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COMPLETE CHARACTERIZATIONS OF WEAKLY *Po* AND RELATED SPACES AND PROPERTIES

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Abstract

In 1936, T_0-identification spaces were introduced. In 2015, T_0-identification spaces were used to define weakly *Po* spaces and properties and T_0-identification P properties. Within this paper, complete characterizations for each of T_0-identification space properties, weakly *Po* spaces and properties, and T_0-identification P properties are given revealing new basic, never before imagined properties and relationships within topology.

1. Introduction and Preliminaries

T_0-identification spaces were introduced in 1936 [12].

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let

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$Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the natural map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

Within a 1936 paper [12], T_0-identification spaces were used to further characterize metrizable spaces: A space is pseudometrizable iff its T_0-identification space is metrizable.

In the 1975 paper [11], the R_1 property and T_0-identification spaces were used to further characterize the T_2 property.

Definition 1.2. A space (X, T) is R_1 iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$ [1].

Theorem 1.1. *A space is R_1 iff its T_0-identification space is T_2* [11].

Since for any topological property P and any space with property P, its T_0-identification space exists, then there are no restrictions on spaces for which its T_0-identification space exists. Thus attention shifts from properties of spaces (X, T) for which its T_0-identification space $(X_0, Q(X, T))$ exists to the properties of the T_0-identification spaces $(X_0, Q(X, T))$ motivating the definition and work below.

Definition 1.3. A topological property P is a T_0-identification space property iff there exists a space (X, T), whose T_0-identification space has property P.

In the 1936 paper [12], it was shown that T_0-identification spaces satisfy the T_0 separation axiom. Thus, for a topological property to be a T_0-identification space property, (P and T_0), denoted by Po, would have to exist. Within a 2007 paper [2], it was shown that a space is T_0 iff the natural map N from the space onto its T_0-identification space is a homeomorphism. Thus, for each topological property P for which Po exists, Po is a T_0-identification space property.

Within a 1977 paper [3], several topological properties, including R_1, were shown to be simultaneously shared by a space and its T_0-identification space. Thus R_1 is a T_0-identification space property that is not T_0, raising the question of precisely which topological properties are T_0-identification space properties.

In the 2015 paper [4], the characterizations of metrizable and T_2 given above motivated the introduction and investigation of weakly *Po* spaces and properties.

Definition 1.4. Let P be a topological property for which *Po* exists. Then a space (X, T) is weakly *Po* iff its T_0-identification space $(X_0, Q(X, T))$ has property P. A topological property *Qo* for which weakly *Qo* exists is called a weakly *Po* property.

Since the T_0-identification space of each space is T_0, then for a topological property Q for which weakly *Qo* exists, a space (X, T) is weakly *Qo* iff $(X_0, Q(X, T))$ has property *Qo*, and, within $(X_0, Q(X, T))$, Q and *Qo* are equivalent.

In the 2015 paper [4], it was shown that R_1 = weakly$(R_1)o$ = weakly T_2 and pseudometrizable = weakly (pseudometizable)*o* = weakly (metrizable). Hence R_1 and pseudometrizable are weakly *Po*, and T_2 and metrizable are weakly *Po* properties. Also, in the 2015 paper [4], it was shown that for a topological property Q for which weakly *Qo* exists, weakly *Qo* is simultaneously shared by both a space and its T_0-identification space, which when combined with the results above, led to the introduction and investigation of T_0-identification P properties.

Definition 1.5. A topological property S is a T_0-identification P property iff S is simultaneously shared by both a space and its T_0-identification space [5].

In the 2015 paper [5], it was shown that for a T_0-identification P property S, S = weakly So. Combining this result with the result above concerning weakly Po, the T_0-identification P properties and the properties that are weakly Po are exactly the same. Since for a space (X, T) which is only one of T_0 and "not-T_0", $(X_0, Q(X, T))$ is T_0, then neither T_0 nor "not-T_0" are weakly Po and there are restrictions on topological properties for which weakly Po exists.

