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THE METHOD OF LINES WITH A FINITE VOLUMES APPROACH FOR TRANSIENT CONVECTION-DIFFUSION PROBLEMS

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Abstract

This paper deals with development and analysis of a discretization by the method of lines with a finite volumes approach for a one-dimensional convection-diffusion equation. This discretization leads to an ordinary differential equation (ODE). We use an explicit scheme to solve the obtained ODE. L^{∞} stability of the approximate solution under appropriate CFL conditions is also obtained. Study is done for constant and discontinuous thermal conductivities. Results of numerical experiments using the present approach are reported.

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1. Introduction

We consider the following convection-diffusion problem:

$$\begin{cases} Q_T := \{0 < x < L,\ 0 < t \le T\}, \\ \mu\rho[u_t(x, t) + \phi\nu\, u_x(x, t)] = (\lambda u_x)_x(x, t) + f(x, t), \quad (x, t) \in Q_T, \\ u(0, t) = g_0(t), \quad t \in]0, T], \\ u(L, t) = g_L(t), \quad t \in]0, T], \\ u(x, 0) = u^0(x), \quad x \in]0, L[, \end{cases} \tag{1.1}$$

where u denotes the temperature, u_t the time derivative of u, u_x the derivative with respect to the space variable, μ is the specific heat, ρ is the mass density $(\mu\rho > 0)$, λ is the thermal conductivity, $\phi\,(0 < \phi < 1)$ denotes the porosity (the ratio of liquid volume to the total volume), ν is the velocity $(\nu > 0)$, t is a time, $f(x, t)$ is the capacity of internal heat sources. All thermophysical parameters in (1.1) are assumed to be known. λ is assumed to be constant or discontinuous such that $\underline{\lambda} \le \lambda \le \overline{\lambda}$ with $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}_+^*$ and all of the others parameters are assumed to be constant.

The discontinuity of λ is justified since the conductivities of the different components of the medium may be quite different.

The problem (1.1) can be written in the form:

$$\begin{cases} Q_T := \{0 < x < L,\ 0 < t \le T\}, \\ u_t(x, t) = (\alpha u_x)_x(x, t) - \phi\nu\, u_x(x, t) + \dfrac{f(x, t)}{\mu\rho}, \quad (x, t) \in Q_T, \\ u(0, t) = g_0(t), \quad t \in]0, T], \\ u(L, t) = g_L(t), \quad t \in]0, T], \\ u(x, 0) = u^0(x), \quad x \in]0, L[, \end{cases} \tag{1.2}$$

where $\alpha = \frac{\lambda}{\mu\rho}$. Study of this problem without discontinuities of λ was done for instance in [7] and the authors use the finite difference method. Similar model was also studied in [9, 12]. One finds also an example of convection-diffusion equation in [4] where the authors combine Laplace transform method and homotopy perturbation method to determine analytical solutions.

In this paper, we perfom a finite volume method (see, e.g., [2, 3]) semi-discretization only in space. Hence we get an ODE in time. This approach is also called method of lines (MOL). An explicit scheme is used to solve the obtained ODE and stability criteria are formulated. The MOL has been already used in many problems (see, e.g., [1, 5, 6, 9, 10]) with often a semi-discretization by a finite difference method.

The outline of the remainder of this paper is as follows. In next section, we present the semi-discrete approximation of the problem. Section 3 is devoted to the fully discrete approximation. Finally in Section 4, numerical results using this approach are shown and compared sometimes with the exact solution (EXACT).

2. Semi-Discrete Approximation

By semi-discretization we mean discretization only in space, not in time. We discretize space into N equal size grid cells of size $h = L/N$, and define $x_i = h/2 + ih$, $i = 0, \ldots, N-1$, so that x_i is the center of cell $I_i = (x_{i-1/2}, x_{i+1/2})_{i=0,\ldots,N-1}$ with $x_{-\frac{1}{2}} = 0$, $x_{N-\frac{1}{2}} = L$.

Definition 2.1. Let $\mathcal{T} = (I_i)_{i=0,\ldots,N-1}$, where I_i is given above. $\mathcal{T}$ is called admissible uniform mesh of $(0, L)$.

In a finite volume method, the unknowns approximate the average of the solution over a grid cell. More precisely, we let $u_i(t)$ be the approximation

$$u_i(t) \approx \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,\, t)dx.$$

Integrating (1.2) over the cell I_i and dividing par h we get

$$\frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} u_t(x,\, t)dx = \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} (\alpha u_x)_x(x,\, t)dx - \frac{\phi\nu}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} u_x(x,\, t)dx$$

$$+ \frac{(\mu\rho)^{-1}}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} f(x,\, t)dx. \tag{2.1}$$

2.1. Constant conductivities case

Let us suppose that λ is constant, so α is constant. Hence we have

$$\int_{x_{i-1/2}}^{x_{i+1/2}} (\alpha u_x)_x(x,\, t)dx = \alpha(u_x(x_{i+1/2},\, t) - u_x(x_{i-1/2},\, t)). \tag{2.2}$$

