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INFINITELY MANY NEW CHARACTERIZATIONS OF URYSOHN, T_3, $T_{3\frac{1}{2}}$, WEAKLY URYSOHN, REGULAR, AND COMPLETELY REGULAR OBTAINED BY APPLYING PROPERTIES OF T_0-IDENTIFICATION SPACES, T_0, AND T_0-IDENTIFICATION P PROPERTIES

CHARLES DORSETT

Department of Mathematics
Texas A&M University-Commerce
Commerce, Texas 75429
USA
e-mail: charles.dorsett@tamuc.edu

Abstract

Within this paper, established properties of T_0-identification spaces, T_0, and T_0-identification P properties are used to give infinitely many new characterizations of the Urysohn, T_3, $T_{3\frac{1}{2}}$, weakly Urysohn, regular, and completely regular properties.

1. Introduction and Preliminaries

In the 1936 paper [8], T_0-identification spaces were introduced and used to further characterize the metric property.

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Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the natural map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

Theorem 1.1. *A space is pseudometrizable iff its T_0-identification space is metrizable.*

Within the 1936 paper [8], it was shown that for a space, its T_0-identification space is T_0. The further investigation of T_0-identification spaces in 1978 [1] established that the natural map $N : (X, T) \to (X_0, Q(X, T))$ has very strong properties: continuity, onto, closed, open, $N^{-1}(N(O)) = O$ for all $O \in T$, and $N^{-1}(N(C)) = C$ for all closed sets in (X, T). Thus, T_0-identification spaces were further revealed as a valuable, powerful mathematical tool that could be used to extend a non-T_0 space to a strongly related T_0 space, whose properties could then be established for the initial space using N^{-1}. In the 2007 paper [2], T_0-identification spaces, with their strong natural map, were used to further characterize T_0 spaces: a space is T_0 iff it is homeomorphic to its T_0-identification space. Within a recent paper [3], the results above were used to give infinitely many new characterizations of T_0, which was used to reveal additional properties of T_0-identification spaces.

Definition 1.2. Let (X, T) be a topological space, let $(X_1, Q_1(X, T))$ be the T_0-identification space of (X, T), for each natural number $n \geq 2$, let $(X_n, Q_n(X, T))$ be the T_0-identification space of the space $(X_{n-1}, Q_{n-1}(X, T))$, and let $(\mathcal{X}, \mathcal{T}) = \{(X, T)\} \cup \{(X_n, Q_n(X, T)) | n$ is a natural number$\}$.

Within this paper, the notation given in Definition 1.2 will be repeatedly used.

Theorem 1.1. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is* T_0, (b) *for each natural number* n, $(X_n, Q_n(X, T))$ *is homeomorphic to* (X, T), (c) *for some natural number* p, (X, T) *is homeomorphic to* $(X_p, Q_p(X, T))$, (d) *each element in* $(\mathcal{X}, \mathcal{T})$ *is* T_0 *and all elements of* $(\mathcal{X}, \mathcal{T})$ *are topologically equivalent*, (e) *each element of* $(\mathcal{X}, \mathcal{T})$ *is* T_0, (f) *all the elements of* $(\mathcal{X}, \mathcal{T})$ *are homeomorphic and, thus, all the elements of* $(\mathcal{X}, \mathcal{T})$ *are topologically equivalent, and* (g) *for a fixed natural number* p, $(X_p, Q_p(X, T))$ *is homeomorphic to* (X, T).

Theorem 1.2. *Let* (X, T) *be a space. Then each element of* $\{(X_n, Q_n(X, T)) | n$ *is a natural number*$\}$ *is* T_0 *and all elements of* $\{(X_n, Q_n(X, T)) | n$ *is a natural number*$\}$ *are topologically equivalent*.

The use of T_0-identification spaces in the 1936 paper [8] to further characterize the metric property raised questions about other properties that could be similarly characterized motivating a 1977 paper [4]. Included within the 1977 paper [4] were new characterizations of the regular and the completely regular properties using T_0-identification spaces.

Theorem 1.3. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is regular* (*completely regular*), (b) $(X_0, Q(X, T))$ *is regular* (*completely regular*), *and* (c) $(X_0, Q(X, T))$ *is* $T_3(T_{3\frac{2}{2}})$.

The regular property was introduced in 1921 [10].

Definition 1.3. A space (X, T) is regular iff for each closed set C in X and each $x \notin C$, there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$. A regular T_1 space is denoted by T_3.

The completely regular property was introduced in 1925 [9].

Definition 1.4. A space (X, T) is completely regular iff for each closed set C in (X, T) and each $x \notin C$, there exists a continuous function $f : (X, T) \to (I, U)$ such that $f(x) = 0$ and $f(C) = 1$, where $I = [0, 1]$ and U is the usual metric topology on I. A completely regular T_1 space is denoted by $T_{3\frac{1}{2}}$.

