ON GORENSTEIN DIAGONAL SUBALGEBRAS

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Abstract

Let \( R \) be a multigraded ring over a local ring. In this paper, we study the Gorenstein property of a diagonal subalgebra of \( R \). Then we give a sufficient condition for this algebra to be Gorenstein. As an application, we apply our result to multi-Rees algebra and show that the diagonal subalgebra of a Gorenstein multi-Rees algebra is Gorenstein in the bigraded case.

1. Introduction

Let \( R = \bigoplus_{\alpha \in \mathbb{Z}^n} R_\alpha \) be a Noetherian \( \mathbb{Z}^n \)-graded ring. For a subgroup \( H \) of \( \mathbb{Z}^n \), we define \( R^{(H)} = \bigoplus_{\alpha \in H} R_\alpha \) and call it the diagonal subalgebra of \( R \) (w.r.t. \( H \)). In this paper, we are interested in the relationship between the Gorenstein property of \( R \) and that of its diagonal subalgebra \( R^{(H)} \).

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Many authors have been studied the Cohen-Macaulay property of a diagonal subalgebra and gave a necessary and sufficient condition of \( R^{(H)} \) to be Cohen-Macaulay in a limited extent [8, 9, 17, 21, 18]. But, in general, \( R^{(H)} \) is not Cohen-Macaulay (resp., Gorenstein), even if \( R \) is Cohen-Macaulay (resp., Gorenstein). The study of the Cohen-Macaulay property of \( R \) and \( R^{(H)} \) in general situation can be found in Hyry [13]. He have been studied the local cohomology modules of \( R \) and gave some condition for the Cohen-Macaulayness of \( R \) and \( R^{(H)} \).

Our aim of this paper is to give a sufficient condition for the Gorensteinness of \( R^{(H)} \) using some information of cosets of \( \mathbb{Z}^n/H \) similar to conic divisors of [3] and we give some applications of our result for multi-Rees algebras.

To describe our result, we give some notions as follows. Suppose that \( R = R_0[x_1, \ldots, x_m] \) for homogeneous elements \( x_1, \ldots, x_m \in R \) and \( R_0 \) is a local ring. For a \( \mathbb{Z}^n \)-graded \( R \)-module \( M \), we denote by \( \text{deg}(m) = \alpha \) for \( m \in M_\alpha \) and put \( \text{Deg}(M) = \{ \alpha \in \mathbb{Z}^n \mid M_\alpha \neq (0) \} \). Without loss of generality, we may assume that \( \mathbb{Z}^n = \mathbb{Z}\text{Deg}(R) \). Throughout this paper, we always assume that \( \text{Deg}(R) \) is positive, i.e., \( \text{Deg}(R) \cap -\text{Deg}(R) = (0) \). Under this assumption, if \( R \) is Gorenstein, then there exist unique integral vector \( \alpha \in \mathbb{Z}^n \) such that \( K_{R} \cong R(\alpha) \) ([4], [7]). We denote this vector \( \alpha \) by \( a(R) \).

Define a surjective semigroup homomorphism \( \phi : \mathbb{Z}^m \rightarrow \mathbb{Z}^n \) by \( \phi(e_i) = \text{deg}(x_i) \) for \( 1 \leq i \leq m \). For a subgroup \( G' \subset \mathbb{Z}^n \) and \( \gamma \in \mathbb{Z}^n \), we put \( H(G') = \mathbb{Z}_0^m \cap \phi^{-1}(G') \) and \( H(\gamma, G') = \mathbb{Z}_0^m \cap \phi^{-1}(\gamma + G') \), where \( \mathbb{Z}_0 \) is the set of non-negative integers. Finally, we put \( \text{supp}(H(G')) = \{ i \in [m] = \{1, \ldots, m\} \mid \alpha(i) > 0 \text{ for some } \alpha \in H(G') \} \). Then our main result describe as follows.
Theorem 3.1. Let $G'$ be a subgroup of $G$ such that $\dim(R^{(G')}) = \dim(R) - rk(\mathbb{Z}^n/\mathbb{Z}H(G'))$. Suppose that $\text{Deg}(K_R) \subset \mathbb{Z}_0^n$ and $\mathbb{Z}[H(-\alpha, G')]$ is a Cohen-Macaulay module for $\alpha \in \mathbb{Z}_0^n$ such that $\alpha \leq \sum_{i=1}^m \text{deg}(x_i)$. Then we have the following:

(1) If $R$ is Cohen-Macaulay, then so is $R^{(G')}$. 

(2) If $R$ is Gorenstein and $a(R) - \sum_{j \notin \text{supp}(H(G'))} \text{deg}(x_j) \in G'$, then so is $R^{(G')}$. 

Direct consequence of this result, we have the following corollary.

Corollary 4.2. Let $A$ be a Noetherian local ring. Let $I_1, I_2$ be ideals of positive height. If $R(I_1, I_2)$ is Cohen-Macaulay (resp., Gorenstein), then so is $R(I_1 I_2)$. 

By the same argument of Corollary 2.10 of Hyry [13], we have the following general statement.