Since the value in the midpoint of the cell is a second order approximation of average, we have for smooth u

$$u_x(x_{i-1/2},\, t) \approx \frac{1}{h}[u(x_i,\, t) - u(x_{i-1},\, t) + O(h^2)],$$

and

$$u_x(x_{i+1/2},\, t) \approx \frac{1}{h}[u(x_{i+1},\, t) - u(x_i,\, t) + O(h^2)]. \tag{2.3}$$

We have also

$$\int_{x_{i-1/2}}^{x_{i+1/2}} u_x(x,\, t)dx = u(x_{i+1/2}) - u(x_{i-1/2}) \approx u(x_i) - u(x_{i-1}). \tag{2.4}$$

From (2.1)-(2.4), we get the following numerical scheme for inner points $1 \le i \le N-2$:

$$\frac{du_i(t)}{dt} = \frac{\alpha}{h^2}[(u_{i+1}(t) - u_i(t)) - (u_i(t) - u_{i-1}(t))] - \frac{\phi\nu}{h}[u_i(t) - u_{i-1}(t)] + (\mu\rho)^{-1} f_i(t).$$

So

$$\frac{du_i}{dt} = \left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2}\right)u_{i-1} - \left(\frac{\phi\nu}{h} + \frac{2\alpha}{h^2}\right)u_i + \frac{\alpha}{h^2}u_{i+1} + (\mu\rho)^{-1}f_i, \text{ for } i = 1, \ldots, N-2. \tag{2.5}$$

To complete the scheme (2.5) we need update formula also for the boundary points $i = 0$ and $i = N-1$. These must be derived by taking the boundary conditions into account. We introduce the ghost cells I_{-1} and I_N which located just outside the domain.

The boundary conditions are used to fill these cells with values u_{-1} and u_N, based on the values u_i in the interior cells. The same update formula (2.5) as before can then be used also for $i = 0$ and $i = N-1$. Let us consider our boundary conditions $u(0, t) = g_0(t)$ and $u(L, t) = g_L(t)$.

Since the center of cells I_0 and I_{N-1} are not on the boundary, we take the average of two cells to approximate the value in between,

$$g_0(t) = u(0, t) \approx \frac{u(x_0, t) + u(x_{-1}, t)}{2} + O(h^2) = \frac{u_0(t) + u_{-1}(t)}{2} + O(h^2),$$

and

$$g_L(t) = u(L, t) \approx \frac{u(x_{N-1}, t) + u(x_N, t)}{2} + O(h^2) = \frac{u_{N-1}(t) + u_N(t)}{2} + O(h^2),$$

leading to the approximations

$$u_{-1}(t) = 2g_0(t) - u_0(t) \text{ and } u_N(t) = 2g_L(t) - u_{N-1}(t).$$

We now insert this into the update (2.5) for $i = 0$ and $i = N-1$, we get

$$\frac{du_0}{dt} = \frac{\alpha}{h^2}u_1 - \left(\frac{\phi\nu}{h} + \frac{2\alpha}{h^2}\right)u_0 + \left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2}\right)u_{-1} + (\mu\rho)^{-1}f_0,$$

$$\frac{du_0}{dt} = -\left(2\frac{\phi\nu}{h} + \frac{3\alpha}{h^2}\right)u_0 + \frac{\alpha}{h^2}u_1 + 2g_0(t)\left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2}\right) + (\mu\rho)^{-1}f_0, \tag{2.6}$$

and

$$\frac{du_{N-1}}{dt} = \frac{\alpha}{h^2} u_N - \left(\frac{\phi\nu}{h} + \frac{2\alpha}{h^2}\right) u_{N-1} + \left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2}\right) u_{N-2} + (\mu\rho)^{-1} f_{N-1},$$

$$\frac{du_{N-1}}{dt} = \left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2}\right) u_{N-2} - \left(\frac{\phi\nu}{h} + \frac{3\alpha}{h^2}\right) u_{N-1} + 2g_L(t)\frac{\alpha}{h^2} + (\mu\rho)^{-1} f_{N-1}. \tag{2.7}$$

We put all the Equations (2.5), (2.6), (2.7) together and write them in matrix form, then we get the linear ODE system:

$$\frac{du(t)}{dt} = Au(t) + F(t), \tag{2.8}$$

where the matrix A is given by

$$\begin{cases} A_{0\,0} = -\left(2\frac{\phi\nu}{h} + \frac{3\alpha}{h^2}\right), \quad A_{0\,1} = \frac{\alpha}{h^2}, \\ A_{i\,i-1} = \left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2}\right), \quad 1 \le i \le N-2, \\ A_{i\,i+1} = \frac{\alpha}{h^2}, \quad 1 \le i \le N-2, \\ A_{i\,i} = -\left(\frac{\phi\nu}{h} + \frac{2\alpha}{h^2}\right), \quad 1 \le i \le N-2, \\ A_{N-1\,N-2} = \left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2}\right), \quad A_{N-1\,N-1} = -\left(\frac{\phi\nu}{h} + \frac{3\alpha}{h^2}\right), \end{cases} \tag{2.9}$$

$$\frac{du(t)}{dt} = \begin{bmatrix} \frac{du_0(t)}{dt} \\ \frac{du_1(t)}{dt} \\ \cdot \\ \cdot \\ \cdot \\ \frac{du_{N-2}(t)}{dt} \\ \frac{du_{N-1}(t)}{dt} \end{bmatrix} \text{ and } F(t) = (\mu\rho)^{-1} \begin{bmatrix} f_0(t) \\ f_1(t) \\ \cdot \\ \cdot \\ \cdot \\ f_{N-2}(t) \\ f_{N-1}(t) \end{bmatrix} + \begin{bmatrix} 2g_0(t)\left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2}\right) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 2g_L(t)\frac{\alpha}{h^2} \end{bmatrix}.$$

Here, in this semi-discretization the time-dependent partial differential equation has been approximated by a system of ODEs.

