

CONVERGING POLYGONS BY A DYNAMIC PROCESS TO A PERFECT POLYGON

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Abstract

We give an interesting description of a dynamic process in which a polygon inscribed in a circle converges to a perfect polygon, and also of the convergence of a triangle where in a stage-by-stage manner one constructs a triangle, the length of whose sides are the averages of the side lengths of the previous triangle, which monotonically converges by a dynamic process to an equilateral triangle. Mathematical proofs are given to the convergence to a perfect polygon, and the stages are presented visually using computerized dynamic applets which one can reach by links that exist in the paper.

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1. Introduction

Geometry has a history of hundreds and thousands of years in which many properties have gained the status of theorems, and they form the basis of the “toolbox” for proving and solving different tasks.

Today as well, geometry is a wonderful and challenging world which by means of multidirectional research built on the foundations laid by prominent mathematicians through the generations allows us to discover new and surprising properties as answers to the questions “what else?”.

In the modern era, the computerized technological tool joined the traditional tools of research.

Dynamic geometry software (DGS) is a technological tool that allows the representation of mathematical equations and the construction of mathematical objects in a manner that provides constant feedback to the user.

This tool allows us to make different hypotheses and to immediately check their correctness by dynamic investigation. However, after computerized confirmation of the correctness of the property, one must also provide a mathematical proof, as required [1-5].

Articles appear from time to time in journals with interesting discoveries regarding new geometrical properties [6-8].

In the previous paper [9], a complete proof was given to the property obtained in the following process:

In the triangle $\Delta A_1B_1C_1$ that is inscribed in a circle, we draw the angle bisectors and they intersect the circle at the points A_2 , B_2 , and C_2 , producing a new triangle $\Delta A_2B_2C_2$ (see Figure 1). When again we draw the angle bisectors of the new triangle, we again obtain a new triangle $\Delta A_3B_3C_3$. As we repeat this process over and over again, we obtain a triangle that tends to an equilateral triangle.

One may reach a GeoGebra applet that presents this property in a dynamic fashion via the link.

Link 1: to the dynamic applet.

<https://www.geogebra.org/m/r5cTDT6h>

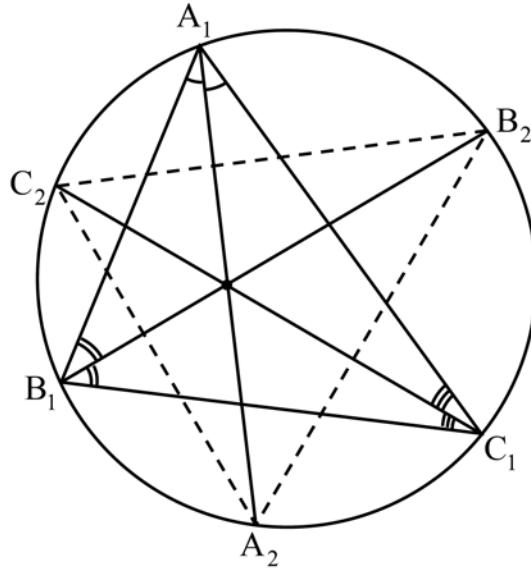


Figure 1. The triangle formed by connecting the points of intersection of the angle bisectors of the original triangle with the circle.

The interesting property of convergence to an equilateral triangle raises the question of whether a similar action of drawing the angle bisectors in a quadrilateral will eventually result in the original quadrilateral's convergence to a square.

In this case already after the first stage of the transition from the quadrilateral $A_1B_1C_1D_1$ to the new quadrilateral $A_2B_2C_2D_2$, it turns out that the latter is a rectangle. When one continues to draw angle bisectors, a congruent rectangle is obtained which is perpendicular to the first rectangle. And thus it continues, where at each stage a rectangle is formed that is congruent and perpendicular to the previous one.

Conclusion: The convergence property that was found in the case where the angle bisectors were drawn in a triangle inscribed in a circle does not hold in a quadrilateral that is inscribed in a circle.

Another direction to be considered is the fact that when the angle bisectors of a triangle inscribed in a circle are drawn, the continuations of the angle bisectors bisect each of the arcs between the vertices. In other words, in the new triangle, the location of the vertices is at the midpoint of the arcs of the previous triangle.

In line with this direction, it was decided to examine the quadrilateral that is formed by connecting the midpoints of the arcs of a quadrilateral that is inscribed in a circle, as seen in Figure 2.

A GeoGebra applet allowing the creation of a sequence of quadrilaterals was prepared for the dynamic activity of creating a sequence of quadrilaterals, the vertices of each of which are located at the middles of the arcs of the previous quadrilateral. For each quadrilateral, the values of the angles of the new quadrilateral obtained, and their standard deviations appear on the screen. The standard deviation indeed decreases monotonically, which suggests that the angles converge to 90° . However, it must be proven mathematically that at the end of the process the quadrilateral converges to a square and not to a rectangle.

