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IS THE CONVERSE CLAIM TRUE? AND IF IT IS, HOW IS IT PROVEN?

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Abstract

Theorems and claims in Mathematics in general, and in geometry in particular, are an inseparable part of the material and of the mathematical education imparted to high school students. Proving claims is a central component in geometry classes. Students acquire knowledge on the properties of shapes and bodies and make use of these properties in proving different claims. For each claim, one can formulate the converse claim, which in some cases may be right and in others not. Teachers do not always make use of the opportunity to formulate with the students the converse claim and to examine its truth by

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achieving a proof or finding an example to the contrary if it is not true. The present paper presents eight interesting mathematical claims in geometry, were for each claim and converse claim complete proofs are given using different mathematical tools, including indirect proofs. Teachers can make use of such and other examples to expose the students to the claim and the converse claim and to the methods for dealing with various methods of proof and of preventing mistakes.

1. Introduction

Since the eighties of the last century, methodological aspects of solving mathematical problems moved towards adding a style of question - “the art of asking questions”, such as: overloading data – what if more (WIM)?, replacing data – what if instead (WII)?, removing data – what if not (WIN)? [1-5], general case – is generalization possible? It is recommended to suggest to the students after proving a theorem or solving an interesting question, to ask whether the converse claim is true.

It is important to emphasize, particular in Euclidean geometry, that the concepts: direct and converse theorems are interrelated as clearly cited in [6].

Converse of a theorem- a theorem obtained by reversing the roles of premise and conclusion of the initial (direct) theorem. Any theorem is the converse of its converse. Thus, a theorem and its converse are converses of each other. In general, the validity of a theorem does not imply the validity of its converse. The means used to prove a theorem may be insufficient to prove its valid converse. If a theorem and its converse are valid, then the premise of either theorem is not only sufficient but also necessary for the validity of the conclusion.

In the methodological process, it is very important to strengthen the relationship between them and also to explain the concepts: *necessary and sufficient conditions* or “*if*” and “*if and only if*”, in order to prevent students further mistakes.

Extensively on these subjects, with a lot of examples, can be found in books [7, 8] and also in a research [9].

In the present paper, we present eight opportunities, beyond the basic ones, which we as teachers and educators in mathematics, may use in order to enrich knowledge and to increase the drive and the enjoyment from learning mathematics.

The following is a list of the presented cases:

First case: Midline in the trapezoid.

Second case: The Pythagorean theorem.

Third case: An equivalent theorem of the Pythagorean theorem.

Fourth case: A generalization of the Pythagorean theorem for parallelograms.

Fifth date: A property of the rectangle.

Sixth case: Point in an equilateral triangle.

Seventh case: Fixed sum of distances inside a quadrilateral.

Eighth case: The relation between the radius of the incircle and the radius of the circumcircle of a triangle.

2. Methods of Proof for the Different Cases

The methods of proof for the different cases are mainly based on the use of *Euclidean geometry*, and in some cases on the use of trigonometry. It should be noted here that there are mathematical tasks that can be solved using different methods either by mathematical tools from the same field or using tools from another field, or by combining tools from several fields. The more extensive the "toolbox" available to the student, the higher is their chance to deal with the mathematical tasks and their ability to find a solution using the shortest and simplest method. By employing a large variety of mathematical tools, one may discover unorthodox solutions or proofs which accentuate the beauty of mathematics and increase motivation and enjoyment from teaching and from the study of the subject [10, 11].

The theorem and the converse theorem

From a methodical viewpoint, it is important that teachers stress the fact that the direct theorem and the converse theorem are not always satisfied together. As in the first case that will be presented concerning the midline in the trapezoid, it must be stressed that "a segment drawn from the middle of the side of the triangle to the second side, and which equals half of the third side" is not necessarily a midline, and therefore cannot be relied upon when proving parallelism. It was found that many students (about 40%), and even a small part of the teachers, think that the converse theorem is always true. This requires us to make the students used to the fact that the converse theorem is not always true, and if one wants to use it, one must make sure it is true. It must also be stressed to the students that even in the cases when the direct theorem and the converse theorem are true, sometimes the proof of the theorems in both directions is easy and sometimes the proof in one direction is simple and the proof in the other direction is difficult. For example, the converse theorem: "*A triangle in which the lengths of the two angle bisectors are equal is an isosceles triangle*" is known in the literature as the *Steiner-Lehmus theorem*. This theorem is also called the "*Internal bisectors problem*" and "*Lehmus' theorem*".