2.2. Piecewise constant conductivities case

Let us suppose that the coefficient λ is discontinuous (more precisely piecewise constant). So α is discontinuous. Let $\mathcal{T} = (I_i)_{i=0,\dots,N-1}$ be an admissible uniform mesh of $(0, L)$ in the sense of Definition 2.1 such that the discontinuities of α coincide with the interfaces of the mesh. First integral in the right-hand term of (2.1) becomes

$$\int_{x_{i-1/2}}^{x_{i+1/2}} (-\alpha u_x)_x dx = (\alpha u_x)(x_{i-1/2}, t) - (\alpha u_x)(x_{i+1/2}, t). \tag{2.10}$$

In order that the scheme be conservative, the discretization of the flux $-\alpha u_x$ at $x_{i+\frac{1}{2}}$ should have the same value on I_i and I_{i+1}. To this purpose, one introduces as in [2, 3, 8], the auxiliary unknown $u_{i+\frac{1}{2}}$ (approximation of u at $x_{i+\frac{1}{2}}$). Since on I_i and I_{i+1}, α is constant, the approximation of $-\alpha u_x$ may be perform on each side of $x_{i+\frac{1}{2}}$ by using the finite difference principle:

$$F_{1+\frac{1}{2}}(t) = -\alpha u_x(x_{1+\frac{1}{2}}) \approx -\alpha_i \frac{u_{i+\frac{1}{2}}(t) - u_i(t)}{\frac{h}{2}} \text{ on } I_i,$$

$$F_{1+\frac{1}{2}}(t) = -\alpha u_x(x_{1+\frac{1}{2}}) \approx -\alpha_{i+1} \frac{u_{i+1}(t) - u_{i+\frac{1}{2}}(t)}{\frac{h}{2}} \text{ on } I_{i+1}, \tag{2.11}$$

where α_i (respectively, α_{i+1}) is the value of α on I_i (respectively, on I_{i+1}). Requiring the two above approximation of $(-\alpha u_x)\left(x_{i+\frac{1}{2}}, t\right)$ to be equal (conservativity of the flux) yields the value of $u_{1+\frac{1}{2}}(t)$ (for $i = 0, \dots, N-1$) :

$$u_{1+\frac{1}{2}}(t) = \frac{(-\alpha_{i+1}u_{i+1}(t) + \alpha_i u_i(t))}{\alpha_{i+1} + \alpha_i},$$

which, in turn, allows to give expression of the approximation $F_{1+\frac{1}{2}}(t)$ of $(-\alpha u_x)\left(x_{i+\frac{1}{2}}, t\right)$:

$$F_{1+\frac{1}{2}}(t) = -\tau_{i+\frac{1}{2}}(u_{i+1}(t) - u_i(t)), \tag{2.12}$$

with

$$\tau_{i+\frac{1}{2}} = \frac{2\alpha_i\alpha_{i+1}}{h(\alpha_{i+1} + \alpha_i)}, \quad i = 0, \dots, N-1. \tag{2.13}$$

From (2.1), (2.10), (2.12), and (2.13), we get the following numerical scheme for inner points $1 \le i \le N-2$:

$$\frac{du_i(t)}{dt} = \frac{1}{h}\left[\tau_{i+\frac{1}{2}}(u_{i+1}(t) - u_i(t)) - \tau_{i-\frac{1}{2}}(u_i(t) - u_{i-1}(t))\right] - \frac{\phi\nu}{h}[u_i(t) - u_{i-1}(t)] + (\mu\rho)^{-1}f_i(t).$$

So

$$\frac{du_i}{dt} = \frac{1}{h}\left[\left(\phi\nu + \tau_{i-\frac{1}{2}}\right)u_{i-1} - \left(\phi\nu + \tau_{i-\frac{1}{2}} + \tau_{i+\frac{1}{2}}\right)u_i + \left(\tau_{i+\frac{1}{2}}\right)u_{i+1}\right] + (\mu\rho)^{-1}f_i,$$

$$\text{for } i = 1, \dots, N-2. \tag{2.14}$$

Using the ghost cells I_{-1} and I_N and the limit conditions as the previous section, one gets:

$$\frac{du_0}{dt} = \frac{1}{h}\left[\left(\phi\nu + \tau_{-\frac{1}{2}}\right)u_{-1} - \left(\phi\nu + \tau_{-\frac{1}{2}} + \tau_{\frac{1}{2}}\right)u_0 + \left(\tau_{\frac{1}{2}}\right)u_1\right] + (\mu\rho)^{-1}f_0,$$

