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ON GORENSTEIN DIAGONAL SUBALGEBRAS

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Abstract

Let R be a multigraded ring over a local ring. In this paper, we study the Gorenstein property of a diagonal subalgebra of R. Then we give a sufficient condition for this algebra to be Gorenstein. As an application, we apply our result to multi-Rees algebra and show that the diagonal subalgebra of a Gorenstein multi-Rees algebra is Gorenstein in the bigraded case.

1. Introduction

Let $R = \oplus_{\alpha \in \mathbb{Z}^n} R_\alpha$ be a Noetherian $\mathbb{Z}^n$-graded ring. For a subgroup H of $\mathbb{Z}^n$, we define $R^{(H)} = \oplus_{\alpha \in H} R_\alpha$ and call it the diagonal subalgebra of R (w.r.t. H). In this paper, we are interested in the relationship between the Gorenstein property of R and that of its diagonal subalgebra $R^{(H)}$.

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Many author have been studied the Cohen-Macaulay property of a diagonal subalgebra and gave a necessary and sufficient condition of $R^{(H)}$ to be Cohen-Macaulay in a limited extent [8, 9, 17, 21, 18]. But, in general, $R^{(H)}$ is not Cohen-Macaulay (resp., Gorenstein), even if R is Cohen-Macaulay (resp., Gorenstein). The study of the Cohen-Macaulay property of R and $R^{(H)}$ in general situation can be found in Hyry [13]. He have been studied the local cohomology modules of R and gave some condition for the Cohen-Macaulayness of R and $R^{(H)}$.

Our aim of this paper is to give a sufficient condition for the Gorensteinness of $R^{(H)}$ using some information of cosets of $\mathbb{Z}^n/H$ similar to conic divisors of [3] and we give some applications of our result for multi-Rees algebras.

To describe our result, we give some notions as follows. Suppose that $R = R_0[x_1, \cdots, x_m]$ for homogeneous elements $x_1, \cdots, x_m \in R$ and R_0 is a local ring. For a $\mathbb{Z}^n$-graded R-module M, we denote by $\deg(m) = \alpha$ for $m \in M_\alpha$ and put $Deg(M) = \{\alpha \in \mathbb{Z}^n \mid M_\alpha \neq (0)\}$. Without loss of generality, we may assume that $\mathbb{Z}^n = \mathbb{Z}Deg(R)$. Throughout this paper, we always assume that $Deg(R)$ is positive, i.e., $Deg(R) \cap -Deg(R) = (0)$. Under this assumption, if R is Gorenstein, then there exist unique integral vector $\alpha \in \mathbb{Z}^n$ such that $K_R \cong R(\alpha)$ ([4], [7]). We denote this vector α by $a(R)$.

Define a surjective semigroup homomorphism $\varphi : \mathbb{Z}^m \to \mathbb{Z}^n$ by $\varphi(\mathbf{e}_i) = \deg(x_i)$ for $1 \leq i \leq m$. For a subgroup $G' \subset \mathbb{Z}^n$ and $\gamma \in \mathbb{Z}^n$, we put $H(G') = \mathbb{Z}_0^m \cap \varphi^{-1}(G')$ and $H(\gamma, G') = \mathbb{Z}_0^m \cap \varphi^{-1}(\gamma + G')$, where $\mathbb{Z}_0$ is the set of non-negative integers. Finally, we put $\operatorname{supp}(H(G')) = \{i \in [m] = \{1, \cdots, m\} \mid \alpha(i) > 0$ for some $\alpha \in H(G')\}$. Then our main result describe as follows.

Theorem 3.1. *Let* G' *be a subgroup of* G *such that* $\dim(R^{(G')}) = \dim(R) - rk(\mathbb{Z}^m/\mathbb{Z}H(G'))$. *Suppose that* $Deg(K_R) \subset \mathbb{Z}_0^n$ *and* $\mathbb{Z}[H(-\alpha, G')]$ *is a Cohen-Macaulay module for* $\alpha \in \mathbb{Z}_0^n$ *such that* $\alpha \leq \sum_{i=1}^m \deg(x_i)$. *Then we have the following*:

(1) *If* R *is Cohen-Macaulay, then so is* $R^{(G')}$.

(2) *If* R *is Gorenstein and* $a(R) - \sum_{j \notin \mathrm{supp}(H(G'))} \deg(x_j) \in G'$, *then so is* $R^{(G')}$.

□

Direct consequence of this result, we have the following corollary.

Corollary 4.2. *Let* A *be a Noetherian local ring. Let* I_1, I_2 *be ideals of positive height. If* $R(I_1, I_2)$ *is Cohen-Macaulay* (*resp.*, *Gorenstein*), *then so is* $R(I_1 I_2)$. □

By the same argument of Corollary 2.10 of Hyry [13], we have the following general statement.

Corollary 4.3 (Corollary 2.10 of Hyry [13])**.** *Let* A *be a Noetherian local ring. Let* $I_1, \cdots, I_n$ *be ideals of positive height. If* $R(I_1, \cdots, I_n)$ *is Cohen-Macaulay* (*resp.*, *Gorenstein*), *then so is* $R(I_1 \cdots I_n)$. □

As another application, we give an expected statement for some special case of polynomial ideals.

