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INFINITELY MANY TOPOLOGICAL PROPERTIES IN WHICH T_0, T_1, T_2, URYSOHN, T_3, AND $T_{3\frac{1}{2}}$ ARE EQUIVALENT AND INFINITELY MANY NEW CHARACTERIZATIONS OF THE $T_{3\frac{1}{2}}$ PROPERTY

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Abstract

The ((completely regular) and T_1) topological property, commonly denoted by $T_{3\frac{1}{2}}$, is a long-known, long-studied, useful separation axiom. It is well-known that $T_{3\frac{1}{2}}$ implies T_3, which implies Urysohn, which implies T_2, which implies T_1, which implies T_0 and examples are known showing the implications are not reversible. Thus questions concerning topological properties for which the six separation axioms are equivalent arise. In this paper, a new category of topological properties is introduced and used to give infinitely many topological properties for which the six separation axioms are equivalent, and earlier results from the study of weakly *Po* spaces and properties are used to give infinitely many new characterizations of the $T_{3\frac{1}{2}}$ separation axiom.

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1. Introduction and Preliminaries

In a 1925 paper [17], three new separation axioms were added to the study of topology: completely regular, ((completely regular) and T_1) $= T_{3\frac{1}{2}}$, and Urysohn.

Definition 1.1. A space (X, T) is completely regular iff for each closed set C and each $x \notin C$, there exists a continuous function $f : (X, T) \to ([0, 1], U)$, where U is the usual relative metric on $[0, 1]$, such that $f(x) = 0$ and $f(C) = 1$. Completely regular and T_1 is denoted by $T_{3\frac{1}{2}}$.

Definition 1.2. A space (X, T) is Urysohn iff for distinct elements x and y in X, there exist open sets U and V such that $x \in U$, $y \in V$, and $Cl(U) \cap Cl(V) = \phi$.

From past studies of separation axioms, it is known that $T_{3\frac{1}{2}}$ implies T_3, which implies Urysohn, which implies T_2, which implies T_1, which implies T_0 and examples are known showing the implications are not reversible. Thus questions concerning topological properties for which the six separation axioms are equivalent arise, leading to the results in this paper. In addition, the results in this paper are combined with known results for weakly *Po* spaces and properties to give infinitely many new characterizations of $T_{3\frac{1}{2}}$.

Below preliminary properties and results needed to move forward in this paper are given.

In a 1943 paper [15], the R_0 separation axiom was introduced.

Definition 1.3. A space (X, T) is R_0 iff for each closed set C and each $x \notin C$, $C \cap Cl(\{x\}) = \phi$.

In a 1961 paper [1], the R_0 separation axiom was revisited and used to further characterize T_1 and the R_1 separation axiom was introduced and used to further characterize T_2.

Definition 1.4. A space (X, T) is R_1 iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 1.1. *A space is* T_i *iff it is* (T_{i-1} *and* R_{i-1}), $i = 1, 2$, *respectively*.

Below, the results from the 1961 paper [1] are combined with long-known properties to give one answer to a question above.

One Answer: In a completely regular space (X, T), T_0, T_1, T_2, Urysohn, T_3, and $T_{3\frac{1}{2}}$ are equivalent.

Proof. Since completely regular implies R_0 [2], then (X, T) is R_0. Suppose (X, T) is T_0. Then (X, T) is ((completely regular) and T_0) = (((completely regular) and R_0) and T_0) = ((completely regular) and (R_0 and T_0)) = ((completely regular) and T_1) = $T_{3\frac{1}{2}}$, and since $T_{3\frac{1}{2}}$ implies T_3, which implies Urysohn, which implies T_2, which implies T_1, which implies T_0, then all six separation axioms are equivalent in the completely regular space (X, T).

Are there other topological properties for which the six separation axioms are equivalent?

In the 1975 paper [14], T_0-identification spaces and the R_1 separation axiom were used to further characterize T_2.

Theorem 1.2. *A space is* R_1 *iff its* T_0*-identification space is* T_2.

