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SOME COPLANAR POINTS IN TETRAHEDRON

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Abstract

In this work, we determine the conditions for coplanarity of the vertices, the incenter, the excenters, and the symmedian point of a tetrahedron.

1. Introduction and Preliminaries

In geometry, the barycentric coordinate system is a coordinate system in which the location of a point of a simplex (a triangle, tetrahedron, etc.) is specified as the center of mass, or barycenter, of usually unequal masses placed at its vertices. Coordinates also extend outside the simplex, where one or more coordinates become negative. The system was introduced in 1827 by August Ferdinand Möbius [1, 2].

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In a given triangle ABC, every point P is coordinatized by a triple of numbers $(u : v : w)$ in such a way that the system of masses u at A, v at B, and w at C will have its balance point at P. These masses can be taken in the proportions of the areas of triangle PBC, PCA, and PAB. Allowing the point P to be outside the triangle, we use signed areas of oriented triangles. The homogeneous barycentric coordinates of P with reference to ABC is a triple of numbers $(x : y : z)$ such that

$$x : y : z = PBC : PCA : PAB.$$

The centroid G, the incenter I and the symmedian point K have the homogeneous barycentric coordinates $(1, 1, 1), (a, b, c),$ and (a^2, b^2, c^2), respectively [3]. The coordinates of vertices of triangle ABC are $A(1, 0, 0), B(0, 1, 0), C(0, 0, 1)$ and the excenters are $J_A(-a, b, c), J_B(a, -b, c), J_C(a, b, -c)$. Some collinearities of these points in the heptagonal triangle are discussed [2].

Barycentric coordinates may be easily extended to three dimensions. The 3D simplex is a tetrahedron, a polyhedron having four triangular faces and four vertices. Let P be a point inside the tetrahedron. It divides the tetrahedron into four sub-tetrahedrons.

Let the volumes of sub-tetrahedrons be $PBCD = V_A$, $PACD = V_B$, $PABD = V_C$, and $PABC = V_D$. Using the signed volumes of the oriented sub-tetrahedrons the homogeneous barycentric coordinates of P with the reference tetrahedron $ABCD$ is (x, y, z, t) such that

$$x : y : z : t = V_A : V_B : V_C : V_D.$$

Respect to the tetrahedron $ABCD$, the barycentric coordinates of the vertices are $A(1, 0, 0, 0)$, $B(0, 1, 0, 0)$, $C(0, 0, 1, 0)$, and $D(0, 0, 0, 1)$ [6]. The incenter of tetrahedron is the intersection point of planes that bisect the angles between the tetrahedron faces. It is the center of insphere which is tangent to four faces. In a tetrahedron $ABCD$, let S_a, S_b, S_c, S_d be the area of the faces *BCD, ACD*, *ABD*, and *ABC*, respectively. The barycentric coordinate of the incenter of *ABCD* is $I(S_a, S_b, S_c, S_d)$ and the barycentric coordinates of the exsphere centers are $J_A(-S_a, S_b, S_c, S_d)$, $J_B(S_a, -S_b, S_c, S_d)$, $J_C(S_a, S_b, -S_c, S_d)$, $J_D(S_a, S_b, S_c, -S_d)$ [1].

In the reference tetrahedron *ABCD*, two planes α and β through *AB* are said to be isogonal conjugates if they are equally inclined from the sides that form the dihedral angle between the planes of the triangles *ABC* and *ABD*. α is called the isogonal conjugate of β and vice versa. If a point *X* of *ABCD* is joined to the vertex *A* and the vertex *B*, the plane through *XA* and *XB* has an isogonal conjugate at *A*. Similarly, joining *X* to vertices *B* and *D*, *D* and *C*, *A* and *C*, *B* and *C*, *C* and *D*, produce five more conjugate planes. Let *M* be the midpoint of *CD*. The plane containing *AB* and that is isogonal to the plane of triangle *ABM* is called a symmedian plane of tetrahedron *ABCD*. Taking the midpoints of the six sides of the tetrahedron *ABCD* and forming the associated symmedian planes, we call the intersection point of these symmedian planes the symmedian point of the tetrahedron. The centroid *G* of the tetrahedron has barycentric coordinates $G(1, 1, 1, 1)$ and symmedian point *K* has coordinates $K(S_a^2, S_b^2, S_c^2, S_d^2)$ [4].

In this work, we determine the coplanarity relations among the vertices *A*, *B*, *C*, *D*, the incenter *I*, the excenters J_A, J_B, J_C, J_D, and the symmedian point *K* in the reference tetrahedron by using the barycentric coordinate system.

