Journal of Mathematical Sciences: Advances and Applications
Volume 44, 2017, Pages 111-117
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100121797

UNIQUE, FOUNDATIONAL PROPERTIES OF COMPLETELY REGULAR, NORMAL, AND RELATED PROPERTIES

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Abstract

In this paper, additional unique, foundational roles for each of completely regular, $T_{3\frac{1}{2}}$, normal, normal T_0, normal R_0, and T_4 are established.

1. Introduction and Preliminaries

The normal separation axiom was introduced in 1923 [9].

Definition 1.1. A space (X, T) is normal iff for disjoint closed sets C and D, there exist disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$. A normal T_1 space is denoted by T_4.

In the 1925 paper [10], completely regular and $T_{3\frac{1}{2}}$ were introduced.

2010 Mathematics Subject Classification: 54A05, 54A10, 54D10, 54D15.
Keywords and phrases: weakly *Po* spaces and properties, completely regular, $T_{3\frac{1}{2}}$, normal, T_4.

Received March 14, 2017

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Definition 1.2. A space (X, T) is completely regular iff for each closed set C and each $x \notin C$, there exists a continuous function $f : (X, T) \to ([0, 1], U)$, where U is the usual relative metric on $[0, 1]$, such that $f(x) = 0$ and $f(C) = 1$. A completely regular T_1 space is denoted by $T_{3\frac{1}{2}}$.

The further investigation of $T_{3\frac{1}{2}}$ revealed that the requirement of T_1 in the definition of $T_{3\frac{1}{2}}$ can be replaced by the weaker requirement of T_0 [1], but, not so, for T_4. Thus a space is $T_{3\frac{1}{2}}$ iff it is (completely regular and T_0) giving each of completely regular and T_0 an important role in defining $T_{3\frac{1}{2}}$, but, at the same time raising additional questions about their roles: (1) Are there topological properties other than completely regular and $T_{3\frac{1}{2}}$, which together with T_0, equals $T_{3\frac{1}{2}}$? (2) If there are others, what are they? (3) If there are others, is there a least one? (4) If there are others, is there a strongest one? Completely regular and $T_{3\frac{1}{2}}$ are long-defined, long-investigated, and long-used topological properties, but, as given above, there continues to be unanswered questions concerning each of them.

As stated above, T_1 in the definition of T_4 can not be replaced by the weaker condition T_0. Thus, the question of what property P, if any, can be added to T_0 in order for a space to be T_4 iff it is (normal and P and T_0)? The answer was revealed in the paper [2], where it was proven that the separation axiom R_0 is such a property.

The R_0 separation axiom was introduced in the 1943 paper [7].

Definition 1.3. A space (X, T) is R_0 iff for each closed set C and $x \notin C$, $C \cap Cl(\{x\}) = \phi$.

Theorem 1.1. *A space is* T_4 *iff it is* (*normal* R_0) *and* T_0 [2].

Hence, normal, R_0, and T_0 have an important role in T_4, but precisely what are their roles?

The continued investigation of completely regular and $T_{3\frac{1}{2}}$ revealed that $T_{3\frac{1}{2}}$ is a weakly *Po* property with completely regular = weakly (completely regular)*o* = weakly $T_{3\frac{1}{2}}$ [1]. The continued investigation of normal, normal R_0, and T_4 revealed that normal T_0 and T_4 are a weakly *Po* properties with normal = (weakly (normal)*o* = weakly (normal and T_0) and normal R_0 = weakly (normal R_0)*o* = weakly T_4 [3]. Thus, weakly *Po* spaces and properties and their established properties provided a vehicle for use in addressing the many questions above.

Weakly *Po* spaces and properties were introduced in 2015 [4].

Definition 1.4. Let *P* be a topological property for which *Po* = (*P* and T_0) exists. Then a space is weakly *Po* iff its T_0-identification space has property *P*. A topological property *Po* for which weakly *Po* exists is called a weakly *Po* property.

T_0-identification spaces were introduced in the 1936 paper [8].