Since the first proof by Steiner and Lehmus, dozens of different proofs for this theorem have been suggested both using geometrical tools and using tools from different fields of mathematics or by a combination between different tools [12, 13]. This theorem is an example of a theorem and a converse theorem both of which are true, but where the proof in one direction is simple and immediate, while the proof in the other direction is much more difficult.

First case: Midline in the trapezoid

Direct claim: If MN is a midline in the trapezoid $ABCD$, there holds $MN = \frac{AB + DC}{2}$ (see Figure 1).

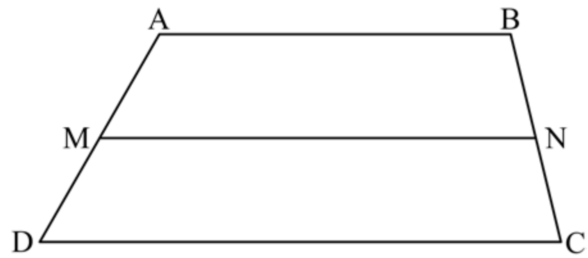

Figure 1.

Converse claim: If in the quadrilateral *ABCD*, *M* is the middle of *AD* and *N* is the middle of *BC*, and there holds $MN = \frac{AB + DC}{2}$, then *ABCD* is a trapezoid $(AB \| CD)$, as shown in Figure 2.

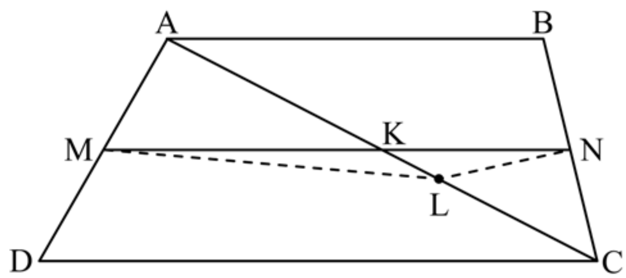

Figure 2.

Proof. We draw the diagonal *AC* and mark by *K* the intersection of *AC* and *MN*. If *AB* is not parallel to *CD*, then the point *K* is not on the middle of *AC*, because otherwise $KM \| DC$ and $KN \| AB$, in which case *AB* is parallel to *DC*. Therefore, we denote by *L* the middle of *AC*.

There holds: $\frac{1}{2} AB = LN$, $\frac{1}{2} DC = ML$.

Therefore: $\frac{AB + DC}{2} = ML + LN$, but also: $\frac{AB + DC}{2} = MN$, which is a contradiction because $ML + LN > MN$. In the other words, the assumption that *AB* is not parallel to *CD* leads to a contradiction. Therefore, $AB \| CD$.

In this activity significant added knowledge was acquired, unnoticed.

In the quadrilateral $ABCD$, there always holds $MN \leq \frac{AB + DC}{2}$, and the quantity holds if and only if $AB \| CD$.

Second case: The Pythagorean theorem

The direct claim: In the right-angled triangle ABC $(\angle A = 90^\circ)$, there holds $AB^2 + AC^2 = BC^2$, as shown in Figure 3.

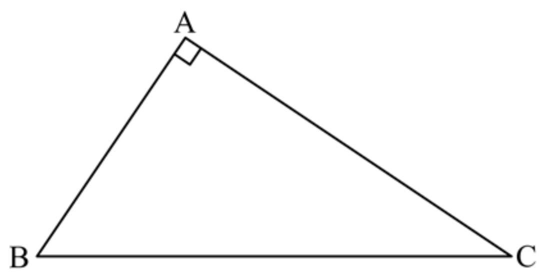

Figure 3.

The converse claim: If in the triangle ABC there holds $AB^2 + AC^2 = BC^2$, then the angle $\angle A$ is a right angle.

Note. usually, when the theorem is proven for students at the age 14-15, the teacher notes that the converse is also true, but the proof would be given upon acquiring proper knowledge in trigonometry – the Law of Cosines.

Proof (without trigonometry).

If $BC^2 = AB^2 + AC^2$, then $\angle A$ is the largest angle in the triangle ABC. We draw the angle $\angle CAE = \angle B$ and $\angle BAD = \angle C$, as shown in Figure 4.

