$

(14)

The above equation can be splitted into

$$I_1 = \int_{(1+\alpha x)^\beta}^{\infty} (1 + y)^{\frac{r}{\beta}} e^{-y} dy;$$

(15)

$$I_2 = -\lambda \int_{(1+\alpha x)^\beta}^{\infty} (1 + y)^{\frac{r}{\beta}} e^{-y} dy;$$

(16)

$$I_3 = 2\lambda \int_{(1+\alpha x)^\beta}^{\infty} (1 + y)^{\frac{r}{\beta}} e^{-2y} dy.$$

(17)

Expanding $(1 + y)^{\frac{r}{\beta}}$ using the Taylor's series will yield

$$(1 + y)^{\frac{r}{\beta}} = \sum_{l=0}^{\infty} \binom{\frac{r}{\beta}}{l} y^l.$$

(18)

Then expanding Equations (15), (16) and (17) using Equation (18), we obtain

$$I_4 = \sum_{l=0}^{\infty} \binom{\frac{r}{\beta}}{l} \int_{(1+\alpha x)^{\beta}}^{\infty} y^l e^{-y} dy = \sum_{l=0}^{\infty} \binom{\frac{r}{\beta}}{l} \Gamma\left(l+1, (1+\alpha t_i)^{\beta}\right); \qquad (19)$$

$$I_5 = -\lambda \sum_{l=0}^{\infty} \binom{\frac{r}{\beta}}{l} \int_{(1+\alpha x)^{\beta}}^{\infty} y^l e^{-y} dy = -\lambda \left\{ \sum_{l=0}^{\infty} \binom{\frac{r}{\beta}}{l} \Gamma\left(l+1, (1+\alpha t_i)^{\beta}\right) \right\}; \quad (20)$$

$$I_6 = 2\lambda \sum_{l=0}^{\infty} \binom{\frac{r}{\beta}}{l} \int_{(1+\alpha x)^{\beta}}^{\infty} y^l e^{-2y} dy = 2\lambda \left\{ \sum_{l=0}^{\infty} \binom{\frac{r}{\beta}}{l} \Gamma\left(l+1, 2(1+\alpha t_i)^{\beta}\right) \right\}. \quad (21)$$

Therefore combining Equations (19), (20), and (21), we have

$$E(1+\alpha x)^r = (1-\lambda) \left\{ \sum_{l=0}^{\infty} \binom{\frac{r}{\beta}}{l} \Gamma\left(l+1, (1+\alpha t_i)^{\beta}\right) \right\}$$

$$+ 2\lambda \left\{ \sum_{l=0}^{\infty} \binom{\frac{r}{\beta}}{l} \Gamma\left(l+1, 2(1+\alpha t_i)^{\beta}\right) \right\}. \qquad (22)$$

To obtain the mean of TNHED, we set $r = 1$ in Equation (22), then we have

$$E(1+\alpha x) = (1-\lambda) \left\{ \sum_{l=0}^{\infty} \binom{\frac{1}{\beta}}{l} \Gamma\left(l+1, (1+\alpha t_i)^{\beta}\right) \right\}$$

$$+ 2\lambda \left\{ \sum_{l=0}^{\infty} \binom{\frac{1}{\beta}}{l} \Gamma\left(l+1, 2(1+\alpha t_i)^{\beta}\right) \right\}. \qquad (23)$$

Then we have,

$$E(x) = (1 - \lambda)\left\{\sum_{l=0}^{\infty}\binom{\frac{1}{\beta}}{l}\Gamma\left(l + 1,\ (1 + \alpha t_i)^{\beta}\right)\right\}$$

$$+ 2\lambda\left\{\sum_{l=0}^{\infty}\binom{\frac{1}{\beta}}{l}\Gamma\left(l + 1,\ 2(1 + \alpha t_i)^{\beta}\right)\right\} - \frac{1}{\alpha}. \tag{24}$$

Also setting $r = 2$ in Equation (22)

$$1 + 2\alpha E(X) + \alpha^2 E(X^2) = (1 - \lambda)\left\{\sum_{l=0}^{\infty}\binom{\frac{2}{\beta}}{l}\Gamma\left(l + 1,\ (1 + \alpha t_i)^{\beta}\right)\right\}$$

$$+ 2\lambda\left\{\sum_{l=0}^{\infty}\binom{\frac{2}{\beta}}{l}\Gamma\left(l + 1,\ 2(1 + \alpha t_i)^{\beta}\right)\right\}.$$