Proposition 5.3. *Let* $S_i = k[x_{i1}, \cdots, x_{in_i}]$ *be a polynomial ring over a field* k *for* $1 \leq i \leq r$ *and put* $S = S_1 \otimes_k \cdots \otimes_k S_r$. *Let* $I_i \subset S_i$ *be an ideal of* $\mathrm{ht}(I_i) \geq 2$ *contained in* $(x_{i1}, \cdots, x_{in_i})$ *and* $I = I_1 \cdots I_n S$. *The following conditions are equivalent*:

(1) $R(I)$ *is Cohen-Macaulay* (*resp.*, *Gorenstein*).

(2) $R(I_i)$ *is Cohen-Macaulay* (*resp.*, *Gorenstein*) *for* $1 \leq i \leq r$.

(3) $R(I_1, \cdots, I_r)$ *is Cohen-Macaulay* (*resp.*, *Gorenstein*).

□

2. Preliminary

In this section, we give some terminology and basic facts of G-graded rings from [4, 6, 7, 16] while will be used in this paper. Let $G = \mathbb{Z}^n$ and $R = \oplus_{\alpha \in G} R_\alpha$ be a Noether G-graded ring such that R_0 is local. For G' be a subgroup of G, we define a diagonal subalgebra $R^{(G')}$ of G' by $R^{(G')} = \oplus_{\alpha \in G'} R_\alpha$ and put $R^{(\gamma, G')} = \oplus_{\beta \in G'} R_{\gamma+\beta}$ for $\gamma \in G$. Then $R^{(G')}$ is a G'-graded ring and $R^{(\gamma, G')}$ is a G'-graded $R^{(G')}$-module for $\gamma \in G$. Furthermore, if $\{\gamma_i\}_i$ is a system of representatives of G / G', then $R = \oplus_i R^{(\gamma_i, G')}$ and this decomposition gives a G/G'-graded structure of R.

For $\gamma \in G$ and M a G-graded R-module, we define a G-graded R-module $M(\gamma)$ by $M = M(\gamma)$ as the underlying R-module and is graded by $[M(\gamma)]_\delta = M_{\gamma+\delta}$ for all $\delta \in G$. The following formula is frequently used in this paper.

Remark 2.1. (1) If $\gamma = \gamma_i + \delta$ for $\delta \in G'$, then $R^{(\gamma, G')} = R^{(\gamma_i, G')}(\delta)$.

(2) If $\gamma + \gamma' = \gamma_i + \delta$ for $\delta \in G'$, then

$$R(\gamma)^{(\gamma', G')} = \bigoplus_{\delta' \in G'} R_{\gamma+\gamma'+\delta'} = R^{(\gamma_i, G')}(\delta).$$

□

Suppose that $R = R_0[x_1, \cdots, x_m]$ for homogeneous elements $x_1, \cdots, x_m \in R$. Let S_0 be a regular local ring and $S = S_0[t_1, \cdots, t_m]$ be a polynomial ring over S_0 with m variables. We set $\deg(t_i) = \mathbf{e}'_i \in \mathbb{Z}^m$ for $1 \le i \le m$ and regards S as a $\mathbb{Z}^m$-graded ring where $\mathbf{e}'_1, \cdots, \mathbf{e}'_m$ are standard basis of $\mathbb{Z}^m$. We define $\varphi : \mathbb{Z}^m \to \mathbb{Z}^n$ by $\varphi(\mathbf{e}'_i) = \deg(x_i)$ for $1 \le i \le m$. Then S is also regarded as a G-graded ring via φ.

For a subgroup $G' \subset G$, we put $L = \varphi^{-1}(G')$ and denote by $\{\mathbf{a}_i\}_i$ a system of representatives of $\mathbb{Z}^m / L (\cong G / G')$. Clearly, we have $S^{(L)} = S_0[H(G')]$ and $S^{(\mathbf{a}, L)} = S_0[H(\varphi(\mathbf{a}), G')]$ for $\mathbf{a} \in \mathbb{Z}^m$. A decomposition of a $\mathbb{Z}^m / L$-grading of S is given by $S = \oplus_i S^{(\mathbf{a}_i, L)} = \oplus_i S_0[H(\varphi(\mathbf{a}_i), G')]$.

Proposition 2.2. *$S_0[H(G')]$ is normal Cohen-Macaulay and if $H(-\varphi(\mathbf{a}_i), G') \neq \emptyset$, then $S_0[H(\varphi(\mathbf{a}_i), G')]$ is divisorial for all i.*

Proof. For $\mathbf{b} \in \mathbb{Z}H(G')$, if $k\mathbf{b} \in H(G')$, then $\varphi(\mathbf{b}) \in G'$ and $k\mathbf{b} \ge \mathbf{0}$. Thus $\mathbf{b} \ge \mathbf{0}$ and $\mathbf{b} \in H(G')$. Namely, $H(G')$ is a normal semigroup and $S_0[H(G')]$ is normal Cohen-Macaulay since S_0 is a regular domain.

Let $\mathbf{a} = -\mathbf{a}_i + \mathbf{b} \in H(-\varphi(\mathbf{a}_i), G')$ for some $\mathbf{b} \in \mathbb{Z}H(G')$. Then the monomial $m := t^{\mathbf{a}}$ is in S and we have the exact sequence $0 \to S(-\mathbf{a}) \xrightarrow{m} S$. Taking a degree G'-part of this sequence, we have $0 \to S_0[H(\varphi(\mathbf{a}_i), G')](-\mathbf{b}) \to S_0[H(G')]$. Thus $S_0[H(\varphi(\mathbf{a}_i), G')]$ is a fractional ideal. Also, if we denote by H^1 the set of all height 1 graded prime ideals of $S_0[H(G')]$, then $S = \bigcap_{\mathfrak{p} \in H^1} S_{(\mathfrak{p})} = \oplus_i \bigcap_{\mathfrak{p} \in H^1} S_0[H(\varphi(\mathbf{a}_i), G')]_{(\mathfrak{p})}$, This implies that $S_0[H(G')]$ is divisorial. □

Next, let us recall some facts of normal semigroups and its semigroup rings from Goto and Watanabe [7], Stanley [22], and Isida [15].