Corollary 4.3 (Corollary 2.10 of Hyry [13]). Let $A$ be a Noetherian local ring. Let $I_1, \ldots, I_n$ be ideals of positive height. If $R(I_1, \ldots, I_n)$ is Cohen-Macaulay (resp., Gorenstein), then so is $R(I_1 \cdots I_n)$. 

As another application, we give an expected statement for some special case of polynomial ideals.

Proposition 5.3. Let $S_i = k[x_{i1}, \ldots, x_{in_i}]$ be a polynomial ring over a field $k$ for $1 \leq i \leq r$ and put $S = S_1 \otimes_k \cdots \otimes_k S_r$. Let $I_i \subset S_i$ be an ideal of $\text{ht}(I_i) \geq 2$ contained in $(x_{i1}, \ldots, x_{in_i})$ and $I = I_1 \cdots I_n S$. The following conditions are equivalent:
(1) \( R(I) \) is Cohen-Macaulay (resp., Gorenstein).

(2) \( R(I_i) \) is Cohen-Macaulay (resp., Gorenstein) for \( 1 \leq i \leq r \).

(3) \( R(I_1, \ldots, I_r) \) is Cohen-Macaulay (resp., Gorenstein).

2. Preliminary

In this section, we give some terminology and basic facts of \( G \)-graded rings from [4, 6, 7, 16] while will be used in this paper. Let \( G = \mathbb{Z}^n \) and \( R = \oplus_{a \in G} R_a \) be a Noether \( G \)-graded ring such that \( R_0 \) is local. For \( G' \) be a subgroup of \( G \), we define a diagonal subalgebra \( R^{(G')} \) of \( G' \) by \( R^{(G')} = \oplus_{a \in G'} R_a \) and put \( R^{(\gamma, G')} = \oplus_{\beta \in G'} R_{\gamma + \beta} \) for \( \gamma \in G \). Then \( R^{(G')} \) is a \( G' \)-graded ring and \( R^{(\gamma, G')} \) is a \( G' \)-graded \( R^{(G')} \)-module for \( \gamma \in G \). Furthermore, if \( \{ \gamma_i \}_{i} \) is a system of representatives of \( G / G' \), then \( R = \oplus_i R^{(\gamma_i, G')} \) and this decomposition gives a \( G/G' \)-graded structure of \( R \).

For \( \gamma \in G \) and \( M \) a \( G \)-graded \( R \)-module, we define a \( G \)-graded \( R \)-module \( M(\gamma) \) by \( M = M(\gamma) \) as the underlying \( R \)-module and is graded by \( [M(\gamma)]_\delta = M_{\gamma + \delta} \) for all \( \delta \in G \). The following formula is frequently used in this paper.

**Remark 2.1.** (1) If \( \gamma = \gamma_i + \delta \) for \( \delta \in G' \), then \( R^{(\gamma, G')} = R^{(\gamma_i, G')} R^{(\delta)} \).

(2) If \( \gamma + \gamma' = \gamma_i + \delta \) for \( \delta \in G' \), then

\[
R^{(\gamma, G')} = \bigoplus_{\delta \in G'} R_{\gamma + \gamma' + \delta} = R^{(\gamma_i, G')} (\delta).
\]
Suppose that $R = R_0[x_1, \ldots, x_m]$ for homogeneous elements $x_1, \ldots, x_m \in R$. Let $S_0$ be a regular local ring and $S = S_0[t_1, \ldots, t_m]$ be a polynomial ring over $S_0$ with $m$ variables. We set $\deg(t_i) = e_i' \in \mathbb{Z}^m$ for $1 \leq i \leq m$ and regards $S$ as a $\mathbb{Z}^m$-graded ring where $e_1', \ldots, e_m'$ are standard basis of $\mathbb{Z}^m$. We define $\varphi : \mathbb{Z}^m \to \mathbb{Z}^n$ by $\varphi(e_i') = \deg(x_i)$ for $1 \leq i \leq m$. Then $S$ is also regarded as a $G$-graded ring via $\varphi$.

For a subgroup $G' \subset G$, we put $L = \varphi^{-1}(G')$ and denote by $\{a_i\}_i$ a system of representatives of $\mathbb{Z}^m/L(\cong G/G')$. Clearly, we have $S^{(L)} = S_0[H(G')]$ and $S^{(a_i,L)} = S_0[H(\varphi(a_i), G')]$ for $a \in \mathbb{Z}^m$. A decomposition of a $\mathbb{Z}^m/L$-grading of $S$ is given by $S = \bigoplus_i S^{(a_i,L)} = \bigoplus_i S_0[H(\varphi(a_i), G')]$.

**Proposition 2.2.** $S_0[H(G')]$ is normal Cohen-Macaulay and if $H(-\varphi(a_i), G') \neq 0$, then $S_0[H(\varphi(a_i), G')]$ is divisorial for all $i$.

**Proof.** For $b \in \mathbb{Z}H(G')$, if $kb \in H(G')$, then $\varphi(b) \in G'$ and $kb \geq 0$. Thus $b \geq 0$ and $b \in H(G')$. Namely, $H(G')$ is a normal semigroup and $S_0[H(G')]$ is normal Cohen-Macaulay since $S_0$ is a regular domain.