$$\frac{du_0}{dt} = \frac{1}{h}\left[-\left(2\phi\nu + 2\tau_{-\frac{1}{2}} + \tau_{\frac{1}{2}}\right)u_0 + \left(\tau_{\frac{1}{2}}\right)u_1\right] + \frac{2g_0(t)}{h}\left(\phi\nu + \tau_{-\frac{1}{2}}\right) + (\mu\rho)^{-1}f_0,$$

$$\frac{du_{N-1}}{dt} = \frac{1}{h}\left[\left(\phi\nu + \tau_{N-\frac{3}{2}}\right)u_{N-2} - \left(\phi\nu + \tau_{N-\frac{3}{2}} + 2\tau_{N-\frac{1}{2}}\right)u_{N-1}\right] + \frac{2g_L(t)\tau_{N-\frac{1}{2}}}{h} + (\mu\rho)^{-1} f_{N-1}.$$

As in the previous section one gets the linear ODE system (2.8), where the matrix A is given by

$$\begin{cases} A_{0\,0} = -\frac{1}{h}\left(2\phi\nu + 2\tau_{-\frac{1}{2}} + \tau_{\frac{1}{2}}\right), \quad A_{0\,1} = \frac{\tau_{\frac{1}{2}}}{h}, \\ A_{i\,i-1} = \frac{1}{h}\left(\phi\nu + \tau_{i-\frac{1}{2}}\right), \quad 1 \le i \le N-2, \\ A_{i\,i+1} = \frac{\tau_{i+\frac{1}{2}}}{h}, \quad 1 \le i \le N-2, \\ A_{i\,i} = -\frac{1}{h}\left(\phi\nu + \tau_{i-\frac{1}{2}} + \tau_{i+\frac{1}{2}}\right), \quad 1 \le i \le N-2, \\ A_{N-1\,N-2} = \frac{1}{h}\left(\phi\nu + \tau_{N-\frac{3}{2}}\right), \quad A_{N-1\,N-1} = -\frac{1}{h}\left(\phi\nu + \tau_{N-\frac{3}{2}} + 2\tau_{N-\frac{1}{2}}\right), \end{cases} \tag{2.15}$$

and

$$F(t) = (\mu\rho)^{-1}\begin{bmatrix} f_0(t) \\ f_1(t) \\ \cdot \\ \cdot \\ \cdot \\ f_{N-2}(t) \\ f_{N-1}(t) \end{bmatrix} + \begin{bmatrix} \frac{2g_0(t)}{h}\left(\phi\nu + \tau_{-\frac{1}{2}}\right) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \frac{2g_L(t)\tau_{N-\frac{1}{2}}}{h} \end{bmatrix}.$$

3. Fully Discrete Approximation

The system (2.8) can be solved by the forward Euler method. Thus, let $\Delta t = \frac{T}{M} (M > 0)$ be a time step, one has:

$$U^{n+1} = U^n + \Delta t\,[AU^n + F(t_n)],\; U^n \approx U(t_n),\; t_n = n\Delta t,\; n = 0, \ldots, M, \tag{3.1}$$

where

$$U(t) = \begin{bmatrix} u_0(t) \\ u_1(t) \\ \cdot \\ \cdot \\ \cdot \\ u_{N-2}(t) \\ u_{N-1}(t) \end{bmatrix}.$$

This is the method we used, others methods can be used. As for any ODE method, there arises the question of stability whose the answer is given by the following propositions.

Proposition 3.1. *Under the CFL condition*:

$$CFL_1 := \Delta t\left(\frac{\phi\nu}{h} + \frac{2\alpha}{h^2}\right) \leq 1,$$

the explicit scheme (3.1), *where the matrix A is given by* (2.9) *is* L^∞ *stable.*

Proof. We write scheme (3.1) in the following form:

$$u_i^{n+1} = \Delta t\left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2}\right)u_{i-1}^n + \left[1 - \Delta t\left(\frac{\phi\nu}{h} + \frac{2\alpha}{h^2}\right)\right]u_i^n + \Delta t\,\frac{\alpha}{h^2}\,u_{i+1}^n + \Delta t(\mu\rho)^{-1} f_i^n. \tag{3.2}$$

The key point of the proof is that, under the condition CFL_1, the right-hand side of the Equation (3.2) is a convex combination of u_{i-1}^n, u_i^n, and u_{i+1}^n, plus the term $\Delta t(\mu\rho)^{-1} f_i$. From (3.2), one can deduce

$$|u_i^{n+1}| \le \Delta t \left| \left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2} \right) u_{i-1}^n \right| + \left| \left[1 - \Delta t \left(\frac{\phi\nu}{h} + \frac{2\alpha}{h^2} \right) \right] u_i^n \right| + \Delta t \left| \frac{\alpha}{h^2} u_{i+1}^n \right|$$

$$+ \Delta t |(\mu\rho)^{-1} f_i^n|, \quad i = 1, \dots, N-2.$$

Using (2.6), (2.7) and the fact that $u(0, t) = g_0(t)$ and $u(L, t) = g_L(t)$, we get the previous inequality for $i = 0$ and $i = N - 1$. One deduces

$$|u_i^{n+1}| \le \left(\Delta t \left(\frac{\phi\nu}{h} + \frac{\alpha}{h^2} \right) + 1 - \Delta t \left(\frac{\phi\nu}{h} + \frac{2\alpha}{h^2} \right) + \Delta t \frac{\alpha}{h^2} \right) \| U^n \|_\infty$$

$$+ \Delta t (\mu\rho)^{-1} \| F^n \|_\infty, \quad i = 0, \dots, N-1.$$

So

$$\| U^{n+1} \|_\infty \le \| U^n \|_\infty + \Delta t (\mu\rho)^{-1} \max_{0 \le k \le M} \| f^k \|.$$