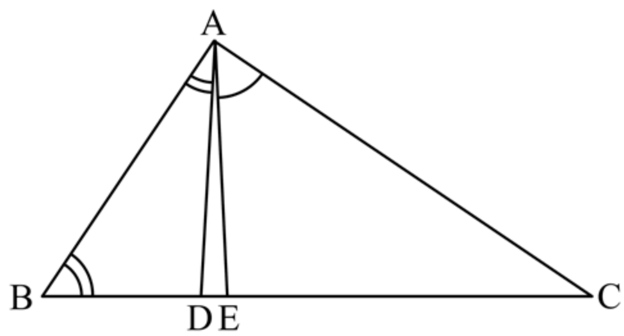

Figure 4.

$$\begin{array}{ccccc} \Delta EAC & \sim & \Delta ABC & \sim & \Delta DBA \\ & \downarrow & & \downarrow & \\ \end{array}$$

$$AC^2 = EC \cdot BC \qquad AB^2 = BD \cdot BC$$

Therefore, $AB^2 + AC^2 = BC(EC + BD)$.

However, it is given that $BC^2 = AB^2 + AC^2$, therefore E and D coincide.

Hence, $2 \cdot \angle B + 2 \cdot \angle C = 180^\circ$, and it follows that $\angle A = 90^\circ$.

Notes: In Euclid's "Elements" the Pythagorean theorem and the converse theorem were of course proven without the use of trigonometry.

Third case: Theorem equivalent to the Pythagorean theorem

The direct claim: ABC is a right-angled triangle $(\angle A = 90^\circ)$. The point M is located on the hypotenuse BC. Then there holds $AB^2 \cdot MC^2 + AC^2 \cdot BM^2 = AM^2 \cdot BC^2$, see Figure 5.

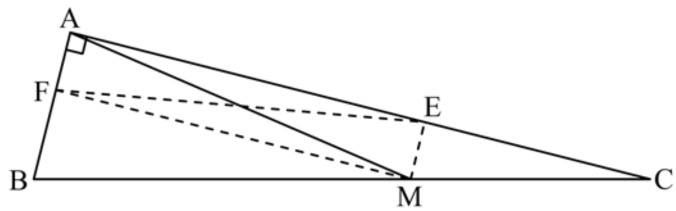

Figure 5.

Proof. We draw $ME \perp AC$ and $MF \perp AB$, therefore $AEMF$ is a rectangle; $ME \| AB$ and $MF \| AC$. From Thales' theorem, there holds: $\frac{ME}{AB} = \frac{MC}{BC}$, therefore $ME = \frac{AB \cdot MC}{BC}$.

In the same manner, $MF = \frac{AC \cdot BM}{BC}$.

It is also clear that $AM^2 = EF^2 = ME^2 + MF^2$.

Therefore, $AM^2 = \frac{AB^2 \cdot MC^2 + AC^2 \cdot BM^2}{BC^2}$, and the claim is proven.

The converse claim: In a triangle ABC, if the point M is located on BC, and there holds: $AB^2 \cdot MC^2 + AC^2 \cdot BM^2 = AM^2 \cdot BC^2$, then the angle A is a right angle (see Figure 6).

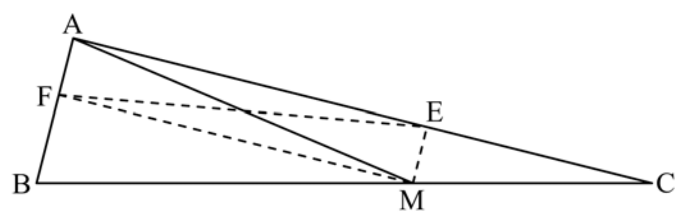

Figure 6.

Proof. We construct $ME \| AB,\ MF \| AC$.

Hence, it follows that $ME^2 = \dfrac{AB^2 \cdot MC^2}{BC^2}$ and $MF^2 = \dfrac{AC^2 \cdot BM^2}{BC^2}$,

in the other words (from the data), $ME^2 + MF^2 = \dfrac{AB^2 \cdot MC^2}{BC^2} + \dfrac{AB^2 \cdot MC^2}{BC^2}$

$= AM^2$, and hence:

$$ME^2 + MF^2 = AM^2 \Rightarrow AF^2 + MF^2 = AM^2 \Rightarrow \angle AFM = 90^\circ.$$

In the other words, a rectangle is obtained, and it follows that $\angle A = 90^\circ$.

Fourth case: A generalization of the Pythagorean theorem for parallelograms

The direct claim: In a parallelogram the sum of the squares of the sides equals the sum of the squares of its diagonals, see Figure 7.