The study of weakly Po spaces and properties has been a fruitful study revealing the importance of the long-neglected topological property "not-P" in the study of topology, where P is a topological property for which "not-P" exists. Thus far, the addition and use of "not-P" in the study of topology has led to the discovery of the never before imagined least of all topological properties $L = (T_0$ or "not-T_0"$)$ [6] and that there is no strongest topological property [7]. As is expected, the existence of the never before imagined topological property L revealed needed changes for product [8] and subspace properties [9] leading to new, meaningful, never before imagined properties and examples for each of those two properties, expanding and changing the study of topology forever.

Below, each of the T_0-identification space properties, the topological properties that are weakly Po, and the topological properties that are weakly Po properties are completely characterized revealing additional basic, fundamental, foundational, never before imagined properties and relationships in the area of topology.

2. The Complete Characterization of the Weakly *Po* Properties

Let n be a natural number, $n \geq 2$.

Definition 2.1. Let Q be a topological property for which Qo exists. A space (X, T) is $Q(1, n)$ iff there exist n distinct elements $a_1, \cdots, a_n$ all of whose closures are equal, and for all other $x \in X$, $Cl(\{x\}) = Cl(\{y\})$ iff $x = y$, and the T_0-identification space of (X, T) is Qo.

Below the existence of $Q(1, n)$ is established.

Theorem 2.1. *Let* $f : X \to Y$. *Then* f *is onto iff for each* $O \subseteq Y$, $f(f^{-1}(O)) = O$. *The straightforward proof is omitted.*

Theorem 2.2. *There exists a space* (X, T) *with property* $Q(1, n)$.

Proof. Let (Y, S) be a space with property Qo. Let $a_1 \in Y$, let $a_2, \cdots, a_n$ be distinct elements not in Y, let $X = Y \cup \{a_2, \cdots, a_n\}$, let $f : X \to Y$ such that f restricted to Y is the identity function and $f(a_i) = a_1$; $i = 2, \cdots, n$, and let $T = \{f^{-1}(O) \mid O \in S\}$. Then T is a topology on X. Let $f^{-1}(V) \in T$ containing one of $a_1, \cdots, a_n$. Then $a_1 \in V$ and all of $a_1, \cdots, a_n$ are in $f^{-1}(V)$. Thus every open set in X containing one of $a_1, \cdots, a_n$ contains all of them and $Cl(\{a_1\}) = \cdots = Cl(\{a_n\})$ and (X, T) is "not-T_0".

Let $x \in X$ different from each of $a_1, \cdots, a_n$. Suppose there exists a $y \in X$ different from x such that $Cl(\{x\}) = Cl(\{y\})$. Then $f(x)$ and $f(y)$ are distinct elements in Y and since (Y, S) is T_0, there exists an open set U in Y containing only one of $f(x)$ and $f(y)$. Then $f^{-1}(U)$ is open in X containing only one of x and y and $Cl(\{x\}) \neq Cl(\{y\})$, which is a contradiction. Thus in X, $Cl(\{x\}) = Cl(\{y\})$ iff $x = y$.

By definition of T, $f : (X, T) \to (Y, S)$ is continuous. Let $O \in T$. Let $U \in S$ such that $O = f^{-1}(U)$. Then $f(O) = f(f^{-1}(U))$, which, since f is onto, equals U. Hence f is open.

For each $x \in X$, let C_x be the element of X_0 containing x. Then $C_{a_i} = \{a_1, \cdots, a_n\}$; $i = 1, \cdots, n$, and for all other $x \in X$, $C_x = \{x\}$. Let $C_x \in X_0$. If $x = a_i$ for some $i \in \{1, \cdots, n\}$, then $C_x \cap Y = a_1$. If $x \neq a_i$; $i = 1, \cdots, n$, then $C_x \cap Y = \{x\}$. Thus for each $C_x \in X_0$, $C_x \cap Y$ is a singleton set. Let g be the relation in $X_0 \times Y$ defined by $g = \{(C_x, y) \mid C_x \cap Y = \{y\}\}$. If $C_u = C_v$, then $C_u \cap Y = C_v \cap Y$. Hence g is a function. Since for each $x \in Y$, $x \in C_x$, then $g(C_x) = x$ and g is onto. Let $C_u, C_v \in X_0$ such that $g(C_u) = g(C_v)$. If $g(C_u) = a_1$, then $C_u = \{a_1, \cdots, a_n\} = C_v$. If $g(C_u) = x \neq a_1$, then $C_u = C_x = C_v$. Hence g is one-to-one.