Let $a = -a_i + b \in H(-\varphi(a_i), G')$ for some $b \in \mathbb{Z}H(G')$. Then the monomial $m := t^a$ is in $S$ and we have the exact sequence $0 \to S(-a^m) \to S$. Taking a degree $G'$-part of this sequence, we have $0 \to S_0 \to S_0[H(\varphi(a_i), G')](-b) \to S_0[H(G')]$. Thus $S_0[H(\varphi(a_i), G')]$ is a fractional ideal. Also, if we denote by $H^1$ the set of all height 1 graded prime ideals of $S_0[H(G')]$, then $S = \bigcap_{p \in H^1} S_{(p)} = \bigoplus_{p \in H^1} S_0[H(\varphi(a_i), G')]_{(p)}$. This implies that $S_0[H(G')]$ is divisorial. □

Next, let us recall some facts of normal semigroups and its semigroup rings from Goto and Watanabe [7], Stanley [22], and Isida [15].
Let $a$ be a $\mathbb{Z}^m$-graded ideal of $S_0[H(G')]$. We consider $aS = Q_1 \cap \cdots \cap Q_l$ the $\mathbb{Z}^m$-graded primary decomposition of $a$ in $S$ and put $\sqrt{Q_i} = P_i$ for $1 \leq i \leq l$. Then $Q_i$, $P_i$ are $\mathbb{Z}^m$-graded ideals, namely, monomial ideals of $S$. Since $H(G')$ is normal (i.e., full in sense of [11]), $a = aS \cap S_0[H(G')]$ and $a = \bigcap_{i=1}^l Q_i^{(G')} = \bigcap_{i=1}^l Q_i \cap S_0[H(G')]$ is a primary decomposition of $a$ in $S_0[H(G')]$ (not necessarily minimal). Also, we have $\sqrt{Q_i} \cap S_0[H(G')] = P_i \cap S_0[H(G')]$. In particular, if $a$ is prime, then $a = P_i \cap S_0[H(G')]$ for some $i$.

For $F \subset [m] = \{1, \ldots, m\}$, we set $P_F = (t_i \mid i \in F) \subset S$ a prime ideal of $S$ generated by variables. Then $\text{Spec}_{\mathbb{Z}^m}(S_0[H(G')])$ the set of all $\mathbb{Z}^m$-graded prime ideals of $S_0[H(G')]$ is given by $\text{Spec}_{\mathbb{Z}^m}(S_0[H(G')]) = \{p + P_F \cap S_0[H(G')] \mid p \in \text{Spec}(S_0) \text{ and } F \subset [m]\}$.

Put $p_F = P_F \cap S_0[H(G')]$ for $F \subset [m]$ and call it monomial prime of $S_0[H(G')]$. We define a equivalence relation $\sim$ on $\{F \subset [m]\}$ by $F \sim F'$ iff $p_F = p_{F'}$ and denote the equivalent class by $F(p) = \{F \subset [m] \mid p_F = p\}$ for a monomial prime $p \subset S_0[H(G')]$.

For $t^a \in S_0[H(G')]$, $t^a \in p_F$ if and only if $a(i) > 0$ for some $i \in F$ by definition. Here we denote $i$-th coordinate of $a$ by $a(i)$. Therefore, for all $F$, $F' \in F(p)$, $p_{F \cup F'} = (t^a \in S_0[H(G')] \mid a(i) > 0 \text{ for some } i \in F \cap F') = p_F + p_{F'} = p$. Thus $F(p)$ has the maximal element $F(p) := \bigcup_{F \in F(p)} F$. Note that if $p_F = (0)$, then $F \cap \text{supp}(H(G')) = \emptyset$ for $F \subset F(p)$.

Finally, with this notation, we can explain the difference between conic divisors in sense of [3] and $S_0[H(\varphi(a), G')]$. Let $H^1(H(G'))$ be a set of all $\mathbb{Z}^m$-graded prime ideal of $S_0[H(G')]$ of height 1 and $\text{Div}(H(G')) = \{...\}$.
\( \oplus_{p \in H^1(G')} \mathbb{Z}[p] \). We put \( e(p) = \sum_{i \in F(p) \cap \text{supp}(H(G'))} e'_i \in \mathbb{Z}^m \) for \( p \in H^1(H(G')) \).

Then we have the following commutative diagram:

\[
\begin{array}{c}
0 \rightarrow ZH(G') \xrightarrow{\text{div}} D(H(G')) \rightarrow Cl(S_0[H(G')]) \rightarrow 0 \\
0 \rightarrow \mathbb{Z}H(G') \rightarrow \mathbb{Z}^m \rightarrow G/\mathbb{Z}H(G') \rightarrow 0
\end{array}
\]

It follows that we have the exact sequence

\[
0 \rightarrow Cl(S_0[H(G')]) \rightarrow G/\mathbb{Z}H(G') \rightarrow \mathbb{Z}^m / \bigoplus_{p \in H^1(H(G'))} \mathbb{Z}e(p) \rightarrow 0.
\]

In the following, we describe the classical result of Gorenstein normal semigroup rings due to Stanley [22] in terms of \( H(\alpha, G') \).