Let $\mathfrak{a}$ be a $\mathbb{Z}^m$-graded ideal of $S_0[H(G')]$. We consider $\mathfrak{a}S = Q_1 \cap \cdots \cap Q_l$ the $\mathbb{Z}^m$-graded primary decomposition of $\mathfrak{a}$ in S and put $\sqrt{Q_i} = P_i$ for $1 \leq i \leq l$. Then Q_i, P_i are $\mathbb{Z}^m$-graded ideals, namely, monomial ideals of S. Since $H(G')$ is normal (i.e., full in sense of [11]), $\mathfrak{a} = \mathfrak{a}S \cap S_0[H(G')]$ and $\mathfrak{a} = \bigcap_{i=1}^{l} Q_i^{(G')} = \bigcap_{i=1}^{l} Q_i \cap S_0[H(G')]$ is a primary decomposition of $\mathfrak{a}$ in $S_0[H(G')]$ (not necessary a minimal). Also, we have $\sqrt{Q_i \cap S_0[H(G')]} = P_i \cap S_0[H(G')]$. In particular, if $\mathfrak{a}$ is prime, then $\mathfrak{a} = P_i \cap S_0[H(G')]$ for some i.

For $F \subset [m] = \{1, \cdots, m\}$, we set $P_F = (t_i \mid i \in F) \subset S$ a prime ideal of S generated by variables. Then $\mathrm{Spec}^{\mathbb{Z}^m}(S_0[H(G')])$ the set of all $\mathbb{Z}^m$-graded prime ideals of $S_0[H(G')]$ is given by $\mathrm{Spec}^{\mathbb{Z}^m}(S_0[H(G')]) = \{\mathfrak{p} + P_F \cap S_0[H(G')] \mid \mathfrak{p} \in \mathrm{Spec}(S_0) \text{ and } F \subset [m]\}$.

Put $\mathfrak{p}_F = P_F \cap S_0[H(G')]$ for $F \subset [m]$ and call it monomial prime of $S_0[H(G')]$. We define a equivalence relation $\sim$ on $\{F \subset [m]\}$ by $F \sim F'$ iff $\mathfrak{p}_F = \mathfrak{p}_{F'}$ and denote the equivalent class by $\mathbf{F}(\mathfrak{p}) = \{F \subset [m] \mid \mathfrak{p}_F = \mathfrak{p}\}$ for a monomial prime $\mathfrak{p} \subset S_0[H(G')]$.

For $t^{\mathbf{a}} \in S_0[H(G')]$, $t^{\mathbf{a}} \in \mathfrak{p}_F$ if and only if $\mathbf{a}(i) > 0$ for some $i \in F$ by definition. Here we denote i-th coordinate of $\mathbf{a}$ by $\mathbf{a}(i)$. Therefore, for all $F, F' \in \mathbf{F}(\mathfrak{p})$, $\mathfrak{p}_{F \cup F'} = (t^a \in S_0[H(G')] \mid \mathbf{a}(i) > 0 \text{ for some } i \in F \cap F')$ $= \mathfrak{p}_F + \mathfrak{p}_{F'} = \mathfrak{p}$. Thus $\mathbf{F}(\mathfrak{p})$ has the maximal element $F(\mathfrak{p}) := \bigcup_{F \in \mathbf{F}(\mathfrak{p})} F$. Note that if $\mathfrak{p}_F = (0)$, then $F \cap \mathrm{supp}(H(G')) = \emptyset$ for $F \subset F(\mathfrak{p})$.

Finally, with this notation, we can explain the difference between conic divisors in sense of [3] and $S_0[H(\varphi(\mathbf{a}), G')]$. Let $H^1(H(G'))$ be a set of all $\mathbb{Z}^m$-graded prime ideal of $S_0[H(G')]$ of height 1 and $Div(H(G')) =$

$\oplus_{\mathfrak{p}\in H^1(G')}\mathbb{Z}[\mathfrak{p}]$. We put $e(\mathfrak{p}) = \sum_{i\in F(\mathfrak{p})\cap \mathrm{supp}(H(G'))}\mathbf{e}'_i \in \mathbb{Z}^m$ for $\mathfrak{p}\in H^1(H(G'))$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}H(G') & \xrightarrow{\mathrm{div}} & Div(H(G')) & \longrightarrow & Cl(S_0[H(G')]) & \longrightarrow & 0 \\ & & \| & & \downarrow e(-) & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}H(G') & \longrightarrow & \mathbb{Z}^m & \longrightarrow & G/\mathbb{Z}H(G') & \longrightarrow & 0 \end{array}$$

It follows that we have the exact sequence

$$0 \to Cl(S_0[H(G')]) \to G\,/\,\mathbb{Z}H(G') \to \mathbb{Z}^m\,/\bigoplus_{\mathfrak{p}\in H^1(H(G'))}\mathbb{Z}e(\mathfrak{p}) \to 0.$$

In the following, we describe the classical result of Gorenstein normal semigroup rings due to Stanley [22] in terms of $H(\alpha, G')$.