T_0-identification spaces were introduced in the 1936 paper [16] and have proven to be a useful tool in the study of topology.

Definition 1.5. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the natural map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

The 1975 paper [14] was a continuation of work began in the 1936 paper [16] in which metrizable was characterized by using T_0-identification spaces and the pseudometrizable separation axiom.

Theorem 1.3. *A space is pseudometrizable iff its T_0-identification space is metrizable.*

Thus the question of whether other topological properties could be characterized using T_0-identification spaces, as in the case of T_2 and metrizable given above, arose, leading to the introduction and investigation of weakly *Po* properties in 2015 paper [3].

Definition 1.6. Let P be a topological property for which $Po = (P$ and $T_0)$ exists. Then (X, T) is weakly *Po* iff its T_0-identification space $(X_0, Q(X, T))$ has property P. A topological property *Qo* for which weakly *Qo* exists is called a weakly *Po* property.

The two models motivating the definition of weakly *P*o spaces and properties are, in fact, weakly *Po* spaces and properties with weakly metrizable = weakly (pseudometrizable)*o* = pseudometrizable and weakly T_2 = weakly $(R_1)o = R_1$ [3]. Thus metrizable was the first known weakly *Po* property and T_2 was the second known weakly *Po* property. As would have to be true to be a model of the relationships given between the properties above, for a topological property P for which weakly *Po* exists, weakly *Po* would not necessarily have to be T_0 and (weakly *Po*)*o* would have to be *Po*, which is exactly what happens [3].

In the 2015 paper [3], it was proven that for a topological property P for which weakly Po exists, weakly Po is a unique, topological property. Using the important and useful fact that the T_0-identification space for each space is T_0 [16], it was shown in the 2015 paper [3] that for a topological property Q for which weakly Qo exists, the following are equivalent: (a) a space (X, T) has property weakly Qo, (b) the T_0-identification space of (X, T) has property Qo, and (c) the T_0-identification space of (X, T) has property weakly Qo. The fact that topological properties other than weakly Qo are simultaneously shared by both a space and its T_0-identification space led to the introduction and investigation of T_0-identification Q properties.

Definition 1.7. A topological property Q is a T_0-identification P property iff both a space and its T_0-identification space simultaneously share property Q [4].

In the 2015 paper [3], the search for topological properties that are not weakly Po led to the need and use of T_0 and "not-T_0". For the most part, "not-P", where P is a topological property, had been ignored in the study of topology. The need and use of "not-T_0" in the investigation of weakly Po illustrated an importance of "not-P" in the study of topology, leading to several investigations of "not-P" properties that further illustrated the importance of "not-P" properties.

Prior to the investigations of "not-P", the existence of the least of all topological property was not even imagined. However, within a study of "not-P" [5], it was proven that $L = (T_0$ or "not-T_0") is the least topological property and that L can also be given by $L = (P$ or "not-P"), where P is a topological property for which "not-P" exists. As shown in [5] and [6], the known existence of the least topological property necessitated a change in the definition of product properties and in the definition of subspace properties, respectively, and the use of "not-P" in each of those investigations gave many new and unknown properties that do and that do not satisfy each property, changing the study of product spaces and subspaces forever.

If, as is now known, there is a least topological property, is there a strongest topological property? Within a recently submitted paper [7], "not-P" was used to give a simple, straightforward answer to this fundamental topological question: There is no strongest topological property. Thus, the introduction and investigation of weakly Po spaces and properties, which established an importance of "not-P", has greatly changed the study of topology and will continue to do so, as illustrated below.

Because of the special role played by T_0 in the investigation and understanding of weakly Po spaces and properties, questions of what would happen if in the definition of weakly Po, T_0 is replaced by T_1 or T_2 arose, leading to the introduction and investigation of weakly $P1$ and weakly $P2$ spaces and properties.