**Proposition 2.3** (Stanley [22]).

(1) \( \text{rel.int}(H(G')) = \{ \beta \in H(G') \mid b(i) > 0 \text{ for } i \in \text{supp}(H(G')) \} \).

(2) \( K_{S_0[H(G')]} \cong S_0[H( - \sum_{i \in \text{supp}(H(G'))} \varphi(e'_i), G')] \) as underlying modules.

(3) \( S_0[H(G')] \) is Gorenstein if and only if \( \sum_{i \in \text{supp}(H(G'))} \varphi(e'_i) \in G' \).

**Proof.** Since \( S_0[H(G')] \subset S_0[t_i \mid i \in \text{supp}(H(G'))] \), we may assume that \( \text{supp}(H(G')) = [m] \). Then the face of \( H(G') \) is given by \( \{ b \in H(G') \mid b(i) = 0 \text{ for } i \in F \} \) for some \( F \subset [m] \) (Ishida [15]). The assumption \( \text{supp}(H(G')) = [m] \) implies that \( \text{rel.int}(H(G')) = H(G') \cap \mathbb{Z}^m_+ = \{ a \in H(G') \mid a(i) > 0 \text{ for all } i \in [m] \} \).

For \( a \in \text{rel.int}(H(G')) \), since \( a - \sum_{i=1}^{m} e'_i \in \mathbb{Z}_0^m \) and \( \varphi(a - \sum_{i=1}^{m} e'_i) \) is contained in \( -\sum_{i=1}^{m} \varphi(e'_i) + G' \), we have \( a - \sum_{i=1}^{m} e'_i \in H(- \sum_{i=1}^{m} \varphi(e'_i), G') \) and

\[
\text{rel.int}(H(G')) = H( - \sum_{i=1}^{m} \varphi(e'_i), G') + \sum_{i=1}^{m} e'_i.
\]
This conclude that \( K_{S_0[H(G^')] \cong S_0[rel.int(H(G^))]} \cong S_0[H(-\sum_{i=1}^m \varphi(e_i), G^')] \)
as underlying \( S_0[H(G^')] \)-modules. The assertion (3) is immediate consequence of this isomorphism.

3. Proof of Main Result

**Theorem 3.1.** Let \( G' \) be a subgroup of \( G \) such that \( \dim(R^{(G^')}) = \dim(R) - \rk(\mathbb{Z}^m / \mathbb{Z}H(G')) \). Suppose that \( R = R_0[x_1, \ldots, x_m] \) and \( \deg(x_1), \ldots, \deg(x_m) \in \mathbb{Z}^n_0 \). Moreover, we assume that \( \text{Deg}(K_R) \subset \mathbb{Z}^n_0 \) and \( \mathbb{Z}[H(-\alpha, G')] \) is a Cohen-Macaulay module over \( \mathbb{Z}[H(G')] \) for \( \alpha \in \mathbb{Z}^n_0 \) such that \( \alpha \leq \sum_{i=1}^m \deg(x_i) \). Then we have the following:

1. If \( R \) is Cohen-Macaulay, then so is \( R^{(G^')} \).
2. If \( R \) is Gorenstein and \( a(R) - \sum_{j \in \text{supp}(H(G'))} \deg(x_j) \in G' \), then so is \( R^{(G^')} \).

**Proof of Theorem 3.1.** We put \( \alpha_i = \deg(x_i) \) for \( 1 \leq i \leq m \) and \( \alpha = \sum_{i=1}^m \alpha_i \).

(1) Through the completion of \( R_0 \) at the maximal ideal, we may assume that \( R = S / I \) for some \( G \)-graded ideal \( I \subset S \). Let \( 0 \to F_p \to \cdots \to F_1 \to F_0 \) be a minimal \( G \)-graded \( S \)-free resolution of \( R \) and we put \( F_i = \oplus_j S(-\alpha_{ij}) \) for \( 0 \leq i \leq p \).

Assume that \( R \) is Cohen-Macaulay. Then \( \text{Hom}(\mathbb{F}, S(-\alpha)) \) gives the minimal \( G \)-graded \( S \)-free resolution of \( K_R \). Note that the i-th component of this resolution is described as \( F^*(-\alpha) = \oplus_j S(-\alpha + \alpha_{P-i}) \). On the other hand, since \( \text{Deg}(K_R) \subset \mathbb{Z}^n_0 \) and is Cohen-Macaulay, all betti degrees of this resolution are contained in \( \mathbb{Z}^n_0 \). Thus \( \sum_{i=1}^m \alpha_i - \alpha_{ij} \geq 0 \).
and $\alpha_{ij} \leq \sum_{i=1}^{m} a_i$. This implies that $F^{(G)}_i$ is the direct sum of 
\{ $S_0[H(- \alpha_{ij}, G')]$ \} for all $i$ and $S_0[H(- \alpha_{ij}, G')]$ is Cohen-Macaulay by 
our assumption. Hence we have the resolution

\[ 0 \to F^{(G)}_p \to \cdots \to F^{(G)}_1 \to F^{(G)}_0 \]

of $R^{(G)}$ consisting of maximal Cohen-Macaulay $S^{(G)}$-modules.