Proposition 2.3 (Stanley [22])**.**

(1) $rel.int(H(G')) = \{\beta \in H(G') \mid \mathbf{b}(i) > 0$ *for* $i \in \mathrm{supp}(H(G'))\}$.

(2) $K_{S_0[H(G')]} \cong S_0[H(-\sum_{i\in\mathrm{supp}(H(G'))}\varphi(\mathbf{e}'_i), G')]$ *as underlying modules.*

(3) $S_0[H(G')]$ *is Gorenstein if and only if* $\sum_{i\in\mathrm{supp}(H(G'))}\varphi(\mathbf{e}'_i) \in G'$.

Proof. Since $S_0[H(G')] \subset S_0[t_i \mid i \in \mathrm{supp}(H(G'))]$, we may assume that $\mathrm{supp}(H(G')) = [m]$. Then the face of $H(G')$ is given by $\{\mathbf{b} \in H(G') \mid \mathbf{b}(i) = 0$ for $i \in F\}$ for some $F \subset [m]$ (Ishida [15]). The assumption $\mathrm{supp}(H(G')) = [m]$ implies that $rel.int(H(G')) = H(G') \cap \mathbb{Z}^m_+ = \{\mathbf{a} \in H(G') \mid \mathbf{a}(i) > 0$ for all $i \in [m]\}$.

For $\mathbf{a} \in rel.int(H(G'))$, since $\mathbf{a} - \sum_{i=1}^m \mathbf{e}'_i \in \mathbb{Z}^m_0$ and $\varphi(\mathbf{a} - \sum_{i=1}^m \mathbf{e}'_i)$ is contained in $-\sum_{i=1}^m \varphi(\mathbf{e}'_i) + G'$, we have $\mathbf{a} - \sum_{i=1}^m \mathbf{e}'_i \in H(-\sum_{i=1}^m \varphi(\mathbf{e}'_i), G')$ and

$$rel.int(H(G')) = H(-\sum_{i=1}^m \varphi(\mathbf{e}'_i), G') + \sum_{i=1}^m \mathbf{e}'_i.$$

This conclude that $K_{S_0[H(G')]} \cong S_0[rel.int(H(G'))] \cong S_0[H(-\sum_{i=1}^m \varphi(\mathbf{e}'_i), G')]$ as underlying $S_0[H(G')]$-modules. The assertion (3) is immediate consequence of this isomorphism. □

3. Proof of Main Result

Theorem 3.1. *Let G' be a subgroup of G such that* $\dim(R^{(G')}) = \dim(R) - rk(\mathbb{Z}^m / \mathbb{Z}H(G'))$. *Suppose that* $R = R_0[x_1, \cdots, x_m]$ *and* $\deg(x_1), \cdots, \deg(x_m) \in \mathbb{Z}_0^n$. *Moreover, we assume that* $Deg(K_R) \subset \mathbb{Z}_0^n$ *and* $\mathbb{Z}[H(-\alpha, G')]$ *is a Cohen-Macaulay module over* $\mathbb{Z}[H(G')]$ *for* $\alpha \in \mathbb{Z}_0^n$ *such that* $\alpha \leq \sum_{i=1}^m \deg(x_i)$. *Then we have the following*:

(1) *If R is Cohen-Macaulay, then so is* $R^{(G')}$.

(2) *If R is Gorenstein and* $a(R) - \sum_{j \notin \mathrm{supp}(H(G'))} \deg(x_j) \in G'$, *then so is* $R^{(G')}$.

Proof of Theorem 3.1. We put $\alpha_i = \deg(x_i)$ for $1 \leq i \leq m$ and $\alpha = \sum_{i=1}^m \alpha_i$.

(1) Through the completion of R_0 at the maximal ideal, we may assume that $R = S / I$ for some G-graded ideal $I \subset S$. Let $0 \to F_p \to \cdots \to F_1 \to F_0$ be a minimal G-graded S-free resolution of R and we put $F_i = \oplus_j S(-\alpha_{ij})$ for $0 \leq i \leq p$.

Assume that R is Cohen-Macaulay. Then $\mathrm{Hom}(\mathbb{F}, S(-\alpha))$ gives the minimal G-graded S-free resolution of K_R. Note that the i-th component of this resolution is described as $F^*(-\alpha) = \oplus_j S(-\alpha + \alpha_{p-ij})$. On the other hand, since $Deg(K_R) \subset \mathbb{Z}_0^n$ and is Cohen-Macaulay, all betti degrees of this resolution are contained in $\mathbb{Z}_0^n$. Thus $\sum_{i=1}^m \alpha_i - \alpha_{ij} \geq \mathbf{0}$

and $\alpha_{ij} \leq \sum_{i=1}^{m} \alpha_i$. This implies that $F_i^{(G')}$ is the direct sum of $\{S_0[H(-\alpha_{ij}, G')]\}_j$ for all i and $S_0[H(-\alpha_{ij}, G')]$ is Cohen-Macaulay by our assumption. Hence we have the resolution

$$0 \to F_p^{(G')} \to \cdots \to F_1^{(G')} \to F_0^{(G')}$$

of $R^{(G')}$ consisting of maximal Cohen-Macaulay $S^{(G')}$-modules.