Definition 1.8. Let P be a topological property for which $P1 = (P$ and $T_1)$ exists. Then a space (X, T) is weakly $P1$ iff its T_0-identification space $(X_0, Q(X, T))$ has property $P1$. A topological property $Q1$ for which weakly $Q1$ exists is called a weakly $P1$ property [8].

Definition 1.9. Let P be a topological property for which $P2 = (P$ and $T_2)$ exists. Then a space (X, T) is weakly $P2$ iff its T_0-identification space $(X_0, Q(X, T))$ has property $P2$. A topological property $Q2$ for which weakly $Q2$ exists is called a weakly $P2$ property [9].

Within recent papers [10] and [11], the weakly P spaces and properties were expanded to include weakly P(Urysohn) and weakly $P3$ spaces and properties, respectively.

Definition 1.10. Let P be a topological property for which P(Urysohn) = (P and Urysohn) exists. Then a space (X, T) is weakly P(Urysohn) iff its T_0-identification space $(X_0, Q(X, T))$ is P(Urysohn). A topological property Q(Urysohn) for which weakly Q(Urysohn) exists is called a weakly P(Urysohn) property [10].

Definition 1.11. A space (X, T) is regular iff for each closed set C and each $x \notin C$, there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$. A regular and T_1 space is denoted by T_3 [18].

Definition 1.12. Let P be a topological property for which $P3$ = (P and T_3) exists. Then a space is weakly $P3$ iff its T_0-identification space is $P3$. A topological property $Q3$ for which weakly $Q3$ exists is called a weakly $P3$ property [11].

Thus far, the study of the weakly P properties given above has been a productive investigation revealing many new properties and results greatly expanding the topics of interest in the study of topology and providing greater insights and understanding of foundational topology. Below, the study of weakly P properties continue with the expansion of weakly P properties to include the completely regular and $T_{3\frac{1}{2}}$ properties.

2. Weakly *P* Properties for Completely Regular and $T_{3\frac{1}{2}}$ Spaces and Properties

Within a 1977 paper [2], it was proven that for a space (X, T), the following are equivalent: (a) (X, T) is completely regular, (b) $(X_0, Q(X, T))$ is completely regular, and (c) $(X_0, Q(X, T))$ is $T_{3\frac{1}{2}}$, giving the following results.

Corollary 2.1. *Completely regular is a* T_0*-identification* P *property and* $T_{3\frac{1}{2}}$ *is a weakly Po property with weakly* $T_{3\frac{1}{2}}$ = *weakly* (*completely regular*)*o* = *completely regular*.

Definition 2.1. Let Q be a topological property for which $Q3\frac{1}{2}$ = (Q and $T_{3\frac{1}{2}}$) exists. Then a space is weakly $Q3\frac{1}{2}$ iff its T_0-identification space has property $Q3\frac{1}{2}$. A topological property $Q3\frac{1}{2}$ for which weakly $Q3\frac{1}{2}$ exists is called a weakly $P3\frac{1}{2}$ property.

Theorem 2.1. *Let* Q *be a topological property for which weakly* $Q3\frac{1}{2}$ *exists. Then weakly* $Q3\frac{1}{2}$ *implies weakly* $Q3$ *and* $Q3$ *is a weakly* $P3$ *property*, *weakly* $Q3\frac{1}{2}$ *implies weakly* Q(*Urysohn*) *and* Q(*Urysohn*) *is a weakly* P(*Urysohn*) *property*, *weakly* $Q3\frac{1}{2}$ *implies weakly* $Q2$ *and* $Q2$ *is a weakly* $P2$ *property*, *weakly* $Q3\frac{1}{2}$ *implies weakly* $Q1$ *and* $Q1$ *is a weakly* $P1$ *property*, *and weakly* $Q3\frac{1}{2}$ *implies weakly* Qo *and* Qo *is a weakly Po property*.