From above, we have

$$\alpha^2 E(X^2) = (1 - \lambda)\left\{\sum_{l=0}^{\infty}\binom{\frac{2}{\beta}}{l}\Gamma\left(l + 1,\ (1 + \alpha t_i)^{\beta}\right)\right\}$$

$$+ 2\lambda\left\{\sum_{l=0}^{\infty}\binom{\frac{2}{\beta}}{l}\Gamma\left(l + 1,\ 2(1 + \alpha t_i)^{\beta}\right)\right\} - 2\alpha E(X) - 1.$$

*On simplification we have

$$E(X^2) = \frac{1}{\alpha^2}\left[(1 - \lambda)\left\{\sum_{l=0}^{\infty}\binom{\frac{2}{\beta}}{l}\Gamma\left(l + 1,\ (1 + \alpha t_i)^{\beta}\right)\right\} + 2\lambda\left\{\sum_{l=0}^{\infty}\binom{\frac{2}{\beta}}{l}\Gamma\left(l + 1,\ 2(1 + \alpha t_i)^{\beta}\right)\right\}\right.$$

$$\left. - (1 - \lambda)2\alpha\left\{\sum_{l=0}^{\infty}\binom{\frac{1}{\beta}}{l}\Gamma\left(l + 1,\ (1 + \alpha t_i)^{\beta}\right)\right\} + 2\lambda\left\{\sum_{l=0}^{\infty}\binom{\frac{1}{\beta}}{l}\Gamma\left(l + 1,\ 2(1 + \alpha t_i)^{\beta}\right)\right\} + 1\right].$$

Then the variance of TNHED is given as

$$\text{Var}(X) = \frac{1}{\alpha^2}\left[(1-\lambda)\left\{\sum_{l=0}^{\infty}\binom{\frac{2}{\beta}}{l}\Gamma\left(l+1,\ (1+\alpha t_i)^{\beta}\right)\right\} + 2\lambda\left\{\sum_{l=0}^{\infty}\binom{\frac{2}{\beta}}{l}\Gamma\left(l+1,\ 2(1+\alpha t_i)^{\beta}\right)\right\}\right.$$

$$-(1-\lambda)2\alpha\left\{\sum_{l=0}^{\infty}\binom{\frac{1}{\beta}}{l}\Gamma\left(l+1,\ (1+\alpha t_i)^{\beta}\right)\right\} + 2\lambda\left\{\sum_{l=0}^{\infty}\binom{\frac{1}{\beta}}{l}\Gamma\left(l+1,\ 2(1+\alpha t_i)^{\beta}\right)\right\} + 1\right]$$

$$-\left[(1-\lambda)\left\{\sum_{l=0}^{\infty}\binom{\frac{1}{\beta}}{l}\Gamma\left(l+1,\ (1+\alpha t_i)^{\beta}\right)\right\} + 2\lambda\left\{\sum_{l=0}^{\infty}\binom{\frac{1}{\beta}}{l}\Gamma\left(l+1,\ 2(1+\alpha t_i)^{\beta}\right)\right\} - \frac{1}{\alpha}\right]^2.$$

$$(25)$$

5. Maximum Likelihood Estimators (MLE)

In this section, we consider maximum likelihood estimators (MLE) of TNHED. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from TNHED, then the log-likelihood function $ll(\Theta)$ for $\Theta = (\alpha, \beta, \lambda)^T$ can be written as

$$ll(\Theta) = n\log(\alpha) + n\log(\beta) + (\beta-1)\sum_{i=1}^{n}\log(1+\alpha x_i) + \sum_{i=1}^{n}\log\left[1-(1+\alpha x_i)^{\beta}\right]$$

$$+ \sum_{n=1}^{\infty}\log\left(1-\lambda+2\lambda e^{1-(1+\alpha x_i)^{\beta}}\right). \tag{26}$$