Thus, there are properties other than metrizable that can be further investigated and characterized using T_0-identification spaces raising the question of whether the process could be generalized allowing all such properties to be studied simultaneously, which led to a 2015 paper [5] in which weakly *Po* spaces and properties were introduced and investigated.

Definition 1.5. Let P be a topological property such that $Po = (P$ and $T_0)$ exists. Then a space (X, T) is weakly *Po* iff $(X_0, Q(X, T))$ has property P. A topological property *Qo* for which weakly *Qo* exists is called a weakly *Po* property.

Within the 2015 paper [5], it was shown that for a weakly *Po* property *Qo*, weakly *Qo* is a topological property, a space is weakly *Qo* iff its T_0-identification space is *Qo*; and a space is weakly *Qo* iff its T_0-identification space is weakly *Qo*. The fact that there are topological properties simultaneously shared by both a space and its T_0-identification space, as in the examples given above, motivated the introduction and investigation of T_0-identification P properties [6].

Definition 1.6. Let S be a topological property. Then S is a T_0-identification P property iff both a space and its T_0-identification space simultaneously share property S.

Within the recent paper [3] cited above, for a T_0-identification P property Q, infinitely many new characterizations for each of Q and *Qo* were given.

Theorem 1.4. *Let* (X, T) *be a space and let* Q *be a* T_0*-identification P property. Then the following are equivalent*: (a) (X, T) *has property* Q, (b) *for each natural number* n, $(X_n, Q_n(X, T))$ *has property* Q *and all elements of* $\{(X_n, Q_n(X, T))|n$ *is a natural number*$\}$ *are topologically equivalent*, (c) *for each natural number* n, $(X_n, Q_n(X, T)))$ *has property* Q, (d) *for a fixed natural number* p, $(X_p, Q_p(X, T))$ *has property* Q, (e) *there exists a natural number* p *such that* $(X_p, Q_p(X, T))$ *has property* Q, (f) *all elements of* $(\mathcal{X}, \mathcal{T})$ *have property* Q, (g) *there is an element of* $(\mathcal{X}, \mathcal{T})$ *with property* Q, (h) *there exists a natural number* p *such that* $(X_p, Q_p(X, T))$ *has property* Qo, (i) *there exists a natural number* p *such that* $(X_p, Q_p(X, T))$ *has property* Qo *and all elements of* $\{(X_n, Q_n(X, T))|n$ *is a natural number*$\}$ *are topologically equivalent*, (j) *for a fixed natural number* p, $(X_p, Q_p(X, T))$ *has property* Qo, (k) *for each natural number* n, $(X_n, Q_n(X, T))$ *has property weakly* Qo *and all elements of* $\{(X_n, Q_n(X, T))|n$ *is a natural number*$\}$ *are topologically equivalent*, (l) *there exists a natural number* p *such that* $(X_p, Q_p(X, T))$ *is weakly* Qo *and all elements in* $\{(X_n, Q_n(X, T))|n$ *is a natural number*$\}$ *are topologically equivalent*, (m) *there exists a natural number* p *such that* $(X_p, Q_p(X, T))$ *is weakly* Qo, (n) *for a fixed natural number* p, $(X_p, Q_p(X, T))$ *has property weakly* Qo, *and* (o) *all elements of* $(\mathcal{X}, \mathcal{T})$ *have property weakly* Qo.

Theorem 1.5. *Let* (X, T) *be a space and let* Q *be a* T_0*-identification P property. Then the following are equivalent*: (a) (X, T) *has property* Qo, (b) *all the elements in* $(\mathcal{X}, \mathcal{T})$ *have property* Qo, (c) (X, T) *is* T_0 *and all elements in* $\{(X_n, Q_n(X, T))|n$ *is a natural number*$\}$ *have property* Qo, (d) (X, T) *is* T_0 *and for each natural number* n, $(X_n, Q_n(X, T)))$ *has property* Q, (e) (X, T) *is* T_0 *and for each natural number* n,