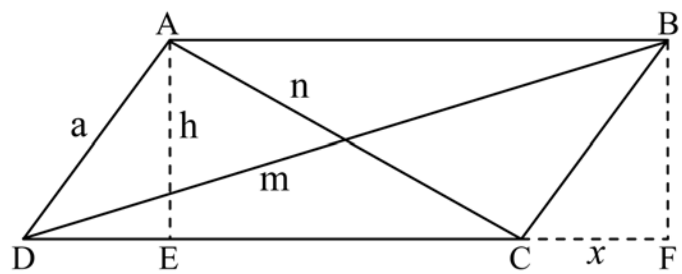

Figure 7.

Note. If the parallelogram is a rectangle, this is exactly the Pythagorean theorem.

Note. This question usually appears during the studies of trigonometry. Here we present a geometric proof.

Proof. We draw $AE \perp DC$ and $BF \perp DC$, and denote:

$$AE = BF = h,\ DE = CF = x,\ AC = n,\ DB = m,\ AB = DC = b,\ AD = BC = a.$$

From the Pythagorean theorem, we have

$$\begin{array}{rl} & n^2 = (b-x)^2 + h^2 \\ + & \\ & \underline{m^2 = (b+x)^2 + h^2} \end{array}$$

$$m^2 + n^2 = 2b^2 + 2x^2 + 2h^2.$$

Therefore, $m^2 + n^2 = 2b^2 + 2a^2$.

The converse claim: If in a convex quadrilateral the sum of the squares of the sides equals the sum of the squares of the diagonals, the quadrilateral is a parallelogram.

The proof is complex and it was suggested by *Euler*, who was the most prolific mathematician throughout all history of mathematics, and who set the way in many fields of "higher" mathematics, but also worked in elementary mathematics. Euler proved the following theorem:

Euler's theorem: In a quadrilateral $ABCD$, M and N are the middle points of the diagonals BD and AC, respectively. Then there holds:

$$AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2 + 4 \cdot MN^2.$$

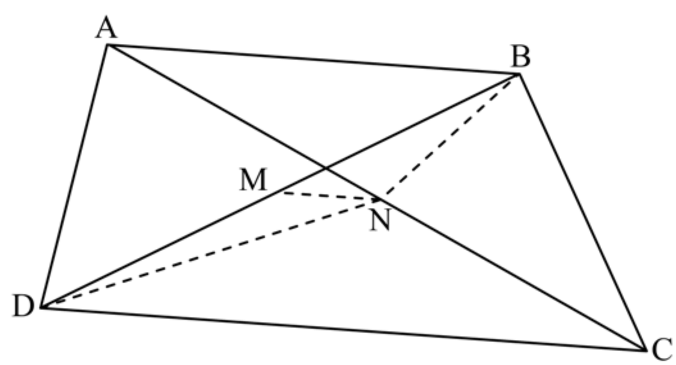

Figure 8.

Euler's theorem proved the converse claim, because if it is given that the sum of the squares of the sides equals the sum of the squares of the diagonals, then $MN = 0$, the diagonals bisect each other *and the quadrilateral is a parallelogram*.

Following is the proof of Euler's theorem.

In the triangle ABC, BN is a median to AC, therefore there holds:

$$(1) \quad BN^2 = \frac{2AB^2 + 2BC^2 - AC^2}{4}.$$

In the triangle ADC, DN is a median to AC, therefore there holds:

$$(2) \quad DN^2 = \frac{2AD^2 + 2DC^2 - AC^2}{4}.$$

In the triangle DNB, MN is a median to DB, therefore there holds:

$$(3) \quad MN^2 = \frac{2BN^2 + 2DN^2 - BD^2}{4}. \qquad (3)$$

We substitute BN^2 from (1) and DN^2 from (2) into (3), and after collecting similar terms we obtain *Euler's formula* (see also [14]).

Fifth case: A property of the rectangle

The direct claim: $ABCD$ is a rectangle and M is a point in it or on its perimeter. Then there holds: $AM^2 + MC^2 = BM^2 + MD^2$, see Figure 9.