The normal equation become

$$\frac{dl}{d\alpha} = \frac{n}{\alpha} + (\beta-1)\sum_{i=1}^{n}\frac{x_i}{(1+\alpha x_i)}$$

$$+ \beta\sum_{i=1}^{n}x_i - 2\lambda\beta\sum_{i=1}^{n}\frac{x_i(1+\alpha x_i)^{\beta-1}\ln(1+\alpha x_i)e^{1-(1+\alpha x_i)^{\beta}}}{1-\lambda+2\lambda e^{1-(1+\alpha x_i)^{\beta}}}; \tag{27}$$

$$\frac{dl}{d\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log(1+\alpha x_i) + \sum_{i=1}^{n}x_i - 2\lambda\sum_{i=1}^{n}\frac{(1+\alpha x_i)^{\beta}\ln(1+\alpha x_i)e^{1-(1+\alpha x_i)^{\beta}}}{1-\lambda+2\lambda e^{1-(1+\alpha x_i)^{\beta}}};$$

$$(28)$$

$$\frac{dl}{d\lambda} = \sum_{i=1}^{n} \frac{1 - 2e^{1-(1+\alpha x_i)^\beta}}{1 - \lambda + 2\lambda e^{1-(1+\alpha x_i)^\beta}}. \tag{29}$$

The MLE of α, β, and λ can be obtain by solving the Equations (27), (28), and (29), using $\frac{dll}{d\alpha} = 0,\ \frac{dll}{d\beta} = 0,\ \text{and}\ \frac{dll}{d\lambda} = 0.$

6. Application of TNHED

In this section, we use a real data set to show that the TNHED can be a better model than one based on the generalized exponential distribution (GED), transmuted exponential distribution (TED), extended exponential distribution (EED), Nadarajah and Haghighi exponential distribution (NHED), and exponential distribution (ED). We consider a data set of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so we have complete data with the exact times of failure. This data are:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

In order to compare the distributions, we consider some other criterion like − 2LL, AIC (Akaike information criterion), AICC (Akaike information criterion corrected), and BIC (Bayesian information criterion) for the real data set. The best distribution corresponds to lower − 2LL, AIC, AICC, and BIC, where

$$AIC = 2P - 2LL,\ AICC = AIC + \frac{2p(p+1)}{n-p-1},\ \text{and}\ BIC = p\log(n) - 2ll,$$

where p is the number of parameters in the statistical model, n is the

sample size and ll is the maximized value of the likelihood function for the estimated model. Table 1 shows the exploratory data analysis of the data. Table 2 shows parameter MLE for each one of the fitted distributions. Table 3 shows the values of $- 2LL$, AIC, AICC, and BIC values.

Table 1. Summary of data on fatigue fracture of Kevlar 373/epoxy at 90% stress level

Min	Lower quartile	median	Upper quartile	Mean	Max.	Variance	Skewness	Kurtosis	Range
0.0251	0.09048	1.7361	2.2960	1.9590	9.0960	2.4774	1.9406	8.1608	9.0709

Table 2. Estimated parameters of the TNHED, TED, GED, EED, NHED, and ED

	Parametrs estimates			
Models	α	β	λ	$- ll$
TNHED	0.12371	$- 3.5367$	$- 13.0643$	39.435
TED	1.3763	–	$- 0.8487$	121.517
GED	0.703	1.709	–	122.244
EED	0.954	6.366	–	121.650
NHED	0.195	2.007	–	124.738
ED	0.510	–	–	127.114

Table 3. Criteria for comparison

Models	$- 2LL$	AIC	AICC	BIC
TNHED	78.87	72.87	71.37	69.88
TED	243.034	247.033	243.198	246.796
GED	244.49	244.49	248.49	103.97
EED	243.30	247.30	247.47	251.96
NHED	249.48	253.48	253.64	258.14
ED	254.23	256.23	256.28	258.56

7. Conclusion

It will be observed from the values of the parameters estimates and also the values of the criterion for comparison, the model that contains the minimum information loss which corresponds to minimum *ll*, AIC, AICC, and BIC is considered to be the best model in the class of models considered. An application of the transmuted Nadarajah and Haghighi exponential distribution to real data shows that the new distribution provides a better fit than the generalized exponential distribution, transmuted exponential distribution, exponentiated exponential distribution, Nadarajah Haghighi exponential distribution, and exponential distribution.

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