Proof. Let (X, T) be weakly $Q3\frac{1}{2}$. Then $(X_0, Q(X, T))$ is $Q3\frac{1}{2}$. Since $T_{3\frac{1}{2}}$ implies T_3, then $(X_0, Q(X, T))$ is $Q3$, which implies (X, T) is weakly $Q3$. Thus weakly $Q3\frac{1}{2}$ implies weakly $Q3$ and $Q3$ is a weakly $P3$ property. Similarly, the remainder of the theorem follows.

Theorem 2.2. *Let* Q *be a topological property for which weakly* $Q3\frac{1}{2}$ *exists. Then* $Q3\frac{1}{2}$ *is a weakly* Po, *a weakly* $P1$ *property, a weakly* $P2$ *property, a weakly* P(*Urysohn*), *and a weakly* $P3$ *property.*

Proof. Since $Q3\frac{1}{2} = (Q3\frac{1}{2})o$, then weakly $(Q3\frac{1}{2})o =$ weakly $Q3\frac{1}{2}$ exists and $Q3\frac{1}{2} = (Q3\frac{1}{2})o$ is a weakly Po property. Similarly, $Q3\frac{1}{2}$ is a weakly $P1$, a weakly $P2$, a weakly P(Urysohn) property, and a weakly $P3$ property.

Combining Theorem 2.2 with results above gives the following result.

Corollary 2.2. *Let* Q *be a topological property for which weakly* $Q3\frac{1}{2}$ *exists. Then weakly* $Q3\frac{1}{2}$ *is a unique topological property that is neither* T_0 *nor "not-*T_0*".*

Theorem 2.3. *Let* Q *be a topological property for which weakly* $Q3\frac{1}{2}$ *exists. Then* $(\textit{weakly } Q3\frac{1}{2})o = Q3\frac{1}{2}$.

Proof. Since $(Q3\frac{1}{2}) = (Q3\frac{1}{2})o$ and $(\text{weakly } Q3\frac{1}{2})o = (\text{weakly } (Q3\frac{1}{2})o)o = (Q3\frac{1}{2})o$, then $(\text{weakly } Q3\frac{1}{2})o = Q3\frac{1}{2}$.

Combined Theorem 2.2 with results above gives the following result.

Corollary 2.3. *Let* $Q3\frac{1}{2}$ *be a weakly* $P3\frac{1}{2}$ *property. Then a space is weakly* $Q3\frac{1}{2}$ *iff its* T_0*-identification space is weakly* $Q3\frac{1}{2}$.

Thus weakly $Q3\frac{1}{2}$ is a T_0-identification P property.

In the paper [12], it was shown that for each weakly *Po* property *Qo*, weakly *Qo* = (*Qo* or ((weakly *Qo*) and "not-T_0")) is a decomposition of weakly *Qo* into two topological properties neither of which are weakly *Po* properties, which when combined with the results above give the following result.

Corollary 2.4. *Let* $Q3\frac{1}{2}$ *be a weakly* $P3\frac{1}{2}$ *property. Then weakly* $Q3\frac{1}{2}$ = $(Q3\frac{1}{2})o$ *or* ((*weakly* $Q3\frac{1}{2}$) *and* "*not-*T_0")) = (($Q3\frac{1}{2}$) *or* ((*weakly* $Q3\frac{1}{2}$) *and* "*not-*T_0")) *is a decomposition of weakly* $Q3\frac{1}{2}$ *into two topological properties neither of which are weakly* $Q3\frac{1}{2}$ *properties.*

Theorem 2.4. *Let Q be a topological property for which weakly* $Q3\frac{1}{2}$ *exists. Then* (*weakly* $Q3\frac{1}{2})3\frac{1}{2} = Q3\frac{1}{2}$.