We put $B_i = \mathrm{Im}(F_i \to F_{i-1})^{(G')}$ for $0 \leq i \leq p$ and $m' = \dim(S^{(G')})$. Then we have isomorphisms $H_{\mathfrak{m}}^i(R^{(G')}) \cong H_{\mathfrak{m}}^{i+1}(B_1) \cong H_M^{i+2}(B_2) \cong \cdots \cong H_{\mathfrak{m}}^{m'-1}(B_{m'-1-i})$ and, thus, $H_{\mathfrak{m}}^i(R^{(G')}) = (0)$, if $p \leq m' - 1 - i$. Since $m' = \dim(S^{(G')}) = \dim(S_0) + rk(\mathbb{Z}H(G'))$, we have equalities

$$\begin{aligned} m' - 1 - p &= \dim(S^{(G')}) - \mathrm{pd}_R(R) - 1 \\ &= \dim(S_0) + rk(\mathbb{Z}H(L)) - (\dim(S_0) + m - \mathrm{depth}(R)) - 1 \\ &= \mathrm{depth}(R) - rk(\mathbb{Z}^m / \mathbb{Z}H(L)) - 1 \\ &= \dim(R) - rk(\mathbb{Z}^m / \mathbb{Z}H(L)) - 1 \\ &< \dim(R^{(G')}). \end{aligned}$$

Hence $H_{\mathfrak{m}}^i(R^{(G')}) = (0)$ for $i < \dim(R^{(G')})$ and $R^{(G')}$ is Cohen-Macaulay.

(2) We put $\alpha' = \sum_{i \in \mathrm{supp}(H(G'))} \alpha_i$ and $\alpha'' = \sum_{j \notin \mathrm{supp}(H(G'))} \alpha_j$. Suppose that R is Gorenstein and $a(R) - \alpha'' \in G'$. Since R is Gorenstein, we have $F_p = S(-\sum_{i=1}^{m} \alpha_i + a(R)) = S(-\alpha' - \alpha'' + a(R))$. Then, by Remark 2.1 (2) and Proposition 2.3 (2), $F_p^{(G')} \cong S_0[H(-\alpha', G')](a(R) - \alpha'') \cong K_{S_0[H(G')]}$. Note that $p = \dim(S) - \dim(R) = \dim(S_0) + m - (\dim(R^{(G')}) - rk(\mathbb{Z}^m / \mathbb{Z}H(G'))$ $= \dim(S_0[H(G')]) - \dim(R^{(G')}) = \mathrm{codim}(R^{(G')})$. Then, by the local duality theorem for maximal Cohen-Macaulay modules, we have

$$\begin{aligned} K_{R^{(G')}} &\cong \operatorname{Ext}^p_{S_0[H(G')]}(R^{(G')}, K_{S_0[H(G')]}) \cong \operatorname{Ext}^{p-1}_{S_0[H(G')]}(B_1, K_{S_0[H(G')]}) \\ &\cong \cdots \\ &\cong \operatorname{Ext}^1_{S_0[H(G')]}(B_{p-1}, K_{S_0[H(G')]}). \end{aligned}$$

It follows that we have the exact sequence

$$\operatorname{Hom}_{S_0[H(G')]}(F^{(G')}_{p-1}, K_{S_0[H(G')]}) \to \operatorname{Hom}_{S_0[H(G')]}(F^{(G')}_p, K_{S_0[H(G')]}) \to K_{R^{(G')}} \to 0.$$

Since $F^{(G')}_p \cong K_{S_0[H(G')]}$ as above, we have

$$\operatorname{Hom}_{S_0[H(G')]}(F^{(G')}_p, K_{S_0[H(G')]}) \cong \operatorname{Hom}_{S_0[H(G')]}(K_{S_0[H(G')]}, K_{S_0[H(G')]}),$$

and is isomorphic to $S_0[H(G')]$ as underling $S_0[H(G')]$-modules. This implies that $K_{R^{(G')}}$ is a cyclic module and $R^{(G')}$ is Gorenstein. □

The proof of Theorem 3.1 based on the following general statement.

Lemma 3.2. *Let* $(A, \mathfrak{m})$ *be a d-dimensional Cohen-Macaulay ring with a canonical module* K_A *and M be a finitely generated A-module of* $\operatorname{codim}(M) = s$. *Suppose that there is a resolution*

$$0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to M \to 0$$

of M such that F_i *is a maximal Cohen-Macaulay A-module for* $0 \le i \le s$. *Then M is a* $(d-s)$*-dimensional Cohen-Macaulay module and has the exact sequence* $K_{F_{s-1}} \to K_{F_s} \to K_M \to 0$.

Proof. The proof is similar to the above. Let $B_i = \operatorname{Im}[F_i \to F_{i-1}]$ for $0 \le i \le s$. We have isomorphisms $H^i_{\mathfrak{m}}(M) \cong H^{i+1}_{\mathfrak{m}}(B_1) \cong \cdots \cong H^{i+k}_{\mathfrak{m}}(B_k)$ for $i + k < d$. Thus, for $i < d - s$, $H^i_{\mathfrak{m}}(M) \cong H^{d-1}_{\mathfrak{m}}(B_{d-i-1}) = (0)$, since $d - i - 1 \ge s$. Hence M is a $(d-s)$-dimensional Cohen-Macaulay module. In particular, since $H^{d-s}_{\mathfrak{m}}(M) \cong H^{d-1}_{\mathfrak{m}}(B_{s-1})$, $K_M \cong \operatorname{Ext}^s_A(M, K_A) \cong H^{d-s}_{\mathfrak{m}}$

$(M)^* \cong H_{\mathfrak{m}}^{d-1}(B_{s-1})^* \cong Ext_A^1(B_{s-1}, K_A)$. Taking the canonical dual of the sequence $0 \to F_s \to F_{s-1} \to B_{s-1} \to 0$, we have the exact sequence $0 \to K_{B_{s-1}} \to K_{F_{s-1}} \to K_{F_s} \to K_M \to 0$. □

Remark 3.3. (1) In general, $H(-\alpha, G')$ is not Cohen-Macaulay, if R is Cohen-Macaulay.