Let $C_x \in X_0$. If $C_x = C_{a_1}$, then $f(N^{-1}(C_x)) = f(\{a_1, \cdots, a_n\}) = a_1 = g(C_x)$. If $C_x \neq C_{a_1}$, then $f(N^{-1}(C_x)) = f(\{x\}) = x = g(C_x)$. Hence, for each $C_x \in X_0$, $f(N^{-1}(C_x)) = g(C_x)$. Let $O \in Q(X, T)$. Then $N^{-1}(O) \in T$ and $f(N^{-1}(O)) = g(O) \in S$. Thus g is open. Let $V \in S$. Then $f^{-1}(V) \in T$ and, since N is open [10], then $N(f^{-1}(V)) = g^{-1}(V) \in Q(X, T)$. Hence g is continuous.

Therefore $g : (X_0, Q(X, T)) \to (Y, S)$ is a homeomorphism. Since (Y, S) is Qo and Qo is a topological property, then $(X_0, Q(X, T))$ is Qo. Hence (X, T) has property $Q(1, n)$ and $Q(1, n)$ exists.

Theorem 2.3. *Let (X, T) and (Y, S) be topological spaces and let $f : (X, T) \to (Y, S)$ be a homeomorphism. Then the relation $f^* = \{(C_x, C_{f(x)}) \mid C_x \in X_0\}$ in $X_0 \times Y_0$ is a homeomorphism.*

Proof. Since f is continuous and onto, then f^* is a continuous function from $(X_0, Q(X, T))$ onto $(Y_0, Q(Y, S))$ [10]. Since $f : (X, T) \to (Y, S)$ is open, then $f^* : (X_0, Q(X, T)) \to (Y_0, Q(Y, S))$ is open [10].

Let $C_u, C_v \in X_0$ such that $f^*(C_u) = f^*(C_v)$. Then $C_{f(u)} = C_{f(v)}$. Let $O \in T$ such that $u \in O$. Then $f(O)$ is open in (Y, S), $C_u \subseteq O$, and $C_{f(u)} \subseteq f(O)$. Since $C_{f(u)} = C_{f(v)}$, then $f(v) \in f(O)$ and $v \in O$. Thus every open set in X containing u contains v. Similarly, every open set in X containing v contains u. Thus $Cl(\{u\}) = Cl(\{v\})$ and $C_u = C_v$. Hence f^* is one-to-one and $f^* : (X_0, Q(X, T)) \to (Y_0, Q(Y, S))$ is a homeomorphism.

Theorem 2.4. *Let Q be a topological property for which Qo exists and let n be a natural number*; $n \geq 2$. *Then* $Q(1, n)$ *is a topological property*.

Proof. Let (X, T) have property $Q(1, n)$, let (Y, S) be a space, and let $f : (X, T) \to (Y, S)$ be a homeomorphism. Let $a_1, \cdots, a_n$ be the n distinct elements in X such that $Cl(\{a_1\}) = Cl(\{a_2\}) = \cdots = Cl(\{a_n\})$ and for all other x in X, $Cl(\{x\}) = Cl(\{y\})$ iff $x = y$. Since $f : (X, T) \to (Y, S)$ is a homeomorphism, then $\{f(a_i) \mid i = 1, \cdots, n\}$ is a set of n distinct elements of Y all of whose closures are equal. Let $y \in Y$ different from $f(a_i)$; $i = 1, \cdots, n$, and let $v \in Y$ different from y. Let $x, u \in X$ such that $f(x) = y$ and $f(u) = v$. Then $Cl(\{x\}) \neq Cl(\{u\})$ and there exists an open set U in X containing only one of x and u. Thus $f(U)$ is open in Y containing only one of x and v and $Cl(\{y\}) \neq Cl(\{v\})$.

Since (X, T) has property $Q(1, n)$, then $(X_0, Q(X, T))$ has property Qo and since $f : (X, T) \to (Y, S)$ is a homeomorphism, then, by Theorem 2.3, $(Y_0, Q(Y, S))$ has property Qo. Hence (Y, S) has property $Q(1, n)$. Therefore $Q(1, n)$ is a topological property.