Proof. Since (weakly $Q3\frac{1}{2})3\frac{1}{2}$ = ((weakly $Q3\frac{1}{2}$) and $T_{3\frac{1}{2}}$) = ((weakly $Q3\frac{1}{2}$) and (T_0 and $T_{3\frac{1}{2}}$)) = (((weakly $Q3\frac{1}{2}$) and T_0) and $T_{3\frac{1}{2}}$) = ((weakly $Q3\frac{1}{2})o$ and $T_{3\frac{1}{2}}$) = ((weakly $Q3\frac{1}{2})o)3\frac{1}{2}$, (weakly $Q3\frac{1}{2})o = Q3\frac{1}{2}$, and $(Q3\frac{1}{2})3\frac{1}{2} = Q3\frac{1}{2}$, then (weakly $Q3\frac{1}{2})3\frac{1}{2}$ = ((weakly$Q3\frac{1}{2})o)3\frac{1}{2}$ = $(Q3\frac{1}{2})3\frac{1}{2} = Q3\frac{1}{2}$.

Theorem 2.5. *Let Q be a topological property for which weakly* $Q3\frac{1}{2}$ *exists. Then weakly* $Q3\frac{1}{2}$ = ($Q3\frac{1}{2}$ *or* (*weakly* $Q3\frac{1}{2}$ *and* "*not-*$T_{3\frac{1}{2}}$")).

Proof. Since weakly $Q3\frac{1}{2}$ = ((weakly $Q3\frac{1}{2}$) and L) = ((weakly $Q3\frac{1}{2}$) and ($T_{3\frac{1}{2}}$ or "not-$T_{3\frac{1}{2}}$")), then weakly $Q3\frac{1}{2}$ = ((weakly $Q3\frac{1}{2}$)$3\frac{1}{2}$ or ((weakly $Q3\frac{1}{2}$) and "not-$T_{3\frac{1}{2}}$")) = ($Q3\frac{1}{2}$ or ((weakly $Q3\frac{1}{2}$) and "not-$T_{3\frac{1}{2}}$")).

Theorem 2.6. *Let Q be a topological property for which weakly $Q3\frac{1}{2}$ exists. Then, in a weakly $Q3\frac{1}{2}$ space, "not-T_0" and "not-$T_{3\frac{1}{2}}$" are equivalent.*

Proof. Let (X, T) be weakly $Q3\frac{1}{2}$. Since weakly $Q3\frac{1}{2}$ = ($Q3\frac{1}{2}$ or ((weakly $Q3\frac{1}{2}$) and "not-T_0")) = ($Q3\frac{1}{2}$ or ((weakly $Q3\frac{1}{2}$) and "not-$T_{3\frac{1}{2}}$")), where each of ($Q3\frac{1}{2}$ and ((weakly $Q3\frac{1}{2}$) and "not-T_0")), and ($Q3\frac{1}{2}$ and ((weakly $Q3\frac{1}{2}$) and "not-$T_{3\frac{1}{2}}$")) do not exist, then ((weakly $Q3\frac{1}{2}$) and "not-T_0") = (weakly $Q3\frac{1}{2}$) $\setminus$ $Q3\frac{1}{2}$ = ((weakly $Q3\frac{1}{2}$) and "not-$T_{3\frac{1}{2}}$"). Thus, since (X, T) is weakly $Q3\frac{1}{2}$, then (X, T) is "not-T_0" iff (X, T) is "not-$T_{3\frac{1}{2}}$".

Theorem 2.7. *Let Q be a topological property for which weakly $Q3\frac{1}{2}$ exists and let (X, T) be weakly $Q3\frac{1}{2}$. Then (X, T) is simultaneously T_0, $T_{3\frac{1}{2}}$, T_3, Urysohn, T_2, and T_1.*

Proof. By the contrapositive of Theorem 2.6, (X, T) is T_0 iff (X, T) is $T_{3\frac{1}{2}}$. Since $T_{3\frac{1}{2}}$ implies T_3, which implies Urysohn, which implies T_2, which implies T_1, which implies T_0, the proof is complete.

Thus, for weakly $Q3\frac{1}{2}$ spaces, all of T_0, T_1, T_2, Urysohn, T_3, and $T_{3\frac{1}{2}}$ are equivalent, raising the new question: For each weakly $Q3\frac{1}{2}$ space, are there related spaces in which all of the six separation axioms are equivalent?