By obvious recurrence, we get

$$\| U^n \|_\infty \le \| U^0 \|_\infty + n\Delta t (\mu\rho)^{-1} \max_{0 \le k \le M} \| f^k \| \le \| U^0 \|_\infty + T(\mu\rho)^{-1} \max_{0 \le k \le M} \| f^k \|,$$

$$\forall n \in \{0, 1, \dots, M\}.$$

Hence the L^∞ stability is proven. □

Proposition 3.2. *Let* $C = \max_{0<i<N-1} \left(\frac{2\alpha_i \alpha_{i+1}}{\alpha_i + \alpha_{i+1}} \right)$ *and* $\mathcal{T} = (I_i)_{i=0,\dots,N-1}$ *be an admissible mesh of* $(0, L)$ *in the sense of Definition* 2.1 *such that, the discontinuities of* λ *coincide with the interfaces of the mesh. Then under the CFL condition*:

$$CFL_2 := \Delta t\left(\frac{\phi\nu}{h} + \frac{2C}{h^2}\right) \leq 1,$$

the explicit scheme (3.1), *where the matrix A is given by* (2.15) *is* L^∞ *stable.*

Proof. Proof is done by a simular way as Proposition 3.1. □

4. Numerical Simulations

In this section, we present numerical results. First test series was done with following data: $f(x, t) = (\lambda - \mu\rho)\cos(x)\sin(t) - \mu\rho\phi\nu\sin(x)\cos(t)$, $g_0(t) = \cos(t)$, $g_L(t) = \cos(L)\cos(t)$, $u^0(x) = \cos(x)$. So the exact solution is $u(x, t) = \cos(x)\cos(t)$. Figure 1 shows that the numerical solution (*NS*) gives a good approximation of the exact solution. Figure 2 gives the temperature distribution after 15, 25, and 35 seconds.

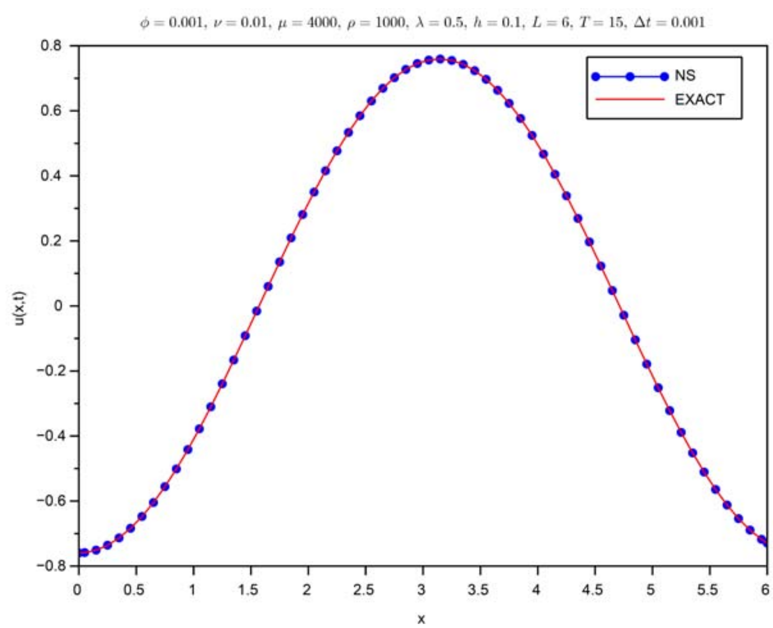

Figure 1. Test problem 1.

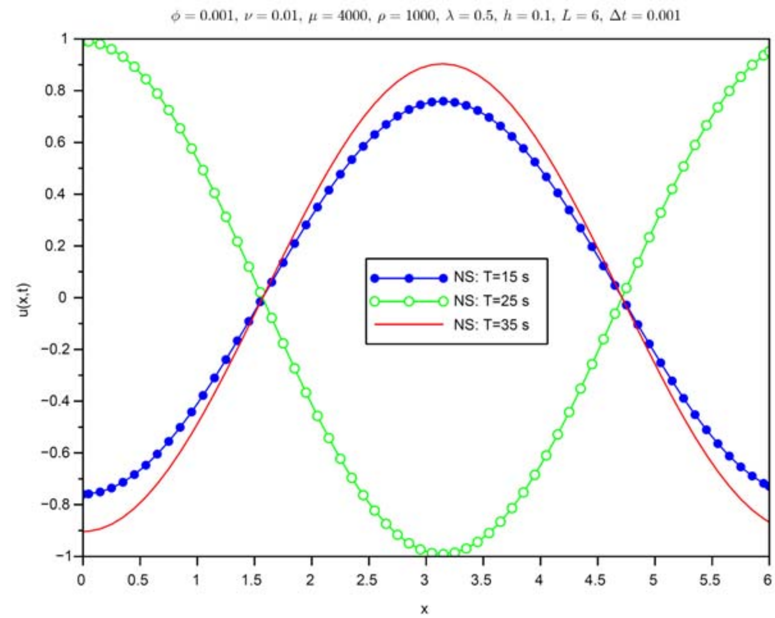

Figure 2. Test problem 1.