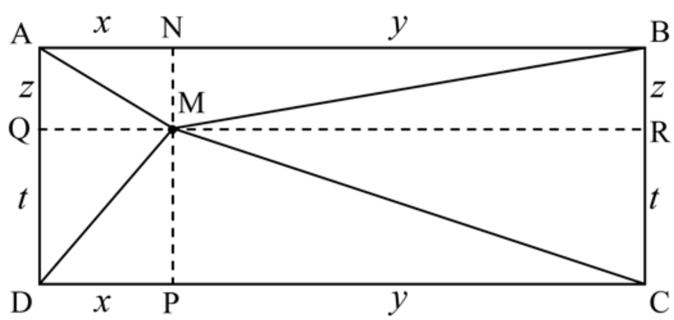

Figure 9.

Proof. We draw a parallel to AD and a parallel to AB through M, and obtain the points N, P, Q, R (shown in Figure 9).

We denote:

$$QD = RC = t, \quad AQ = BR = z, \quad NB = PC = y, \quad AN = DP = x.$$

Then from the Pythagorean theorem, it is clear that:

$$x^2 + y^2 + z^2 + t^2 = AM^2 + MC^2 = DM^2 + MB^2.$$

The converse claim: If in a convex quadrilateral $ABCD$, any point M inside it or on its perimeter satisfies $AM^2 + MC^2 = BM^2 + MD^2$, then the quadrilateral is a *rectangle*, see Figure 10.

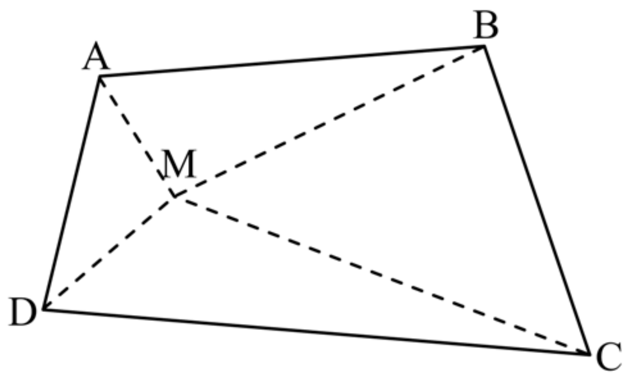

Figure 10.

Proof. The point A satisfies the given property, therefore:

$$(1) \qquad AC^2 = AB^2 + AD^2.$$

Point B satisfies: (2) $BD^2 = AB^2 + BC^2.$

Point C satisfies: (3) $AC^2 = BC^2 + CD^2.$

Point D satisfies: (4) $BD^2 = AD^2 + DC^2.$

From (1) and (3), it follows that:
From (2) and (4), it follows that:
$$\begin{cases} BC^2 + CD^2 = AB^2 + AD^2. \\ AB^2 + BC^2 = AD^2 + DC^2. \end{cases}$$

After subtracting these equalities, we obtain: $CD^2 - AB^2 = AB^2 - CD^2.$

Therefore $AB = CD$. Also, by same method from (1) and (2) and from (3) and (4), the result is $BC = AD$. Therefore, so far $ABCD$ is a parallelogram.

But from (1): $AC^2 = AB^2 + AD^2 = AB^2 + BC^2$, therefore $\angle B = 90^\circ$ and $ABCD$ is a rectangle.

Sixth case: An important property of points in an equilateral triangle

The direct claim: ABC is an equilateral triangle; M is an internal point or a point on the perimeter of the triangle. Then the sum of the distances of the point M from the sides of the triangle is fixed (and is equal to the altitude of the triangle). See Figure 11.

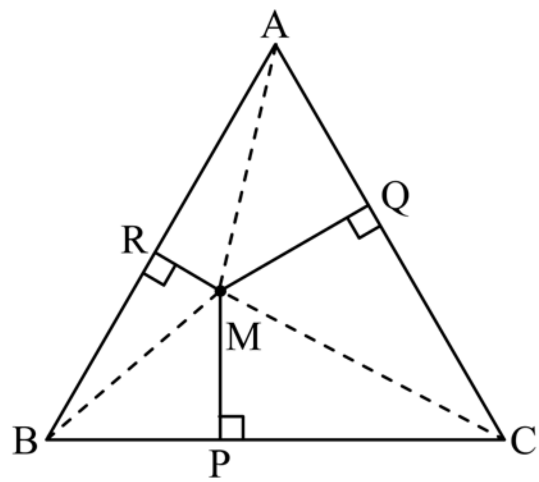

Figure 11.

The proof is well-known.