Within a 2016 paper [12], the following result was proven: "Let Q be a topological property for which weakly Qo exists and let $\mathcal{S} = \{So | S$ is a topological property, So exists, and So implies $Qo\}$. Then $(\text{weakly } Qo)o \in \mathcal{S}$, for each weakly Po property W such that Wo implies Qo, $(\text{weakly } Wo)o \in \mathcal{S}$, each element of $\mathcal{S}$ implies weakly Qo, and there exists the topological property $Q_{\min} = ((\text{weakly } Qo)$ or "not-T_0"), where "not-T_0" is the negation of T_0, weaker than weakly Qo such that $(Q_{\min})o \in \mathcal{S}$. $Q_{\min}$, given above, is unique in that it is the least topological property, which together with T_0, is in $\mathcal{S}$ [12].

Below the new question given above is tied to the results above, which is then used to obtain answers to the new question.

Corollary 2.5. *Let Q be a topological property for which weakly* $Q3\frac{1}{2}$ *exists and let* $\mathcal{S}(Q3\frac{1}{2}) = \{So | S$ *is a topological property*, *So exists*, *and So implies* $(Q3\frac{1}{2})o\}$. *Then* $\mathcal{S}(Q3\frac{1}{2}) = \{So | S$ *is a topological property*, *So exists*, *and So implies* $(Q3\frac{1}{2})\}$.

Corollary 2.6. *Let Q be a topological property for which weakly* $Q3\frac{1}{2}$ *exists and let* $\mathcal{S}(Q3\frac{1}{2}) = \{So | S$ *is a topological property, So exists, and So implies* $Q3\frac{1}{2}\}$. *Then* $(\textit{weakly } Q3\frac{1}{2})o \in \mathcal{S}(Q3\frac{1}{2})$, *for each weakly Po property W such that Wo implies* $Q3\frac{1}{2}$, $(\textit{weakly } W3\frac{1}{2})o \in \mathcal{S}(Q3\frac{1}{2})$, *each element of* $\mathcal{S}(Q3\frac{1}{2})$ *implies weakly* $Q3\frac{1}{2}$, *and there exists the topological property* $(Q3\frac{1}{2})_{\min} = ((\textit{weakly } Q3\frac{1}{2})$ *or "not-*T_0*"*$)$ *weaker than weakly* $Q3\frac{1}{2}$ *such that* $((Q3\frac{1}{2})_{\min})o \in \mathcal{S}(Q3\frac{1}{2})$.

In the 2016 paper [12], it was established that for a topological property for which weakly Qo exists, $Q_{\min}$ is the least topological property for which a space has property Qo iff it has property $(Q_{\min}$ and $T_0)$, which is combined with the results above to give the following result.

Corollary 2.7. *Let Q be a topological property for which weakly* $Q3\frac{1}{2}$ *exists. Then* $(Q3\frac{1}{2})_{\min}$ *is the least topological property for which a space has property* $Q3\frac{1}{2}$ *iff it has property* $((Q3\frac{1}{2})_{\min}$ *and* $T_0)$.

Below, for each topological property Q for which weakly $Q3\frac{1}{2}$ exists, the results above are used to give infinitely many topological characterizations of $Q3\frac{1}{2}$.

Let m and n represent natural numbers greater than or equal to 2.

Definition 2.2. Let $A(n)$ represent a set with n distinct elements, let X be a set containing the elements of $A(n)$, and let $T(A(n))$ be the topology on X defined by $T(A(n)) = \{B \subseteq X \mid A(n) \subseteq B \text{ or } B = \phi\}$ [13].

Definition 2.3. A space (X, T) has property $T(n)$ iff there exists a subset $A(n)$ of X such that $T = T(A(n))$ [13].

In the 2016 paper [13], it was shown that each $T(n)$ space is "not-T_0" and not a weakly *Po* property, that $Q(n) =$ (weakly *Qo* or $T(n)$) is a topological property weaker than weakly *Qo* and stronger than $Q_{\min}$ such that a space has property *Qo* iff it has property ($Q(n)$ and T_0), and that for $m < n$, $Q(m)$ and $Q(n)$ are distinct topological properties, which is combined with the results above to give the next result.