Corollary 2.1. *Let Q be a topological property for which Qo exists and let m and n be natural numbers greater than or equal to* 2 *with $m < n$. Then $Q(1, m)$ and $Q(1, n)$ are topologically distinct topological properties.*

Definition 2.2. Let Q be a topological property such that Qo exists. A space (X, T) has property QNO iff (X, T) is "not-T_0" and $(X_0, Q(X, T))$ has property Qo.

Theorem 2.5. *Let Q be a topological property for which Qo exists. Then QNO is a topological property.*

Proof. Let (X, T) have property QNO, let (Y, S) be a homeomorphic image of (X, T), and let $f : (X, T) \to (Y, S)$ be a homeomorphism. Since (X, T) is "not-T_0" and "not-T_0" is a topological property, then (Y, S) is "not-T_0". Since (X, T) has property QNO, then $(X_0, Q(X, T))$ has property Qo. Since Qo is a topological property, then, by Theorem 2.3, $(Y_0, Q(Y, S))$ has property Qo. Thus QNO is a topological property.

Theorem 2.6. *Let Q be a topological property for which Qo exists. Then there are infinitely many topologically distinct topological properties that are QNO.*

Proof. Let n be a natural number $n \geq 2$ and let (X, T) be a space with property $Q(1, n)$. Then (X, T) has property QNO and QNO exists. Since, by Corollary 2.1, for distinct natural numbers m and n each greater than or equal to 2, $Q(1, m)$ and $Q(1, n)$ are topologically distinct topological properties, there are infinitely many topologically distinct topological properties that are QNO.

Theorem 2.7. *Let Q be a topological property for which Qo exists and let W be a topological property such that for each space with property W its T_0-identification space has property Qo. Then $W = (Qo$ or $QNO)$.*

Proof. Since L is the least topological property, then $W =$ (W and L) = (W and (T_0 or "not-T_0") = (Wo or (W and "not-T_0")). Let (X, T) be a space with property W. If (X, T) is T_0, then $W = Wo$, and, since a space is T_0 iff the natural map $N : (X, T) \to (X_0, Q(X, T))$ is a homeomorphism, then $(X_0, Q(X, T))$ has property Wo, which implies $(X_0, Q(X, T))$ is both Wo and Qo and $Wo = Qo$. Thus consider the case that $W =$ (W and "not-T_0"). Then (X, T) is "not-T_0" and $(X_0, Q(X, T))$ has property Qo, which implies (X, T) has property QNO. Thus $W =$ (Qo or QNO).

Corollary 2.2. *Let Q be a topological property for which Qo exists. Then* (Qo *or* QNO) *is the least of all topological properties W for which the T_0-identification space of each space with property W has property Qo.*

Theorem 2.8. *Let Q be a topological property for which Qo exists and let (X, T) be a space. Then (X, T) has property* (Qo *or* QNO) *iff $(X_0, Q(X, T))$ has property* (Qo *or* QNO).

Proof. Suppose (X, T) has property (Qo or QNO). Then $(X_0, Q(X, T))$ has property Qo, which implies its T_0-identification space has property Qo and $(X_0, Q(X, T))$ has a topological property for which its T_0-identification space is Qo. Thus $(X_0, Q(X, T))$ has property (Qo or QNO).

Conversely, suppose $(X_0, Q(X, T))$ has property (Qo or QNO). Since $(X_0, Q(X, T))$ is T_0, then $(X_0, Q(X, T))$ has property (Qo or QNO)$o = Qo$ and (X, T) has property (Qo or QNO).

Using the fact that the T_0-identification P properties are precisely those topological properties Q that are weakly Qo, gives the next result.

Corollary 2.3. *Let Q be a topological property for which Qo exists. Then (Qo or QNO) is a T_0-identification P property, (Qo or QNO) = weakly (Qo or QNO)o = weakly Qo and (Qo or QNO) is a weakly Po property.*

Corollary 2.4. *Let Q be a topological property for which Qo exists. Then (Qo or QNO) is a topological property for which both T_0 and "not-T_0" exist and (Qo or "not-T_0")o = Qo.*

In the 2016 paper [6], it was proven that for a topological property Q for which weakly Qo exists, weakly Qo = (Qo or(weakly Qo and"not-T_0")), which is used with the results above to get the next result.