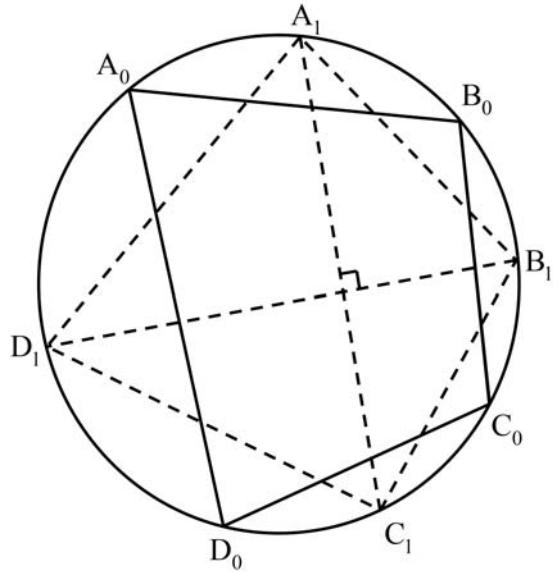


Figure 2. The quadrilateral formed by connecting the midpoints of the arcs of the original triangle.

To illustrate the convergence, an applet was prepared, which may be activated by using the following link:

Link 2: <https://www.geogebra.org/m/MtNnTyqA>

The proof

Given is a circle with the quadrilateral $A_0B_0C_0D_0$ inscribed in it, as shown in Figure 2. We denote:

A_1 is the middle of the arc $\widehat{A_0B_0}$.

B_1 is the middle of the arc $\widehat{B_0C_0}$.

C_1 is the middle of the arc $\widehat{C_0D_0}$.

D_1 is the middle of the arc $\widehat{D_0A_0}$.

The quadrilateral $A_1B_1C_1D_1$ is formed, so that the arc lengths satisfy:

$$\widehat{A_1B_1} = \frac{\widehat{A_0B_0} + \widehat{B_0C_0}}{2}, \quad \widehat{B_1C_1} = \frac{\widehat{B_0C_0} + \widehat{C_0D_0}}{2},$$

$$\widehat{C_1D_1} = \frac{\widehat{C_0D_0} + \widehat{A_0D_0}}{2}, \quad \widehat{D_1A_1} = \frac{\widehat{D_0A_0} + \widehat{A_0B_0}}{2}.$$

It is important to note the famous geometric property that the diagonals of the quadrilateral $A_1B_1C_1D_1$ are perpendicular to each other.

From the quadrilateral $A_1B_1C_1D_1$, one can proceed to the quadrilateral $A_2B_2C_2D_2$, and so on and so forth to $A_nB_nC_nD_n$, with $n \rightarrow \infty$.

From the recurrence formula, one can obtain the side lengths of the quadrilateral $A_{n+1}B_{n+1}C_{n+1}D_{n+1}$:

$$\widehat{A_{n+1}B_{n+1}} = \frac{\widehat{A_nB_n} + \widehat{B_nC_n}}{2}, \quad \widehat{B_{n+1}C_{n+1}} = \frac{\widehat{B_nC_n} + \widehat{C_nD_n}}{2},$$

$$\widehat{C_{n+1}D_{n+1}} = \frac{\widehat{C_nD_n} + \widehat{D_nA_n}}{2}, \quad \widehat{D_{n+1}A_{n+1}} = \frac{\widehat{D_nA_n} + \widehat{A_nB_n}}{2}.$$

It must be proven that for $n \rightarrow \infty$ the quadrilateral $A_nB_nC_nD_n$ tends to a square. For the purpose of the proof, we denote

$$\widehat{A_nB_n} = x_n \quad \text{and} \quad \widehat{A_{n+1}B_{n+1}} = x_{n+1};$$

$$\widehat{B_nC_n} = y_n \quad \text{and} \quad \widehat{B_{n+1}C_{n+1}} = y_{n+1};$$

$$\widehat{C_nD_n} = z_n \quad \text{and} \quad \widehat{C_{n+1}D_{n+1}} = z_{n+1};$$

$$\widehat{D_nA_n} = w_n \quad \text{and} \quad \widehat{D_{n+1}A_{n+1}} = w_{n+1}.$$

Claim 1.

Prove that $x_{n+1}^2 + y_{n+1}^2 + z_{n+1}^2 + w_{n+1}^2 \leq x_n^2 + y_n^2 + z_n^2 + w_n^2$.

Proof. We need to prove that

$$\left(\frac{x_n + y_n}{2}\right)^2 + \left(\frac{y_n + z_n}{2}\right)^2 + \left(\frac{z_n + w_n}{2}\right)^2 + \left(\frac{w_n + x_n}{2}\right)^2 \stackrel{?}{\leq} x_n^2 + y_n^2 + z_n^2 + w_n^2.$$

After opening the parentheses, we obtain

$$x_n y_n + y_n z_n + z_n w_n + w_n x_n \stackrel{?}{\leq} x_n^2 + y_n^2 + z_n^2 + w_n^2.$$

This is in fact true, as this is the famous Cauchy-Schwartz inequality.

Equality holds if $x_n = y_n = z_n = w_n$.

Claim 2.