We denote by a the side length of the triangle, and by h its altitude.

$$S_{\Delta ABC} = \frac{a \cdot MP}{2} + \frac{a \cdot MQ}{2} + \frac{a \cdot RM}{2} = \frac{a \cdot h}{2}.$$

Therefore, $MP + MQ + MR = h$.

(which does not depend on the location of M).

The converse claim: If in the triangle *ABC* the sum of the distances of the point *M* (internal or on the perimeter of the triangle) from the sides of the triangle is fixed (i.e., does not depend on the location of the point *M*), then the triangle is equilateral, see Figure 12.

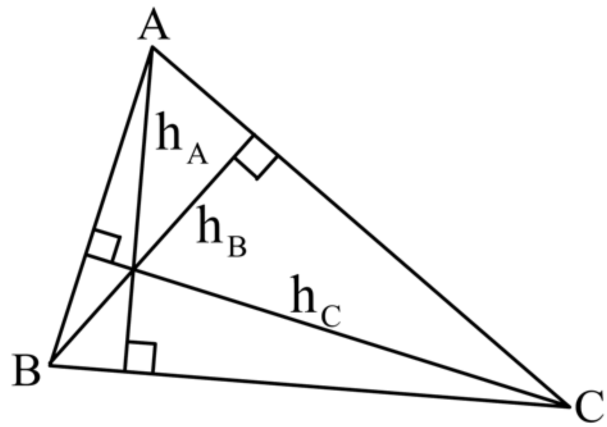

Figure 12.

Proof. We denote the altitudes of the triangle by h_A, h_B, and h_C. The sum of the distances of the point *B* from the sides of the triangle is h_B. The sums of the distances of *C* and A from the sides of the triangle are h_C and h_A. If this sum is fixed, then $h_A = h_B = h_C$, and hence the triangle is *equilateral*.

Seventh case: Fixed sum of distances from points in a quadrilateral to its sides

In the previous section (the sixth case), it was found that *a necessary and sufficient condition* for the sum of some inner point from the sides of the triangle to be fixed (irrespective of the selection of the point's location) is that the triangle is equilateral.

The question is asked whether a fixed sum of distances from the sides of a convex polygon requires the polygon to be perfect? The answer is negative. It is easy to show that if the sides of a polygon are equal, the property holds and the fixed sum is the ratio between the area of the polygon and half of its perimeter.

In the book "Problem-Solving Strategies" [15], it was proven that if the angles of the polygon are equal – the property is satisfied.

Is it necessary for either the sides or the angles to be equal for the property to hold? Again, the answer is negative.

A parallelogram is a simple example for a quadrilateral not all of whose sides and not all of whose angles are equal, in which the property holds.

The direct claim: In a parallelogram $ABCD$, let M be an interior point or a point on its sides. Then the sum of the distances of the point from the sides of the parallelogram is fixed and equals the sum of the two heights of the parallelogram, see Figure 13.

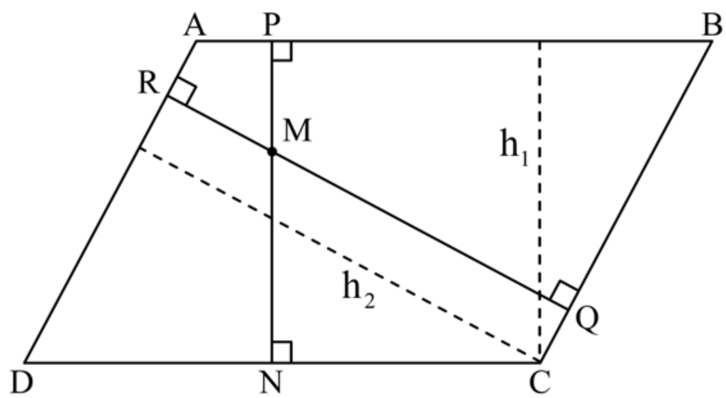

Figure 13.

Proof. Immediate: The points P, M, N are collinear, and the points Q, M, R are also collinear, therefore: $h_1 = MP + MN$, $h_2 = MQ + MR$.

The converse claim: If in a convex quadrilateral the sum of the distances from some point (interior or on the sides of the quadrilateral) to the sides of the quadrilateral is *fixed*, then the quadrilateral is a parallelogram.

Proof. We denote the sides of the quadrilateral by the letters a, b, c, d (see Figure 14), and the sum of the distances from each point in the quadrilateral or on its sides to the sides by K.