Second test series was done with following data: $L = 0.05$m, the thermal conductivity $\lambda = 0.5$W/(mK), the specific heat $\mu = 4000$J/(kgK), the density $\rho = 1000\text{k}\dot{\text{g}}/\text{m}^3$, $u^0(x) = 30$°C, $g_0(t) = 30$°C, $g_L(t) = 30$°C, and

$$f(x, t) = \begin{cases} 10^7, & \text{if } 0.02 \le x \le 0.03 \text{ and } t \le 10, \\ 0, & \text{otherwise.} \end{cases}$$

Mesh step is equal to $h = 0.001$m, time step: $\Delta t = 0.01$s.

Figures 3 & 4 give the temperature distribution after 15, 25, 35, and 45 seconds. Porosity is the same in both cases but the velocities are different. We change the porosity but conserve the same velocity like in Figure 4 to get Figure 5. Figure 6 gives the temperature distribution for different velocities with the same porosity.

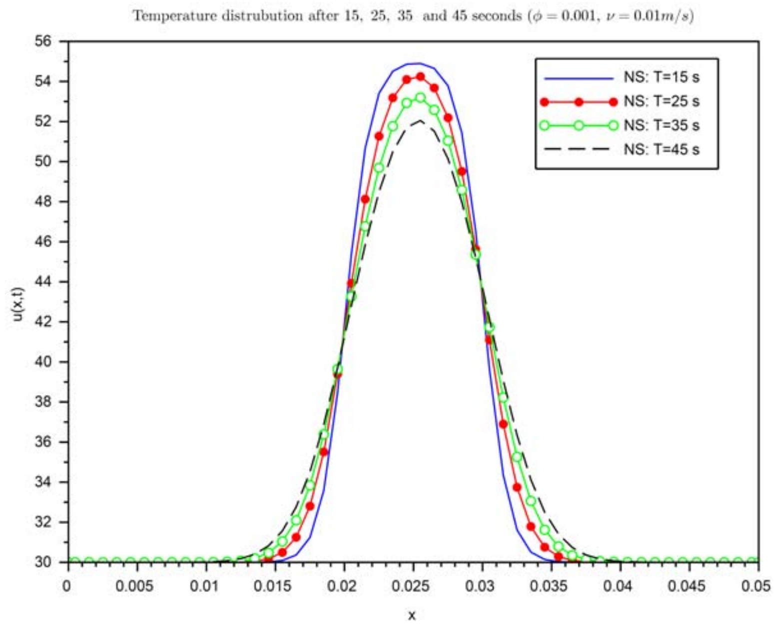

Figure 3. Test problem 2.

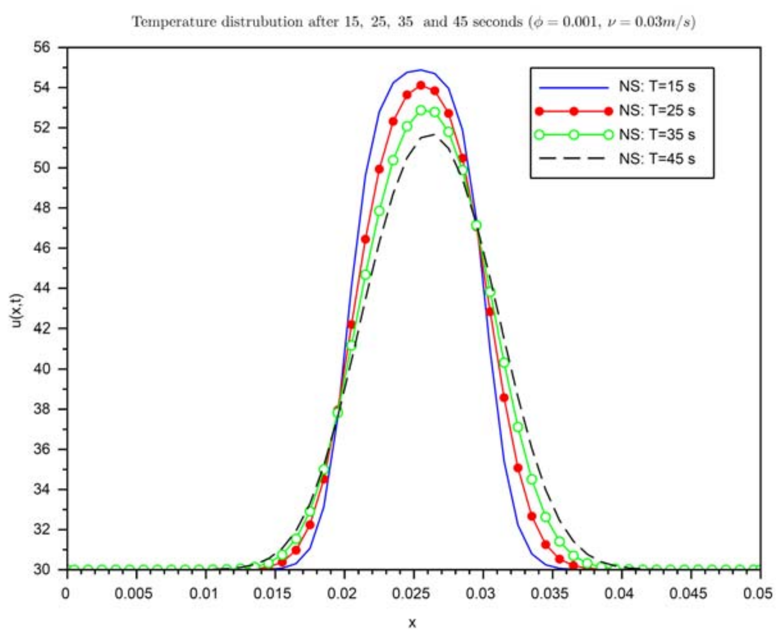

Figure 4. Test problem 2.