(2) The proof of Theorem 3.1 tells us some information about $\operatorname{depth}(R^{(\gamma, G')})$, even if R is not CM. Namely, if all components $\mathbb{F}^{(\gamma, G')}$ is Cohen-Macaulay, then $\operatorname{depth}(R^{(\gamma, G')}) \geq \operatorname{depth}(R) - rk(\mathbb{Z}^m / \mathbb{Z}H(L))$ for $\gamma \in G$.

□

In the following, we give two statements which satisfy the condition of Theorem 3.1.

Corollary 3.4 ([16]). *For a subgroup $G' \subset G$ such that G / G' is torsion*

(1) *If R is Cohen-Macaulay, then so is $R^{(G')}$.*

(2) *If R is Gorenstein and $a(R) \in G'$ then so is $R^{(G')}$.*

□

Corollary 3.5 (Bruns and Guerrieri [2]; Bruns and Gubeladze [3]).

Let $S = S_0[x_1, \cdots, x_n, y_1, \cdots, y_m]$ be a $\mathbb{Z}^2$-graded polynomial ring of $\deg(x_i) = (1, 0)$, $\deg(y_j) = (0, 1)$ for all i, j and $L = \mathbb{Z}(1, 1)$. Then $S_0[H((i, 0), L)]$ and $S_0[H(0, j), L]$ are maximal Cohen-Macaulay modules for $0 \leq i < n$ and $0 \leq j < m$. □

4. Diagonal Subalgebras of Multi-Rees Algebras

In this section, we apply Theorem 3.1 to multi-Rees algebras. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $I_1, \cdots, I_n$ ideals of A. We assume

that $\operatorname{grade}(I_i) > 0$ for $1 \le i \le n$. The multi-Rees algebra of $I_1, \cdots, I_n$ is defined to be the subalgebra $R(I_1, \cdots, I_n) = A[I_1 s_1, \cdots, I_n s_n]$ of a polynomial ring $A[s_1, \cdots, s_n]$, where $s_1, \cdots, s_n$ are in determinates. We regards $R(I_1, \cdots, I_n)$ as a $\mathbb{Z}^n$-graded ring by $\deg(s_i) = \mathbf{e}_i \in \mathbb{Z}^n$. We denote by $I_i = (x_{i1}, \cdots, x_{im_i})$ for $1 \le i \le n$.

Define $\varphi_i : \mathbb{Z}^{m_i} \to \mathbb{Z}^n$ by $\varphi_i(\mathbf{e}_{ij}) = \deg(x_{ij} s_i) = \mathbf{e}_i$ for $1 \le i \le n$ and $1 \le j \le m_i$ and $\varphi = \sum_i \varphi_i : \mathbb{Z}^{m_1 + \cdots + m_n} \to \mathbb{Z}^n$. Let $L = \mathbb{Z}(1, 1, \cdots, 1) \subset \mathbb{Z}^n$. Then the diagonal subalgebra of $R(I_1, \cdots, I_n)$ of L coincides with the usual Rees algebra $R(I_1 \cdots I_n)$ and we are able to replace the statement of Theorem 3.1 as follows.

Proposition 4.1. *For* $\mathbf{0} \le \alpha \le (m_1, \cdots, m_n)$, *we suppose that* $\mathbb{Z}[H(-\alpha, L)]$ *is Cohen-Macaulay. Then we have the following*:

(1) *If* $R(I_1, \cdots, I_n)$ *is Cohen-Macaulay, then so is* $R(I_1 \cdots I_n)$.

(2) *If* $R(I_1, \cdots, I_n)$ *is Gorenstein, then so is* $R(I_1 \cdots I_n)$.

Proof. By results of [13], we have that $\dim(R(I_1 \cdots I_n)) = \dim(A) + 1$ and $\dim(R(I_1, \cdots, I_n)) = \dim(A) + n$. Thus $\dim(R(I_1 \cdots I_n)) = \dim(R(I_1, \cdots, I_n)) - (n-1) = \dim(R(I_1, \cdots, I_n)) - rk(\mathbb{Z}^n / L)$, where $\mathbb{Z}^{m_1 + \cdots + m_n} / \mathbb{Z}H(L) \cong \mathbb{Z}^n / L$ via φ. Also, we have $a(R(I_1, \cdots, I_n)) = (-1, \cdots, -1) \in L$ by [13] and $\operatorname{supp}(H(L)) = [m_1 + \cdots + m_n]$. This shows that a part of condition of Theorem 3.1(2) is always satisfied. Moreover, $Deg(K_{R(I_1, \cdots, I_n)}) \subset \mathbb{Z}_0^n$, since $a(R(I_1, \cdots, I_n)) = (-1, \cdots, -1)$. Finally, $\sum_{i=1}^n \sum_{j=1}^{m_i} \deg(a_{ij} s_i) = (m_1, \cdots, m_n)$ and this conclude that conditions of Theorem 3.1 are satisfied except for the Cohen-Macaulayness of $H(-\alpha, L)$. □

Now, we assume that $n = 2$. Then it is easy to see that the representatives of $\mathbb{Z}^2 / L$ is given by $\{(i, 0), (0, j) \mid 1 \leq i \leq m_1, 1 \leq j \leq m_2\}$. As a direct consequence of the above theorem and Corollary 3.5, the Cohen-Macaulay property and the Gorenstein property of $R(I_1 I_2)$ follows from these of $R(I_1, I_2)$ without any other conditions.