We put $B_i = \text{Im}(F_i \to F_{i-1})^{(G)}$ for $0 \leq i \leq p$ and $m' = \dim(S^{(G)})$. Then we have isomorphisms $H^i_m(R^{(G)}) \cong H^{i+1}_m(B_1) \cong H^{i+2}_m(B_2) \cong \cdots \cong H^{m'-1}_m(B_{m'-1-i})$ and, thus, $H^i_m(R^{(G)}) = 0$, if $p \leq m' - 1 - i$. Since $m' = \dim(S^{(G)}) = \dim(S_0) + rk(\mathbb{Z}H(G'))$, we have equalities

\[ m' - 1 - p = \dim(S^{(G)}) - \text{pd}_R(R) - 1 \]

\[ = \dim(S_0) + rk(\mathbb{Z}H(L)) - (\dim(S_0) + m - \text{depth}(R)) - 1 \]

\[ = \text{depth}(R) - rk(\mathbb{Z}^m / \mathbb{Z}H(L)) - 1 \]

\[ = \dim(R) - rk(\mathbb{Z}^m / \mathbb{Z}H(L)) - 1 \]

\[ < \dim(R^{(G)}). \]

Hence $H^i_m(R^{(G)}) = 0$ for $i < \dim(R^{(G)})$ and $R^{(G)}$ is Cohen-Macaulay.

(2) We put $\alpha' = \sum_{i \in \text{supp}(H(G'))} a_i$ and $\alpha'' = \sum_{j \in \text{supp}(H(G'))} a_j$. Suppose that $R$ is Gorenstein and $\alpha(R) - \alpha'' \in G'$. Since $R$ is Gorenstein, we have $F_p = S(- \sum_{i=1}^{m} a_i + \alpha(R)) = S(- \alpha' - \alpha'' + \alpha(R))$. Then, by Remark 2.1 (2) and Proposition 2.3 (2), $F^{(G)}_p \cong S_0[H(- \alpha', G')](\alpha(R) - \alpha'') \cong K_{S_0[H(G')]}. \]

Note that $p = \dim(S) - \dim(R) = \dim(S_0) + m - (\dim(R^{(G)}) - rk(\mathbb{Z}^m / \mathbb{Z}H(G'))) = \dim(S_0[H(G')]) - \dim(R^{(G)}) = \text{codim}(R^{(G)})$. Then, by the local duality theorem for maximal Cohen-Macaulay modules, we have
\[ K_{R(G')} \cong \Ext_{S_0[H(G')]}^p(R^{(G')}, K_{S_0[H(G')]}(B_1, K_{S_0[H(G')]})) \]
\[ \cong \cdots \]
\[ \cong \Ext_{S_0[H(G')]}^1(B_{p-1}, K_{S_0[H(G')]}). \]

It follows that we have the exact sequence
\[ \Hom_{S_0[H(G')]}(F^{(G')}_p, K_{S_0[H(G')]}(F^{(G')}_p, K_{S_0[H(G')]}(F^{(G')}_p, K_{S_0[H(G')]})), K_{R(G')} \to 0. \]

Since \( F^{(G')}_p \cong K_{S_0[H(G')]} \) as above, we have
\[ \Hom_{S_0[H(G')]}(F^{(G')}_p, K_{S_0[H(G')]}(K_{S_0[H(G')]}), K_{S_0[H(G')]}), \]
and is isomorphic to \( S_0[H(G')] \) as underling \( S_0[H(G')] \)-modules. This implies that \( K_{R(G')} \) is a cyclic module and \( R^{(G')} \) is Gorenstein. \( \square \)

The proof of Theorem 3.1 based on the following general statement.

**Lemma 3.2.** Let \((A, \mathfrak{m})\) be a \(d\)-dimensional Cohen-Macaulay ring with a canonical module \( K_A \) and \( M \) be a finitely generated \( A \)-module of \( \text{codim}(M) = s \). Suppose that there is a resolution
\[ 0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to M \to 0 \]
of \( M \) such that \( F_i \) is a maximal Cohen-Macaulay \( A \)-module for \( 0 \leq i \leq s \). Then \( M \) is a \((d-s)\)-dimensional Cohen-Macaulay module and has the exact sequence \( K_{F_{s-1}} \to K_{F_s} \to K_M \to 0 \).

**Proof.** The proof is similar to the above. Let \( B_i = \text{Im}[F_i \to F_{i-1}] \) for \( 0 \leq i \leq s \). We have isomorphisms \( H^i_m(M) \cong H^i_m(B_1) \cong \cdots \cong H^i_m(B_k) \) for \( i + k < d \). Thus, for \( i < d - s \), \( H^i_m(M) \cong H^{d-1}_{m-s}(B_{d-i-1}) = 0 \), since \( d - i - 1 \geq s \). Hence \( M \) is a \((d-s)\)-dimensional Cohen-Macaulay module.