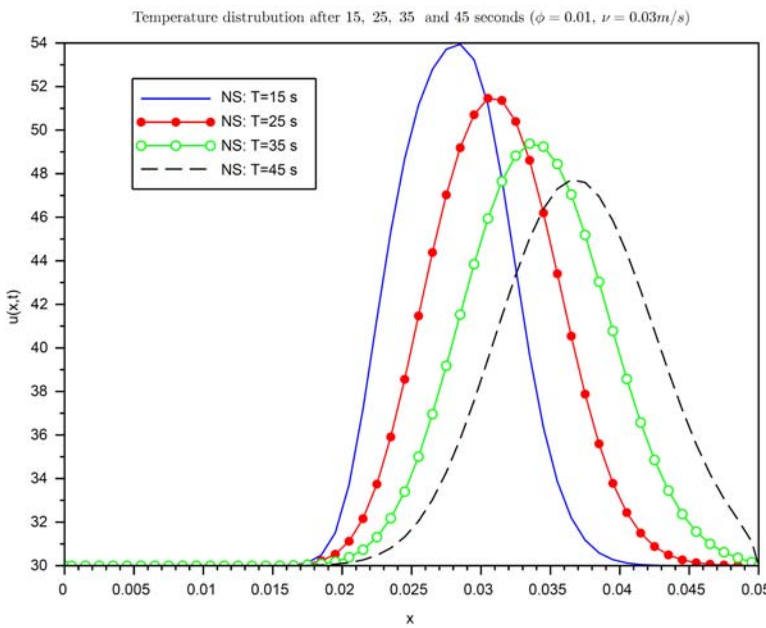

Figure 5. Test problem 2.

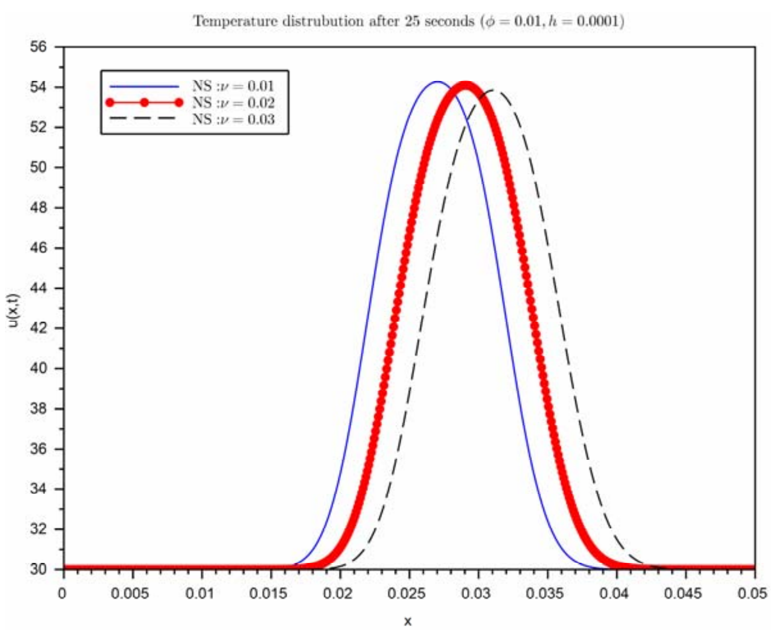

Figure 6. Test problem 2.

Third test series was done with following data: $L = 10\text{m}$, the thermal conductivity specific heat $\mu = 4000\text{J}/(\text{kgK})$, density $\rho = 1000\text{kg}/\text{m}^3$, $T = 15$, $g_0(t) = \cos(t)$, $g_L(t) = \cos(L)\cos(t)$, $u^0(x) = \cos(x)$. So the exact solution is $u(x, t) = \cos(x)\cos(t)$.

$$\lambda = \begin{cases} 10, & \text{if } 0 \le x \le 2.5, \\ 50, & \text{if } 2.5 \le x \le 5, \\ 10, & \text{if } 5 \le x \le 7.5, \\ 50, & \text{if } 7.5 \le x \le 10. \end{cases}$$

One has again a good approximation of the exact solution by the numerical solution with a discontinuous thermal conductivity (see Figure 7).

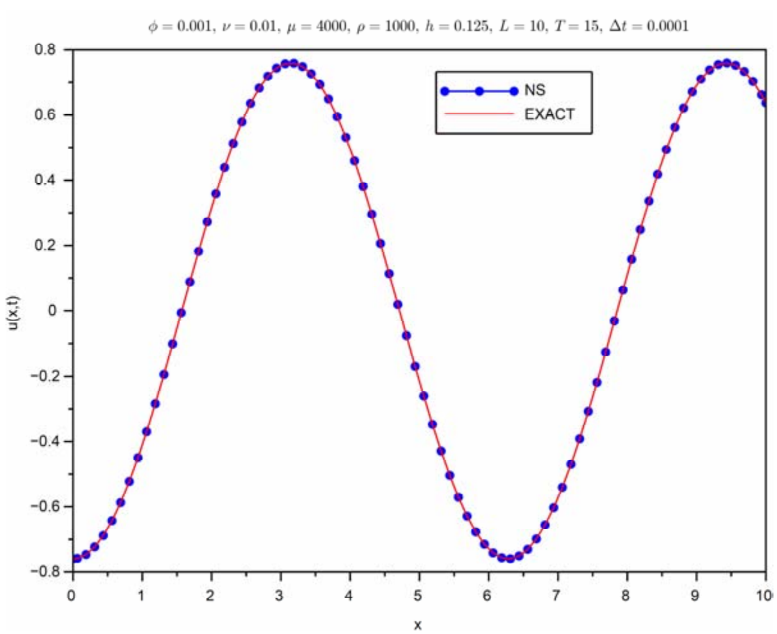

Figure 7. Test problem 3.