Corollary 4.2. *If* $R(I_1, I_2)$ *is Cohen-Macaulay* (*resp.*, *Gorenstein*), *then so is* $R(I_1 I_2)$.

□

Corollary 4.3. *If* $R(I_1, \cdots, I_n)$ *is Cohen-Macaulay* (*resp.*, *Gorenstein*), *then so is* $R(I_1 \cdots I_n)$.

Proof. The proof is the same as the proof of Corollary 2.10 of Hyry [13] and use induction on n.

We put $B = R(I_1, \cdots, I_{n-2})$ and denote $\mathfrak{N}$ a graded maximal ideal of B. Then $R_{B_{\mathfrak{N}}}(I_{n-1}B, I_n B) = R_A(I_1, \cdots, I_n)_{\mathfrak{N}}$ is Cohen-Macaulay (resp., Gorenstein). By Corollary 4.2, this implies that $R(I_1, \cdots, I_{n-2}, I_{n-1}I_n)_{\mathfrak{N}}$ is also Cohen-Macaulay (resp., Gorenstein) and so is $R(I_1, \cdots, I_{n-2}, I_{n-1}I_n)$. Hence, by induction hypothesis, $R(I_1 \cdots I_n)$ is Cohen-Macaulay (resp., Gorenstein). □

As another application of Proposition 4.1, we have the following.

Proposition 4.4. *If* $R(I_1, \cdots, I_n)$ *is Cohen-Macaulay*, *then so is* $R(I_{i_1}, \cdots, I_{i_p})$ *for* $0 < p$ *and* $1 \leq i_1 < \cdots < i_p \leq n$.

Proof. Suppose that $R(I_1, \cdots, I_n)$ is Cohen-Macaulay. To prove this proposition, it is enough to show that $R(I_1, \cdots, I_{n-1})$ is Cohen-Macaulay. Let $M = \{(\alpha, 0) \mid \alpha \in \mathbb{Z}^{n-1}\}$ be the subgroup of $\mathbb{Z}^n$. Then $H(M) = \mathbb{Z}_0^{m_1+\cdots+m_{n-1}} \subset \mathbb{Z}_0^{m_1+\cdots+m_n}$ and $H((0, \cdots, 0, p), G') = \{(\mathbf{a}, \mathbf{b}) \mid$

$\mathbf{a} \in H(M)$, $\mathbf{b} \in \mathbb{Z}_0^{m_n}$ such that $|\mathbf{b}| = p\}$ for all $(0, \cdots, 0, p) \in \mathbb{Z}^n$. Then it is easy to see that $A[H(M)] = A[\mathbb{Z}_0^{m_1+\cdots+m_{n-1}}]$ is the polynomial ring and $A[H((0, \cdots, 0, p), M)]$ is a direct sum of this polynomial ring. This implies that $A[H((0, \cdots, 0, p), M)]$ is Cohen-Macaulay for all $p \geq 0$. The condition of $\dim(R(I_1, \cdots, I_n))$ is clear, hence by Theorem 3.1, we have $R(I_1, \cdots, I_n)^{(M)} = R(I_1, \cdots, I_{n-1})$ is Cohen-Macaulay, if $R(I_1, \cdots, I_n)$ is so.

□

5. Examples

In the previous section, the converse of statements of Corollary 4.2 (or Theorem 3.1) is not true in general. Namely, the Cohen-Macaulayness of $R(I_1 \cdots I_n)$ does not imply the Cohen-Macaulayness of $R(I_1, \cdots, I_n)$, even if $R(I_i)$ are assumed to be Cohen-Macaulay for $1 \leq i \leq n$. In this section, we discuss the expected implications of the Cohen-Macaulay and the Gorenstein property among the multi-Rees algebras.

Let $S = k[x_1, \cdots, x_n]$ be a polynomial ring over the field k. For $F \subset [n] := \{1, \cdots, n\}$, we denote by $\mathfrak{p}_F = (x_i \mid i \in F)$ and by $x_F = \prod_{i \in F} x_i$. For $\mathbf{F} \subset 2^{[n]}$, we define $I_{\mathbf{F}} = \bigcap_{F \in \mathbf{F}} \mathfrak{p}_F$. In particular, for a simplicial complex Δ on $[n]$, if we denote by $\mathbf{F}(\Delta)^c = \{F^c \mid F \in \mathbf{F}(\Delta)\}$, then the Stanley-Reisner ring $k[\Delta] = S / I(\Delta)$ is described as $I(\Delta) = I_{\mathbf{F}(\Delta)^c}$. In this section, we consider the Rees algebra $R(I_{\mathbf{F}})$ and by the argument of multi-Rees algebras, we have the following.

Theorem 5.1. *If* $\mathbf{F} = \coprod_i \mathbf{F}_i$ *decompose into connected components, then* $R(I_{\mathbf{F}})$ *is Cohen-Macaulay iff so is* $R(I_{\mathbf{F}_i})$ *for all i.*

Using this theorem, we produce a lot of examples of Gorenstein Rees algebras.