$(X_n, Q_n(X, T))$ has property weakly Qo, (f) *(X, T) is T_0 and there exists a natural number p such that $(X_p, Q_p(X, T))$ has property weakly Qo*, (g) *(X, T) is T_0 and for a fixed natural number p, $(X_p, Q_p(X, T))$ has property weakly Qo*, (h) *(X, T) is T_0 and for a fixed natural number p, $(X_p, Q_p(X, T))$ has property Q*, (i) *(X, T) has property T_0 and there exists a natural number p such that $(X_p, Q_p(X, T))$ has property Q*, (j) *(X, T) is T_0 and there exists a natural number p such that $(X_p, Q_p(X, T))$ has property Qo*, (k) *(X, T) has property T_0 and for a fixed natural number p, $(X_p, Q_p(X, T))$ has property Qo*, (l) *(X, T) has property Q and is homeomorphic to all elements of $\{(X_n, Q_n(X, T)) | n$ is a natural number$\}$*, (m) *(X, T) has property Q and for a fixed natural number p, (X, T) is homeomorphic to $(X_p, Q_p(X, T))$*, (n) *(X, T) has property Q and there exists a natural number p such that (X, T) is homeomorphic to $(X_p, Q_p(X, T))$*, (o) *(X, T) has property weakly Qo and there exists a natural number p such that (X, T) is homeomorphic to $(X_p, Q_p(X, T))$*, (p) *(X, T) has property weakly Qo and for a fixed natural number p, (X, T) is homeomorphic to $(X_p, Q_p(X, T))$, and* (q) *(X, T) is weakly Qo and is topologically equivalent to all the elements of $\{(X_n, Q_n(X, T)) | n$ is a natural number$\}$.*

Since, by the results above, each of regular and completely regular are T_0-identification P properties, and (regular)$o = T_3$ and (completely regular)$o = T_{3\frac{1}{2}}$, the two results above are applied later in this paper to give infinitely many new characterizations of each of regular, completely regular, T_3, and $T_{3\frac{1}{2}}$.

In addition to the introduction of the completely regular property in the 1925 paper [9], the Urysohn property was introduced.

Definition 1.7. A space (X, T) is Urysohn iff for distinct elements x and y in X, there exist open sets U and V such that $x \in U$, $y \in V$, and $Cl(U) \cap Cl(V) = \phi$.

In the 1988 paper [7], the Urysohn property was generalized to the weakly Urysohn property.

Definition 1.8. A space (X, T) is weakly Urysohn iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist open sets U and V such that $x \in U$, $y \in V$, and $Cl(U) \cap Cl(V) = \phi$.

Within the 1988 paper [7], it was shown that a space (X, T) is weakly Urysohn iff $(X_0, Q(X, T))$ is Urysohn. Below weakly Urysohn is further characterized and the new characterizations are combined with the results above to further characterize weakly Urysohn and Urysohn.

2. Characterizations of Weakly Urysohn, Regular, and Completely Regular

Theorem 2.1. *Let (X, T) be a space. Then (X, T) is Urysohn iff it is (T_0 and weakly Urysohn).*

Proof. Clearly, if (X, T) is Urysohn, then (X, T) is (T_0 and weakly Urysohn).

Conversely, suppose (X, T) is (T_0 and weakly Urysohn). Since (X, T) is T_0, (X, T) is homeomorphic to $(X_0, Q(X, T))$, since (X, T) is weakly Urysohn, $(X_0, Q(X, T))$ is Urysohn, and since Urysohn is a topological property, (X, T) is Urysohn.

Theorem 2.2. *Let (X, T) be a space. Then (X, T) is weakly Urysohn iff $(X_0, Q(X, T))$ is weakly Urysohn.*

Proof. If (X, T) is weakly Urysohn, then $(X_0, Q(X, T))$ is Urysohn, which implies (X, T) is weakly Urysohn.

Conversely, suppose $(X_0, Q(X, T))$ is weakly Urysohn. Then $(X_0, Q(X, T))$ is (T_0 and weakly Urysohn) = Urysohn and (X, T) is weakly Urysohn.

Corollary 2.1. *Weakly Urysohn is a* T_0*-identification P property and weakly Urysohn = weakly (weakly Urysohn)o = weakly (Urysohn).*

Thus, replacing Q by (weakly Urysohn) and Qo by (Urysohn)o = Urysohn, and weakly Qo by weakly (weakly Urysohn)o in Theorem 1.4 above gives infinitely many new characterizations of weakly Urysohn.