Definition 1.5. Let (X, T) be a space, let R be the equivalence relation on *X* defined by *xRy* iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of *X*, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the natural map *N*. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

Below recent discoveries for weakly *Po* spaces and properties are applied to resolve the questions above about completely regular, normal, normal R_0, $T_{3\frac{1}{2}}$, and T_4.

2. Applications of Weakly *Po* Properties for Completely Regular and $T_{3\frac{1}{2}}$

Within the paper [5], it was proven that for a weakly *Po* property *Qo*, the least topological property, which together with T_0, equals *Qo* is ((weakly *Qo*) or "not-T_0"). Applying this result to the weakly *Po* property $T_{3\frac{1}{2}}$ gives the following result.

Corollary 2.1. (*Completely regular or "not-*T_0*"*) *is the least topological property, which together with* T_0, *equals* $T_{3\frac{1}{2}}$.

The continued investigation of weakly *Po* spaces and properties revealed that for a weakly *Po* property *Qo*, *Qo* is the only topological property stronger than weakly *Qo*, which together with T_0, equals *Qo*; the only topological property weaker than ((weakly *Qo*) and "not-T_0"), which together with T_0, equals *Qo*, and implies weakly *Qo* is weakly *Qo*; there are exactly two topological properties, *Qo* and weakly *Qo*, stronger than or equal to weakly *Qo*, which together with T_0, equals *Qo*; and the only topological properties weaker than weakly *Qo*, which together with T_0, equals *Qo* are (weakly *Qo* or "not-T_0") or (weakly *Qo* or F), where F is a topological property that implies "not-T_0" [6]. Applying these results to completely regular and $T_{3\frac{1}{2}}$ gives the following unique, foundational properties for completely regular and $T_{3\frac{1}{2}}$.

Corollary 2.2. $T_{3\frac{1}{2}}$ *is the only topological property stronger than completely regular*, *which together with* T_0, *equals* $T_{3\frac{1}{2}}$.

Corollary 2.3. *Completely regular is the only topological property weaker than* (*completely regular and "not-*T_0*"*), *which together with* T_0, *equals* $T_{3\frac{1}{2}}$ *and implies completely regular.*

Corollary 2.4. *$T_{3\frac{1}{2}}$ and completely regular are the only two topological properties stronger than or equal to completely regular, which together with T_0, equals $T_{3\frac{1}{2}}$.*

Corollary 2.5. *(Completely regular or "not-T_0") and (completely regular or F), where F is a topological property that implies "not-T_0", are the only topological properties weaker than completely regular, which together with T_0, equals $T_{3\frac{1}{2}}$.*

3. Applications for Normal, Normal R_0, Normal T_0, and T_4

Corollary 3.1. *(Normal or "not-T_0") is the least topological property, which together with T_0, equals normal T_0.*

Corollary 3.2. *Normal T_0 is the only topological property stronger than normal, which together with T_0, equals normal T_0.*

Corollary 3.3. *Normal is the only topological property weaker than (normal and "not-T_0"), which together with T_0, equals normal T_0 and implies normal.*

Corollary 3.4. *Normal T_0 and normal are the only two topological properties stronger than or equal to normal, which together with T_0, equals normal T_0.*

Corollary 3.5. *(Normal or "not-T_0") and (normal or F), where F is a topological property that implies "not-T_0", are the only topological properties weaker than normal, which together with T_0, equals normal T_0.*

Corollary 3.6. *((Normal R_0) or "not-T_0") is the least topological property, which together with T_0, equals T_4.*

Corollary 3.7. *T_4 is the only topological property stronger than normal R_0, which together with T_0, equals T_4.*

Corollary 3.8. *Normal R_0 is the only topological property weaker than ((normal R_0) and "not-T_0"), which together with T_0, equals T_4 and implies normal R_0.*

Corollary 3.9. *T_4 and (normal R_0) are the only two topological properties stronger than or equal to (normal R_0), which together with T_0, equals T_4.*

Corollary 3.10. *((Normal R_0) or "not-T_0") and ((normal R_0) or F), where F is a topological property that implies "not-T_0", are the only topological properties weaker than normal R_0, which together with T_0, equals T_4.*

Thus, as established above, the study of weakly *Po* spaces and properties has been a productive study not only raising questions not asked before, but, also providing a vehicle for resolution of those questions.

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