Theorem 2.9. *Let Q be a topological property for which both Qo and (Q and "not-T_0") exist. Then Q is a T_0-identification P property, QNO = (Q and "not-T_0"), and Q = weakly Qo = (Qo or (Q and "not-T_0")).*

Proof. Let (X, T) be a space. Since (Qo or QNO) is a T_0-identification P property and weakly Po, then (X, T) has property (Qo or QNO) iff $(X_0, Q(X, T))$ has property (Qo or QNO) iff $(X_0, Q(X, T))$ has property (Qo or QNO)o = Qo iff $(X_0, Q(X, T))$ has property Q iff (Qo or QNO") = Q iff (X, T) has property Q. Thus Q is a T_0-identification P property and Q = weakly Qo. Hence Q = (Qo or (Q and "not-T_0")) = weaklyQo = (Qo or QNO) and QNO = ((Qo or QNO) $\setminus Qo$) = ((Qo or (Q and "not-T_0")) $\setminus$ Qo) = (Q and "not-T_0").

Theorem 2.10. *$\{P \mid P$ is a T_0-identification Q property$\}$ = $\{P \mid P$ is a topological property and weakly Po exists $\}$ = $\{P \mid P$ is a topological property and both Qo and (Q and "not-T_0") exist$\}$.*

Proof. By the results above, $\{P \mid P$ is a T_0-identification Q property$\}$ = $\{P \mid P$ is a topological property and weakly Po exists$\}$.

Let $Q \in \{P \mid P$ is a topological property and both Qo and (Q and "not-T_0") exist$\}$. Then Qo exists and, by Theorem 2.9, $Q \in \{P \mid P$ is a topological property and weakly Qo exists$\}$.

Let $Q \in \{P \mid P$ is a topological property and weakly Po exists$\}$. Since weakly Qo exists, then Qo is a weakly Po property and Qo exists. Then, by Theorem 2.9, $Q \in \{P \mid P$ is a topological property and both Qo and (Q and "not-T_0") exist$\}$.

Hence, $\{P \mid P$ is a T_0-identification Q property$\} = \{P \mid P$ is a topological property and weakly Qo exists$\} = \{P \mid P$ is a topological property and both Qo and (Q and "not-T_0") exist$\}$.

Corollary 2.5. *$\{Q \mid Q$ is a T_0-identification P property$\} = \{Qo \mid Q$ is a topological property and Qo is a weakly Po property$\} = \{Qo \mid Q$ is a topological property and Qo exists$\}$.*

3. The Complete Characterization of T_0-identification Space Properties

Theorem 3.1. *$\{P \mid P$ is a T_0-identification space property$\} = \{Po \mid P$ is a topological property and Po exists$\} = \{P \mid P$ is a T_0-identification Q property$\}$.*

Proof. Let Q be a T_0-identification space property. Let (X, T) be a space for which $(X_0, Q(X, T))$ has property Q. Then $(X_0, Q(X, T))$ has property Qo. Thus Qo exists and $\{P \mid P$ is a T_0-identification space property$\} \subseteq \{Po \mid P$ is a topological property and Po exists$\}$.

Suppose Q is a topological property for which Qo exists. Then, by the results above, Qo is a weakly Po property, which is, by definition, a T_0-identification space property. Thus $\{Po \mid P$ is a topological property and Po exists$\} \subseteq \{P \mid P$ is a T_0-identification space property$\}$.

Thus, by the results above, $\{P \mid P$ is a T_0-identification space property $\} = \{Po \mid P$ is a topological property and Po exists $\} = \{P \mid P$ is a T_0-identification Q property$\}$.

As established above, the introduction and investigation of T_0-identification spaces in 1936 [12] added an important, strong tool for use in the continued investigation and expansion of topology. The 1936 characterization of metrizable [12] and the 1975 characterization of T_2 [11] were important papers that led to the introduction and investigation of weakly Po spaces and properties leading to the discovery and use of never before imagined properties, which now includes the never before imagined reality that the characterizations of metrizable and T_2 are special cases of all topological properties P for which Po exists.

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