Corollary 2.1. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *has property* (*weakly Urysohn*), (b) *for each natural number* $n, (X_n, Q_n(X, T))$ *has property* (*weakly Urysohn*) *and all elements of* $\{(X_n, Q_n(X, T)) | n$ *is a natural number*$\}$ *are topologically equivalent*, (c) *for each natural number* $n, (X_n, Q_n(X, T)))$ *has property* (*weakly Urysohn*), (d) *for a fixed natural number* $p, (X_p, Q_p(X, T))$ *has property* (*weakly Urysohn*), (e) *there exists a natural number p such that* $(X_p, Q_p(X, T))$ *has property* (*weakly Urysohn*), (f) *all elements of* $(\mathcal{X}, \mathcal{T})$ *have property* (*weakly Urysohn*), (g) *there is an element of* $(\mathcal{X}, \mathcal{T})$ *with property* (*weakly Urysohn*), (h) *there exists a natural number p such that* $(X_p, Q_p(X, T))$ *has property Urysohn*, (i) *there exists a natural number p such that* $(X_p, Q_p(X, T))$ *has property Urysohn and all elements of* $\{(X_n, Q_n(X, T)) | n$ *is a natural number*$\}$ *are topologically equivalent*, (j) *for a fixed natural number* $p, (X_p, Q_p(X, T))$ *has property Urysohn*, (k) *for each natural number* $n, (X_n, Q_n(X, T))$ *has property weakly* (*weakly Urysohn*)*o and all elements of* $\{(X_n, Q_n(X, T)) | n$ *is a natural number*$\}$ *are topologically equivalent*, (l) *there exists a natural number p* *such that* $(X_p, Q_p(X, T))$ *is weakly* (*weakly Urysohn*)*o and all elements in* $\{(X_n, Q_n(X, T))|n$ *is a natural number*$\}$ *are topologically equivalent*, (m) *there exists a natural number p such that* $(X_p, Q_p(X, T))$ *is weakly* (*weakly Urysohn*)*o*, (n) *for a fixed natural number* p, $(X_p, Q_p(X, T))$ *has property weakly* (*weakly Urysohn*)*o*, *and* (o) *all elements of* $(\mathcal{X}, \mathcal{T})$ *have property weakly* (*weakly Urysohn*)*o*.

In like manner, which is omitted, replacing Q by regular, Qo by (regular)o = T_3, and weakly Qo by weakly (regular)o in Theorem 1.4 gives infinitely many new characterizations of regular and replacing Q by (completely regular), Qo by (completely regular)o = $T_{3\frac{1}{2}}$, and weakly Qo by weakly (completely regular)o in Theorem 1.4 gives infinitely many new characterizations of completely regular.

3. Infinitely Many New Characterizations for Each of Urysohn, T_3, and $T_{3\frac{1}{2}}$

Since, from above, weakly Urysohn is a T_0-identification P property and (weakly Urysohn)o = Urysohn, Theorem 1.5 above is used below to give infinitely many new characterizations of Urysohn.

Corollary 3.1. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is Urysohn*, (b) *all the elements in* $(\mathcal{X}, \mathcal{T})$ *have property Urysohn*, (c) (X, T) *is* T_0 *and all elements in* $\{(X_n, Q_n(X, T))|n$ *is a natural number*$\}$ *have property Urysohn*, (d) (X, T) *is* T_0 *and for each natural number* n, $(X_n, Q_n(X, T)))$ *has property* (*weakly Urysohn*), (e) (X, T) *is* T_0 *and for each natural number* n, $(X_n, Q_n(X, T))$ *has property weakly* (*weakly Urysohn*)*o*, (f) (X, T) *is* T_0 *and there exists a natural number p such that* $(X_p, Q_p(X, T))$ *has property weakly* (*weakly Urysohn*)*o*, (g) (X, T) *is* T_0 *and for a fixed natural number p*,

$(X_p, Q_p(X, T))$ *has property weakly* (*weakly Urysohn*)o, (h) (X, T) *is* T_0 *and for a fixed natural number* p, $(X_p, Q_p(X, T))$ *has property* (*weakly Urysohn*), (i) (X, T) *has property* T_0 *and there exists a natural number* p *such that* $(X_p, Q_p(X, T))$ *has property* (*weakly Urysohn*), (j) (X, T) *is* T_0 *and there exists a natural number* p *such that* $(X_p, Q_p(X, T))$ *has property Urysohn*, (k) (X, T) *has property* T_0 *and for a fixed natural number* p, $(X_p, Q_p(X, T))$ *has property Urysohn*, (l) (X, T) *has property* (*weakly Urysohn*) *and is homeomorphic to all elements of* $\{(X_n, Q_n(X, T)) | n$ *is a natural number*$\}$, (m) (X, T) *has property* (*weakly Urysohn*) *and for a fixed natural number* p, (X, T) *is homeomorphic to* $(X_p, Q_p(X, T))$, (n) (X, T) *has property* (*weakly Urysohn*) *and there exists a natural number* p *such that* (X, T) *is homepmorphic to* $(X_p, Q_p(X, T))$, (o) (X, T) *has property weakly* (*weakly Urysohn*)*o and there exists a natural number* p *such that* (X, T) *is homeomorphic to* $(X_p, Q_p(X, T))$, (p) (X, T) *has property weakly* (*weakly Urysohn*)*o and for a fixed natural number* p, (X, T) *is homeomorphic to* $(X_p, Q_p(X, T))$, *and* (q) (X, T) *is weakly* (*weakly Urysohn*)*o and is topologically equivalent to all the elements of* $\{(X_n, Q_n(X, T)) | n$ *is a natural number*$\}$.

In similar manner, which is omitted, infinitely many new characterizations of each of T_3 and $T_{3\frac{1}{2}}$ can be obtained by using Theorem 1.5.

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