Last test series was done with following data: $L = 0.05$m, the thermal conductivity $\lambda = 0.5$W/(mK), the specific heat $\mu = 4000$J/(kgK), the density $\rho = 1000\text{kg/m}^3$, $u^0(x) = 30^{\circ}$C, $g_0(t) = 30^{\circ}$C, $g_L(t) = 30^{\circ}$C,

$$f(x, t) = \begin{cases} 10^7, & \text{if } 0.025 \le x \le 0.0375 \text{ and } t \le 10, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\lambda = \begin{cases} 10, & \text{if } 0.025 \le x \le 0.0375, \\ 1.5, & \text{otherwise.} \end{cases}$$

Mesh step is equal to $h = 0.00125$m, time step: $\Delta t = 0.001$s. Figure 8 gives the temperature distribution after 10, 30, and 50 seconds and Figure 9 gives the temperature distribution after 15, 25, and 45 seconds. Porosity and the velocity are the same in both cases.

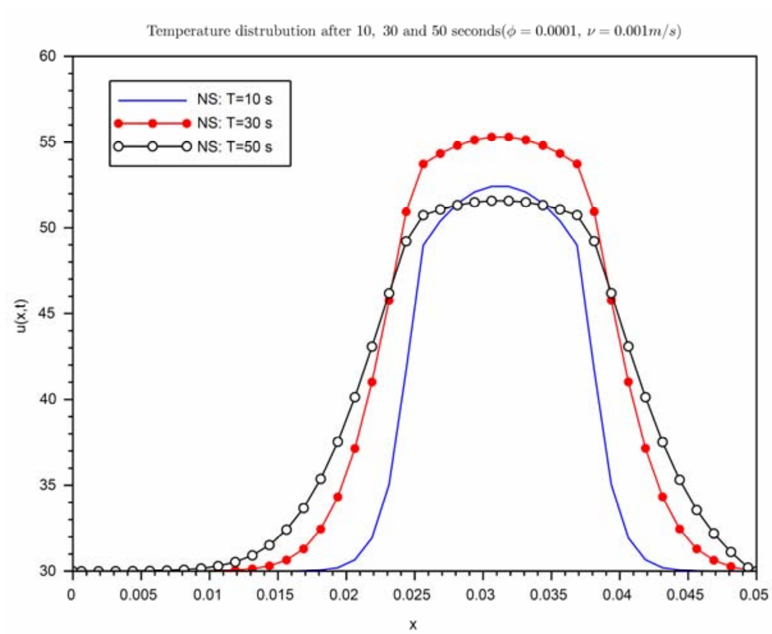

Figure 8. Test problem 4.

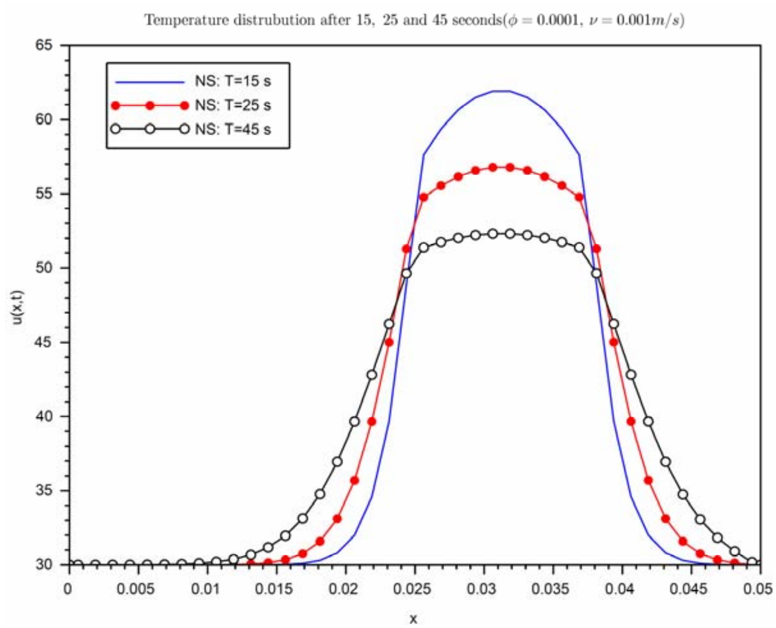

Figure 9. Test problem 4.

5. Concluding Remarks

The purpose of this paper was to apply the method of lines with a finite volumes approach for a one-dimensional convection-diffusion equation. The study was done for both constant and discontinuous conductivities. After proving that the schemes are L^{∞} stable under CFL conditions, we presented numerical resultats indicate that this approach is well adapted to the discretization of this problem. The extension of the present technique to the two-dimensional problem with uniform rectangular grids is straightforward.

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