2. Coplanarity and Barycentric Coordinates in a Tetrahedron

Barycentric coordinates (see, e.g., (Hocking and Young [8]), Chapter 5) are a practical way to parameterize lines, surfaces, etc., for applications that must compute where various geometric objects intersect. In practice, the barycentric coordinate method reduces to specifying two points (x_0, x_1)

on a line, three points (x_0, x_1, x_2) on a plane, four points (x_0, x_1, x_2, x_3) in a volume, etc., and parameterizing the line segment, enclosed triangular area, and enclosed tetrahedral volume, etc., respectively.

Let (a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3), (c_0, c_1, c_2, c_3), and (d_0, d_1, d_2, d_3) be the barycentric coordinates of a, b, c, d with respect to the standard affine frame for affine space, respectively. It is well known that a, b, c, d are coplanar iff

$$\det[a, b, c, d] = 0.$$

Proposition 2.1. *Let the reference tetrahedron be ABCD. Then,*

(a) *Any two vertices and the excenters corresponding to these vertices are coplanar in ABCD.*

(b) *Any vertex, the excenter related to this vertex, the incenter and the symmedian point are coplanar in ABCD.*

Proof. (a) Let the vertices and the excenters of *ABCD* be A, B and J_A, J_B. These four points in terms of the barycentric coordinates are $A = (0, 0, 1, 0)$, $B = (0, 0, 0, 1)$, $J_A = (-S_a, S_b, S_c, S_d)$, $J_B = (S_a, -S_b, S_c, S_d)$. Since $\det[A, B, J_A, J_B]$ is zero, these four points A, B, J_A, J_B of tetrahedron are coplanar. The coplanarity of other vertices and corresponding excenters are proved similarly.

(b) Let the vertex and the excenter corresponding to this vertex of tetrahedron be A and J_A. Since the determinant of the matrix of the barycentric coordinates of the points A, J_A, I, and K is zero, these four points A, J_A, I, and K of the tetrahedron are coplanar. Also, the coplanarity of other vertices and corresponding excenter, I and K of the tetrahedron *ABCD* are proved similarly.

Proposition 2.2. *Let the reference tetrahedron be ABCD. Then any three excenters of ABCD are coplanar neither the symmedian point nor the incenter point.*

Proof. Let any three excenters be J_A, J_B, J_C of the tetrahedron *ABCD*. We assume that J_A, J_B, J_C and the symmedian point K of *ABCD* are coplanar, then the determinant in terms of the barycentric coordinates of J_A, J_B, J_C, K of the tetrahedron is zero. $S_d = S_b + S_c + S_a$ is obtained from this determinant. But the equation $S_d = S_b + S_c + S_a$ contradicts "the sum of the areas of the three faces in a tetrahedron is greater than the area of fourth face" in [5]. So, any three excenters and the symmedian point of a tetrahedron cannot be coplanar.

Similarly, we assume that J_A, J_B, J_C and the incenter point I of *ABCD* are coplanar in order to find a contradiction, then the determinant in terms of the barycentric coordinates of J_A, J_B, J_C, I is zero. From this determinant, the relation $S_a S_b S_c S_d = 0$ is obtained contradicting the fact that $S_a, S_b, S_c, S_d > 0.$ Our claim is proved.

Proposition 2.3. *Let any three excenters be* J_A, J_B, J_C *and the centroid be G in the tetrahedron ABCD. Let* S_a, S_b, S_c, S_d *be the area of the faces BCD, ACD, ABD, and ABC of ABCD. Then the points* $J_A, J_B,$ $J_C,$ *and G are coplanar iff* $\frac{1}{S_a} + \frac{1}{S_b} + \frac{1}{S_c} = \frac{1}{S_d}.$

Proof. Suppose that $J_A, J_B, J_C,$ and G are coplanar. Since the determinant in terms of the barycentric coordinates of $J_A, J_B, J_C,$ and G is zero, $\frac{1}{S_d} = \frac{1}{S_a} + \frac{1}{S_b} + \frac{1}{S_c}$ is obtained.

Conversely, it is seen that $\det[J_A, J_B, J_C, G]$ is zero by using $\frac{1}{S_d} = \frac{1}{S_a} + \frac{1}{S_b} + \frac{1}{S_c}.$ This implies that the points $J_A, J_B, J_C,$ and G are coplanar.

Proposition 2.4. *Let any two excenters be* J_A, J_B, *the incenter be I, the symmedian point be K, and the centroid be G in the tetrahedron ABCD. Let* S_a, S_b, S_c, S_d *be the area of the faces BCD, ACD, ABD, and ABC of ABCD. Then if* $S_c = S_d$, *the points* J_A, J_B, G, I, *and K are coplanar.*

Proof. We assume that $S_c = S_d$. Then it is obtained that the determinant in terms of the barycentric coordinates of J_A, J_B, G, and I is zero. Hence, the points J_A, J_B, G, and I are coplanar.

Similarly, the determinant in terms of the barycentric coordinates J_A, J_B, G, and K is zero, it is seen that the points J_A, J_B, G, and K are coplanar.

So, the points J_A, J_B, G, I, and K are coplanar.

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