Corollary 2.8. *For each* $n \geq 2$ *and each weakly* $P3\frac{1}{2}$ *property* $Q3\frac{1}{2}$, $(Q3\frac{1}{2})(n) = ((\textit{weakly } Q3\frac{1}{2}) \textit{ or } T(n))$ *is a non-weakly* $P3\frac{1}{2}$ *topological property weaker than weakly* $Q3\frac{1}{2}$ *and stronger than* $((Q3\frac{1}{2})_{\min})$ *such that a space has property* $Q3\frac{1}{2}$ *iff it has property* $(((Q3\frac{1}{2})(n))$ *and* $T_0)$.

Thus, for each topological property Q for which weakly $Q3\frac{1}{2}$ exists, there are infinitely many non-weakly $P3\frac{1}{2}$ topological properties W weaker than weakly $Q3\frac{1}{2}$ and stronger than $(Q3\frac{1}{2}_{\min})$ such that a space has property $Q3\frac{1}{2}$ iff it has property (W and T_0), as given above.

Theorem 2.8. *Let P be a topological property such that (P and T_0) exists. Then (P and T_0) implies* $P3\frac{1}{2}$ *iff* T_0, T_1, T_2, *Urysohn*, T_3, *and* $T_{3\frac{1}{2}}$ *are equivalent in P spaces.*

Proof. Suppose (P and T_0) implies $P3\frac{1}{2}$. Thus, if (X, T) is a P space with property T_0, then (X, T) has property $P3\frac{1}{2}$, which implies (X, T) is T_3. Thus (X, T) is Urysohn, which implies T_2, which implies (X, T) is T_1, which implies (X, T) is T_0 and T_0, T_1, T_2, Urysohn, T_3, and $T_{3\frac{1}{2}}$ are equivalent.

Clearly, the converse is true.

Corollary 2.9. *Let Q be a topological property for which weakly $Q3\frac{1}{2}$ exists. Then for each topological property P given above for which (P and T_0) = $Q3\frac{1}{2}$, (P and T_0) is a characterization of $P3\frac{1}{2}$ and within each P spaces, T_0, T_1, T_2, Urysohn, T_3, and $T_{3\frac{1}{2}}$ are equivalent.*

3. Infinitely Many New Characterizations of the $T_{3\frac{1}{2}}$ Separation Axiom

Using (weakly ((weakly (completely regular)*o*)*o*))*o* $= T_{3\frac{1}{2}}$ and the results above give the following infinite number of new topological characterizations of $T_{3\frac{1}{2}}$.

Corollary 3.1. *Let (X, T) be a space. Then the following are equivalent*: (a) (X, T) is $T_{3\frac{1}{2}}$, (b) (X, T) *is* (*weakly* (*completely regular*)*o*)*o*, (c) (X, T) *is* (*weakly* ((*weakly* (*completely regular*)*o*)*o*))*o*, (d) (X, T) *is* $(((T_{3\frac{1}{2}})_{\min})$ *and* $T_0)$, *and* (e) *for each natural number* $n \geq 2$, (X, T) *has property* $(((T_{3\frac{1}{2}})(n))$ *and* $T_0)$, *where* $((T_{3\frac{1}{2}})(n))$ *is as defined above.*

Since $(T_{3\frac{1}{2}})(3\frac{1}{2}) = T_{3\frac{1}{2}}$, then by the equivalences above, the T_0 in Corollary 3.1 can be replaced by T_1 or T_2 or Urysohn or T_3 giving many more new topological characterizations of the $T_{3\frac{1}{2}}$ property.

Thus, as shown above, the introduction and investigation of weakly P properties continues to be a productive study revealing many long overlooked fundamental, foundational properties and providing greater understanding, insights, and knowledge for inclusion in the study of topology and for use in the continued expansion of topology.

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