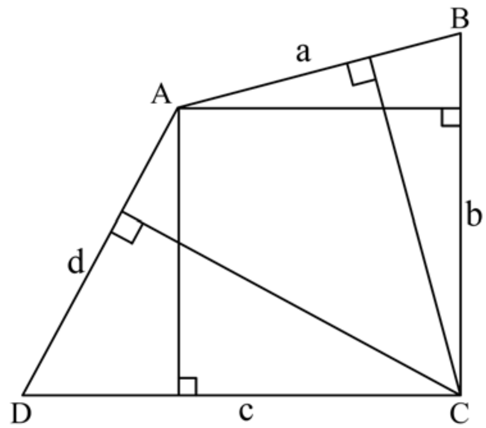

Figure 14.

For the vertex A there holds: (1) $a \sin \angle B + d \sin \angle D = K$.

For the vertex C there holds: (2) $b \sin \angle B + c \sin \angle D = K$.

For the vertex B there holds: (3) $b \sin \angle C + a \sin \angle A = K$.

For the vertex D there holds: (4) $c \sin \angle C + d \sin \angle A = K$.

We express $\sin \angle B$ and $\sin \angle D$ using a, b, c, d and K.

In the systems (1) and (2), we obtain: I $\sin \angle B = \dfrac{K(d - c)}{bd - ac}$;

II $\sin \angle D = \dfrac{K(b - a)}{bd - ac}$.

We express $\sin \angle C$ and $\sin \angle A$ using a, b, c, d and K.

In the systems (3) and (4), we obtain: III $\sin \angle A = \dfrac{K(b - c)}{bd - ac}$;

IV $\sin \angle C = \dfrac{K(d - a)}{bd - ac}$.

Since $\sin \angle B + \sin \angle D = \sin \angle A + \sin \angle C$, we obtain that

$$2 \sin \frac{\angle B + \angle D}{2} \cos \frac{\angle B - \angle D}{2} = 2 \sin \frac{\angle A + \angle C}{2} \cos \frac{\angle A - \angle C}{2},$$

but $\angle A + \angle B + \angle C + \angle D = 360^\circ$, therefore, $\sin\dfrac{\angle B + \angle D}{2} = \sin\dfrac{\angle A + \angle C}{2}$

and $\cos\dfrac{\angle B - \angle D}{2} = \cos\dfrac{\angle A - \angle C}{2}$,

and the following equalities are possible:

$$\angle B - \angle D = \angle A - \angle C,$$

$$\angle B - \angle D = \angle C - \angle A,$$

$$\angle B + \angle C = \angle A + \angle D = 180^\circ,$$

$$\angle B + \angle A = \angle C + \angle D = 180^\circ,$$

and hence $AB \| DC$ or $AD \| BC$.

In the other words, the quadrilateral $ABCD$ is at least a trapezoid. We assume, without loss of generality that $AB \| DC$ (see Figure 15).

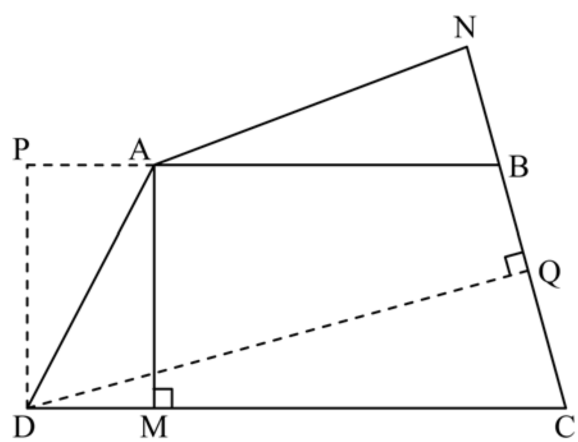

Figure 15.

We construct:

$$DQ \perp BC,\ DP \perp AB,\ AN \perp BC,\ AM \perp DC.$$

It is clear that $AM + AN = PD + PQ$.

If $AB \| CD$, then $PD = AM$, and therefore also $AN = DQ$, but $AN \| DQ$, therefore $ADQN$ is a parallelogram (even a rectangle), therefore also $AD \| BC$.

Thus, we have proven that $ABCD$ is a parallelogram.

Eighth case: The relation between the radius of the incircle and the radius of the circumcircle of an equilateral triangle

The direct claim: In an equilateral triangle, if r is the radius of the incircle and R is the radius of the circumcircle, then $\frac{r}{R} = \frac{1}{2}$ (see Figure 16).