In particular, since \( H^{d-s}_m(M) \cong H^{d-1}_{m-s}(B_{s-1}), K_M \cong \Ext^s_A(M, K_A) \cong H^{d-s}_m \)
As $K_B^\ast \cong \text{Ext}_A^1(B_{s-1}, K_A)$. Taking the canonical dual of the sequence $0 \to F_s \to F_{s-1} \to B_{s-1} \to 0$, we have the exact sequence $0 \to K_{B_{s-1}} \to K_{F_{s-1}} \to K_{F_s} \to K_M \to 0$. 

**Remark 3.3.** (1) In general, $H(\alpha, G')$ is not Cohen-Macaulay, if $R$ is Cohen-Macaulay.

(2) The proof of Theorem 3.1 tells us some information about depth($R^{(\gamma, G')}$), even if $R$ is not CM. Namely, if all components $F^{(\gamma, G')}$ is Cohen-Macaulay, then depth($R^{(\gamma, G')}$) $\geq$ depth($R$) $-$ rank($\mathbb{Z}^m / \mathbb{Z}H(L)$) for $\gamma \in G$.

In the following, we give two statements which satisfy the condition of Theorem 3.1.

**Corollary 3.4** ([16]). For a subgroup $G' \subset G$ such that $G / G'$ is torsion

(1) If $R$ is Cohen-Macaulay, then so is $R^{(G')}$.

(2) If $R$ is Gorenstein and $a(R) \in G'$ then so is $R^{(G')}$.

**Corollary 3.5** (Bruns and Guerrieri [2]; Bruns and Gubeladze [3]).

Let $S = S_0[x_1, \cdots, x_n, y_1, \cdots, y_m]$ be a $\mathbb{Z}^2$-graded polynomial ring of deg($x_i$) = $(1, 0)$, deg($y_j$) = $(0, 1)$ for all $i, j$ and $L = \mathbb{Z}(1, 1)$. Then $S_0[H(i, 0), L]$ and $S_0[H(0, j), L]$ are maximal Cohen-Macaulay modules for $0 \leq i < n$ and $0 \leq j < m$.

**4. Diagonal Subalgebras of Multi-Rees Algebras**

In this section, we apply Theorem 3.1 to multi-Rees algebras. Let $(A, m)$ be a Noetherian local ring and $I_1, \cdots, I_n$ ideals of $A$. We assume
that \( \text{grade}(I_i) > 0 \) for \( 1 \leq i \leq n \). The multi-Rees algebra of \( I_1, \ldots, I_n \) is
defined to be the subalgebra \( R(I_1, \ldots, I_n) = A[I_1s_1, \ldots, I_ns_n] \) of a
polynomial ring \( A[s_1, \ldots, s_n] \), where \( s_1, \ldots, s_n \) are in determinates. We
regards \( R(I_1, \ldots, I_n) \) as a \( \mathbb{Z}^n \)-graded ring by \( \deg(s_i) = e_i \in \mathbb{Z}^n \). We
denote by \( I_i = (x_{i1}, \ldots, x_{im_i}) \) for \( 1 \leq i \leq n \).

Define \( \varphi_i : \mathbb{Z}^{m_i} \to \mathbb{Z}^n \) by \( \varphi_i(e_{ij}) = \deg(x_{ij}s_i) = e_i \) for \( 1 \leq i \leq n \) and
\( 1 \leq j \leq m_i \) and \( \varphi = \sum_i \varphi_i : \mathbb{Z}^{m_1 + \cdots + m_n} \to \mathbb{Z}^n \). Let \( L = \mathbb{Z}(1, \ldots, 1) \subset \mathbb{Z}^n \).

Then the diagonal subalgebra of \( R(I_1, \ldots, I_n) \) of \( L \) coincides with the
usual Rees algebra \( R(I_1 \cdots I_n) \) and we are able to replace the statement
of Theorem 3.1 as follows.

**Proposition 4.1.** For \( 0 \leq \alpha \leq (m_1, \ldots, m_n) \), we suppose that \( \mathbb{Z}[H(-\alpha, L)] \) is Cohen-Macaulay. Then we have the following:

1. If \( R(I_1, \ldots, I_n) \) is Cohen-Macaulay, then so is \( R(I_1 \cdots I_n) \).
2. If \( R(I_1, \ldots, I_n) \) is Gorenstein, then so is \( R(I_1 \cdots I_n) \).