Example 5.2 (Villareal et al. [5]; Herzog et al. [10])**.** Let $\boldsymbol{F}$ be a bipartite graph, i.e., $|F| \leq 2$ for $F \in \mathbf{F}$. Then $R(I_{\mathbf{F}})$ is normal Gorenstein. □

For $1 \leq i \leq r$, we put $S_i = k[x_{i1}, \cdots, x_{in_i}]$ a polynomial ring and $\mathbf{F}_i \subset 2^{[n_i]}$. Let $S = S_1 \otimes_k \cdots \otimes_k S_r = k[\{x_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n_i\}]$ and $\mathbf{F} = \bigcup_{i=1}^r \mathbf{F}_i$. Then $I_{\mathbf{F}} = \bigcap_{i=1}^r I_{\mathbf{F}_i} S = (I_{\mathbf{F}_1} \cdots I_{\mathbf{F}_r})S$ and the diagonal subalgebra of the multi-Rees algebra $R(I_{\mathbf{F}_1} \cdots I_{\mathbf{F}_r})$ coincides with $R(I_{\mathbf{F}})$. Hence Theorem 5.1 is a direct consequence of the following.

Proposition 5.3. *Let* $I_i \subset S_i$ *be an ideal of* $\mathrm{ht}(I_i) \geq 2$ *contained in* $(x_{i1}, \cdots, x_{in_i})$ *and* $I = I_1 \cdots I_n S$. *The following conditions are equivalent*:

(1) $R(I)$ *is Cohen-Macaulay* (*resp.*, *Gorenstein*).

(2) $R(I_i)$ *is Cohen-Macaulay* (*resp.*, *Gorenstein*) *for* $1 \leq i \leq r$.

(3) $R(I_1, \cdots, I_r)$ *is Cohen-Macaulay* (*resp.*, *Gorenstein*).

Proof. Localizing $R(I)$ at $\{x_{ij} \mid 1 \leq i \neq k \leq r, 1 \leq j \leq n_i\}$, we have $R(I)[\{x_{ij}^{-1}\}_{1 \leq i \neq k \leq r,\, 1 \leq j \leq n_i}] \cong R(I_k S_k)[\{x_{ij}^{-1}\}_{1 \leq i \neq k \leq r,\, 1 \leq j \leq n_i}]$. Thus if $R(I)$ is Cohen-Macaulay (resp., Gorenstein), then so is $R(I_i)$ for $1 \leq i \leq r$. This prove the implication (1) $\Rightarrow$ (2). Also, since $R(I_1, \cdots, I_r) \cong R(I_1 S_1) \otimes_k \cdots \otimes_k R(I_r S_r)$, $R(I_1, \cdots, I_r)$ is Cohen-Macaulay (resp., Gorenstein) if and only if so is $R(I_i)$ for $1 \leq i \leq r$. This shows the equivalence of (2) and (3). When this the case, $R(I_{i_1}, \cdots, I_{i_p})$ is also Cohen-Macaulay (resp., Gorenstein) for $1 \leq i_1 < \cdots < i_p \leq r$.

Finally, we prove the implication (3) $\Rightarrow$ (1) by induction on r. Suppose that $R(I_1, \cdots, I_r)$ is Cohen-Macaulay (resp., Gorenstein). Then $R(I_1 \cdots I_{r-1})$ and $R(I_r)$ are Cohen-Macaulay (resp., Gorenstein) by induction hypothesis and (2). We apply the condition (3) for $I_1 \cdots I_{r-1}$ and I_r. Then the multi-Rees algebra $R(I_1 \cdots I_{r-1}, I_r)$ is Cohen-Macaulay (resp., Gorenstein). This complete the proof of (3) $\Rightarrow$ (1) by Corollary 3.5.

□

Example 5.4. It is well known that if $R(I)$ is Gorenstein and I is generically a complete intersection ideal of a Gorenstein local ring, then $\mathrm{ht}(I) = 2$.

Let X_i be a generic $k_i \times (k_i + 1)$ matrix over S_i for $1 \leq i \leq r$. Then, for $1 \leq i \leq r$, the ideal $I_{k_i}(X_i)$ of maximal minor is generated by d-sequences and $R(I_{k_i}(X_i)) \cong Sym(I_{k_i}(X_i))$ is complete intersection by [1], [12]. Hence, we have $R(I_{k_1}(X_1) \cdots I_{k_r}(X_r))$ is Gorenstein by Proposition 5.3. □

Example 5.5. Without the condition on $\mathrm{ht}(I)$ of the above example, the Gorensteinness of $R(I)$ is very complicated (see, for example, [14]). But, using Proposition 5.3, we are able to give Gorenstein Rees algebras of such ideals.

Let $\mathfrak{m}_i = (x_{i1}, \cdots, x_{in_i}) \subset S_i$ for $1 \leq i \leq r$. By the result of [19], $R(\mathfrak{m}_i^{n_i - 1} S_i)$ is Gorenstein for $1 \leq i \leq r$. It is easy to see that the height of $\mathfrak{m}_1^{n_1 - 1} \cdots \mathfrak{m}_r^{n_r - 1} \subset S$ is $\min\{n_1, \cdots, n_r\}$, and by Proposition 5.3, $R(\mathfrak{m}_1^{n_1 - 1} \cdots \mathfrak{m}_r^{n_r - 1})$ is Gorenstein. □

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