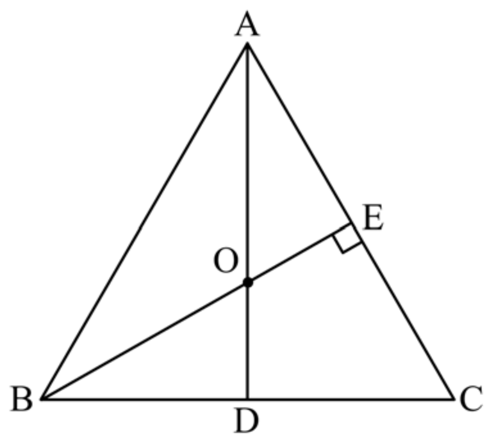

Figure 16.

Proof. In an equilateral triangle, the center of the incircle and the center of the circumcircle coincide at the point O, which is the joint meeting point of the altitudes, the medians, the angle bisectors, and the mid-perpendiculars of the triangle.

If AD is an altitude, then $R = AO$ and $r = OD$. But the medians in a triangle intersect by a ratio of 1 : 2, therefore $\frac{r}{R} = \frac{1}{2}$.

The converse claim: If in a triangle r is the radius of the incircle and R is the radius of the circumcircle, and there holds $\frac{r}{R} = \frac{1}{2}$, then the triangle is *equilateral*.

Proof. Euler proved that if d is the distance between the center of the incircle and the circumcircle in a triangle, then: $d^2 = R(R-2r)$.

If $d = 0$, then $\frac{r}{R} = \frac{1}{2}$ and the center of the incircle coincides with the center of the circumcircle, and therefore the triangle is equilateral.

It remains to prove Euler's formula.

In a triangle ABC, O_2 is the center of the circumcircle, O_1 is the center of the incircle. The continuation of AD_1 (the angle bisector of the angle A) intersects the circumcircle at the point D, and the continuation of DO_2 intersects the circumcircle at the point E.

We construct $AB \perp O_1F$, see Figure 17.

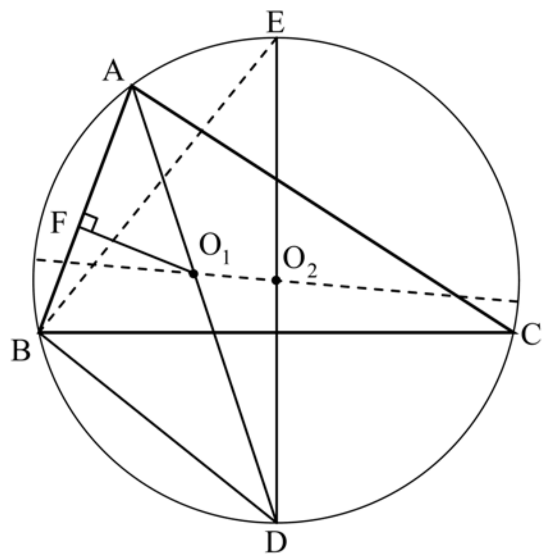

Figure 17.

We now notice the following facts:

(a) The point D is the middle of the arc BC.

(b) $BD = O_1D$ because $\angle DBO_1 = \frac{\angle B}{2} + \frac{\angle A}{2}$, and the angle $\angle DO_1B$, as an exterior angle in the triangle AO_1B, also equals $\frac{\angle A}{2} + \frac{\angle B}{2}$.

(c) $\Delta AO_1F \approx \Delta EDB$ because $\angle EBD = 90^\circ$ (rests on a diameter), and $\angle BAD = \angle BED$.

(d) $DE = 2R$, $O_1F = r$.

From the similarity of the triangles ΔAFO_1 and ΔEBD, it follows that: $\frac{BD}{O_1F} = \frac{DE}{AO_1}$, $O_1D = BD$. In the other words, $AO_1 \cdot O_1D = 2Rr$.

But $AO_1 \cdot O_1D = (R - d)(R + d)$ (from the "circle power" theorem), therefore $R^2 - d^2 = 2Rr$, or: $d^2 = R(R - 2r)$.

Thus Euler's formula is proven.

3. Summary

In this paper, we presented eight theorems students might encounter during their studies, but for which the formulation in the proof of the converse claims of these theorems are not usually dealt with.

It is recommended that when teachers and educators in the field of mathematics encounter a theorem or an interesting problem take advantage of this and the challenge the students in formulating and proving (or disproving) the converse claim of the proven theorem.

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