**Proof.** By results of [13], we have that \( \dim(R(I_1 \cdots I_n)) = \dim(A) + 1 \) and
\( \dim(R(I_1, \ldots, I_n)) = \dim(A) + n \). Thus \( \dim(R(I_1 \cdots I_n)) = \dim\)
\( (R(I_1, \ldots, I_n)) - (n-1) = \dim(R(I_1, \ldots, I_n)) - r(k(\mathbb{Z}^n)/L) \), where \( \mathbb{Z}^{m_1 + \cdots + m_n}
/ \mathbb{Z}H(L) \cong \mathbb{Z}^n/L \) via \( \varphi \). Also, we have \( \alpha(R(I_1, \ldots, I_n)) = (-1, \ldots, -1) \in L \) by [13] and \( \text{supp}(H(L)) = [m_1 + \cdots + m_n] \). This shows that a part of
condition of Theorem 3.1(2) is always satisfied. Moreover,
\( \text{Deg}(K_{R(I_1, \ldots, I_n)}) \subset \mathbb{Z}_0^n \), since \( \alpha(R(I_1, \ldots, I_n)) = (-1, \ldots, -1) \). Finally,
\( \sum_{i=1}^n \sum_{j=1}^{m_i} \deg(a_{ij}s_i) = (m_1, \ldots, m_n) \) and this conclude that conditions of
Theorem 3.1 are satisfied except for the Cohen-Macaulayness of
\( H(-\alpha, L) \). \( \square \)
Now, we assume that \( n = 2 \). Then it is easy to see that the representatives of \( \mathbb{Z}^2 / L \) is given by \( \{(i, 0), (0, j) | 1 \leq i \leq m_1, 1 \leq j \leq m_2\} \).

As a direct consequence of the above theorem and Corollary 3.5, the Cohen-Macaulay property and the Gorenstein property of \( R(I_1I_2) \) follows from these of \( R(I_1, I_2) \) without any other conditions.

**Corollary 4.2.** If \( R(I_1, I_2) \) is Cohen-Macaulay (resp., Gorenstein), then so is \( R(I_1I_2) \).

\( \square \)

**Corollary 4.3.** If \( R(I_1, \ldots, I_n) \) is Cohen-Macaulay (resp., Gorenstein), then so is \( R(I_1 \cdots I_n) \).

**Proof.** The proof is the same as the proof of Corollary 2.10 of Hyry [13] and use induction on \( n \).

We put \( B = R(I_1, \ldots, I_{n-2}) \) and denote \( \mathfrak{N} \) a graded maximal ideal of \( B \). Then \( R_{\mathfrak{N}}(I_{n-1}B, I_nB) = \mathcal{R}_{\mathfrak{N}}(I_1, \ldots, I_n) \) is Cohen-Macaulay (resp., Gorenstein). By Corollary 4.2, this implies that \( R(I_1, \ldots, I_{n-2}, I_{n-1}I_n) \) is also Cohen-Macaulay (resp., Gorenstein) and so is \( R(I_1, \ldots, I_{n-2}, I_{n-1}I_n) \). Hence, by induction hypothesis, \( R(I_1 \cdots I_n) \) is Cohen-Macaulay (resp., Gorenstein).

\( \square \)

As another application of Proposition 4.1, we have the following.

**Proposition 4.4.** If \( R(I_1, \ldots, I_n) \) is Cohen-Macaulay, then so is \( R(I_1i_1, \ldots, I_p) \) for \( 0 < p \) and \( 1 \leq i_1 < \cdots < i_p \leq n \).

**Proof.** Suppose that \( R(I_1, \ldots, I_n) \) is Cohen-Macaulay. To prove this proposition, it is enough to show that \( R(I_1, \ldots, I_{n-1}) \) is Cohen-Macaulay. Let \( M = \{(a, 0) | a \in \mathbb{Z}^{n-1}\} \) be the subgroup of \( \mathbb{Z}^n \). Then \( H(M) = \mathbb{Z}^m_{0, \ldots, 0} \supset \mathbb{Z}^m_{0, \ldots, 0} \) and \( H((0, \ldots, 0, p), G') = \{(a, b) | \)}
In the previous section, the converse of statements of Corollary 4.2 (or Theorem 3.1) is not true in general. Namely, the Cohen-Macaulayness of \( R(I_1 \cdots I_n) \) does not imply the Cohen-Macaulayness of \( R(I_1, \cdots, I_n) \), even if \( R(I_i) \) are assumed to be Cohen-Macaulay for \( 1 \leq i \leq n \). In this section, we discuss the expected implications of the Cohen-Macaulay and the Gorenstein property among the multi-Rees algebras.

Let \( S = k[x_1, \cdots, x_n] \) be a polynomial ring over the field \( k \). For \( F \subset [n] := \{1, \cdots, n\} \), we denote by \( p_F = (x_i \mid i \in F) \) and by \( x_F = \prod_{i \in F} x_i \). For \( F \subset 2^{[n]} \), we define \( I_F = \bigcap_{F \subset p F} p_F \). In particular, for a simplicial complex \( \Delta \) on \([n]\), if we denote by \( F(\Delta)^c = \{ F^c \mid F \in F(\Delta) \} \), then the Stanley-Reisner ring \( k[\Delta] = S / I(\Delta) \) is described as \( I(\Delta) = I_{F(\Delta)^c} \). In this section, we consider the Rees algebra \( R(I_F) \) and by the argument of multi-Rees algebras, we have the following.

**Theorem 5.1.** If \( F = \bigsqcup_i F_i \) decompose into connected components, then \( R(I_F) \) is Cohen-Macaulay iff so is \( R(I_{F_i}) \) for all i.
Using this theorem, we produce a lot of examples of Gorenstein Rees algebras.

**Example 5.2** (Villareal et al. [5]; Herzog et al. [10]). Let $F$ be a bipartite graph, i.e., $|F| \leq 2$ for $F \in \mathcal{F}$. Then $R(I_F)$ is normal Gorenstein.

For $1 \leq i \leq r$, we put $S_i = k[x_{i1}, \ldots, x_{in_i}]$ a polynomial ring and $F_i \subset 2^{[n_i]}$. Let $S = S_1 \otimes_k \cdots \otimes_k S_r = k[[x_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n_i]]$ and $F = \bigcup_{i=1}^{r} F_i$. Then $I_F = \bigcap_{i=1}^{r} I_{F_i} S = (I_{F_1} \cdots I_{F_r}) S$ and the diagonal subalgebra of the multi-Rees algebra $R(I_{F_1} \cdots I_{F_r})$ coincides with $R(I_F)$. Hence Theorem 5.1 is a direct consequence of the following.

**Proposition 5.3.** Let $I_i \subset S_i$ be an ideal of $ht(I_i) \geq 2$ contained in $(x_{i1}, \ldots, x_{in_i})$ and $I = I_1 \cdots I_n S$. The following conditions are equivalent:

1. $R(I)$ is Cohen-Macaulay (resp., Gorenstein).
2. $R(I_r)$ is Cohen-Macaulay (resp., Gorenstein) for $1 \leq i \leq r$.
3. $R(I_1, \ldots, I_r)$ is Cohen-Macaulay (resp., Gorenstein).

**Proof.** Localizing $R(I)$ at $\{x_{ij} \mid 1 \leq i \neq k \leq r, 1 \leq j \leq n_i\}$, we have $R(I)[\{x_{ij}^{-1} \mid 1 \leq i \neq k \leq r, 1 \leq j \leq n_i\}] \cong R(I_k S_k)[\{x_{ij}^{-1} \mid 1 \leq i \neq k \leq r, 1 \leq j \leq n_i\}]$. Thus if $R(I)$ is Cohen-Macaulay (resp., Gorenstein), then so is $R(I_i)$ for $1 \leq i \leq r$. This prove the implication (1) $\Rightarrow$ (2). Also, since $R(I_1, \ldots, I_r) \cong R(I_1 S_1) \otimes_k \cdots \otimes_k R(I_r S_r)$, $R(I_1, \ldots, I_r)$ is Cohen-Macaulay (resp., Gorenstein) if and only if so is $R(I_i)$ for $1 \leq i \leq r$. This shows the equivalence of (2) and (3). When this the case, $R(I_{i_1}, \ldots, I_{i_p})$ is also Cohen-Macaulay (resp., Gorenstein) for $1 \leq i_1 < \cdots < i_p \leq r$. 

Finally, we prove the implication (3) ⇒ (1) by induction on \( r \). Suppose that \( R(I_1, \ldots, I_r) \) is Cohen-Macaulay (resp., Gorenstein). Then \( R(I_1 \cdots I_{r-1}) \) and \( R(I_r) \) are Cohen-Macaulay (resp., Gorenstein) by induction hypothesis and (2). We apply the condition (3) for \( I_1 \cdots I_{r-1} \) and \( I_r \). Then the multi-Rees algebra \( R(I_1 \cdots I_{r-1}, I_r) \) is Cohen-Macaulay (resp., Gorenstein). This complete the proof of (3) ⇒ (1) by Corollary 3.5.

\( \square \)

Example 5.4. It is well known that if \( R(I) \) is Gorenstein and \( I \) is generically a complete intersection ideal of a Gorenstein local ring, then \( \text{ht}(I) = 2 \).

Let \( X_i \) be a generic \( k_i \times (k_i + 1) \) matrix over \( S_i \) for \( 1 \leq i \leq r \). Then, for \( 1 \leq i \leq r \), the ideal \( I_{k_i}(X_i) \) of maximal minor is generated by \( d \)-sequences and \( R(I_{k_i}(X_i)) \cong \text{Sym}(I_{k_i}(X_i)) \) is complete intersection by [1], [12]. Hence, we have \( R(I_{k_1}(X_1) \cdots I_{k_r}(X_r)) \) is Gorenstein by Proposition 5.3.

\( \square \)

Example 5.5. Without the condition on \( \text{ht}(I) \) of the above example, the Gorensteinness of \( R(I) \) is very complicated (see, for example, [14]). But, using Proposition 5.3, we are able to give Gorenstein Rees algebras of such ideals.

Let \( m_i = (x_{i1}, \ldots, x_{in_i}) \subset S_i \) for \( 1 \leq i \leq r \). By the result of [19], \( R(m_i^{n_i-1}S_i) \) is Gorenstein for \( 1 \leq i \leq r \). It is easy to see that the height of \( m_1^{n_1-1} \cdots m_r^{n_r-1} \subset S \) is \( \min\{n_1, \ldots, n_r\} \), and by Proposition 5.3, \( R(m_1^{n_1-1} \cdots m_r^{n_r-1}) \) is Gorenstein.

\( \square \)
References