We denote $Q_\ell = x_\ell^2 + y_\ell^2 + z_\ell^2 + w_\ell^2$.

Prove that the sequence $\{Q_\ell\}_{\ell=1}^{\ell \rightarrow \infty}$ is convergent.

Proof. Based on Claim 1, the sequence $\{Q_\ell\}$ decreases monotonically and is bounded by $4 \cdot (90^\circ)^2$, therefore it is convergent.

Claim 3. Prove that $\{Q_\ell\}_{\ell \rightarrow \infty} = 4 \cdot (90^\circ)^2$.

Proof. It is clear that when $\ell \rightarrow \infty$, there holds: $-Q_{\ell+1} + Q_\ell = 0$.

Therefore, $x_\ell^2 + y_\ell^2 + z_\ell^2 + w_\ell^2 \rightarrow x_\ell y_\ell + y_\ell z_\ell + z_\ell w_\ell + w_\ell x_\ell$.

It is known that

$$x_\ell^2 + y_\ell^2 \geq 2x_\ell y_\ell,$$

$$y_\ell^2 + z_\ell^2 \geq 2y_\ell z_\ell,$$

$$z_\ell^2 + w_\ell^2 \geq 2z_\ell w_\ell,$$

$$w_\ell^2 + x_\ell^2 \geq 2w_\ell x_\ell.$$

Therefore the limit holds only when:

$$\left. \begin{array}{l} x_\ell \rightarrow y_\ell \\ y_\ell \rightarrow z_\ell \\ z_\ell \rightarrow w_\ell \\ w_\ell \rightarrow x_\ell \end{array} \right\} \Rightarrow \{Q_\ell\}_{\ell \rightarrow \infty} = 4 \cdot (90^\circ)^2,$$

and hence $x_\ell = y_\ell = z_\ell = w_\ell$, which implies that all the angles equal 90° .

Note. These considerations are also valid when we divide the circumference of the circle by m points (in the present case $m = 4$), in other words, any initial polygon with m sides inscribed in a circle which will undergo such a sequential process of forming a sequence of polygons whose vertices intersect the arcs of the previous polygon will converge to a perfect polygon with m sides (see Link 3- **pentagon**).

Link 3: <https://www.geogebra.org/m/H8BMvNB5>

Note. It is important to mention that the parameter that is conserved and does not change throughout the entire dynamic process is the radius of the circle that circumscribes the polygon.

2. Convergence to an Equilateral Triangle

Given is a triangle, the lengths of whose sides are a_0 , b_0 , and c_0 , as shown in Figure 3. We calculate the values of the arithmetic mean of every two sides:

$$a_1 = \frac{a_0 + b_0}{2}, \quad b_1 = \frac{b_0 + c_0}{2}, \quad c_1 = \frac{c_0 + a_0}{2}.$$

We construct from them the triangle $\Delta A_1 B_1 C_1$, the lengths of whose sides are a_1 , b_1 , and c_1 . We proceed in this way to construct the triangle

$\Delta A_2B_2C_2$, up to the triangle $\Delta A_nB_nC_n$. When $n \rightarrow \infty$, the triangle $\Delta A_nB_nC_n$ converges to an equilateral triangle.

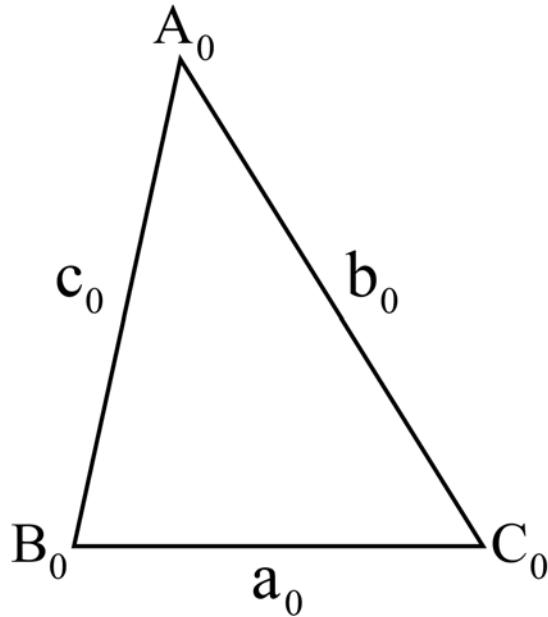


Figure 3. The initial triangle.

Note that all the triangles constructed in such a fashion have the same perimeter, which equals $P_\Delta = a_0 + b_0 + c_0$, the perimeter of the original triangle. And of course, at each stage, the average of the side lengths $\frac{a_n + b_n + c_n}{3} = \frac{P_\Delta}{3}$ is fixed, and the same can be said as to the average of the angles, $\frac{\alpha_n + \beta_n + \gamma_n}{3} = 60^\circ$.

To illustrate the process we construct an applet with a ruler n , allowing one to change the number of the stages in the process. During each stage the lengths of the sides appear on screen, together with the perimeter and the area of the triangle, the radius of the incircle and the standard deviations of its side lengths and angles.

The applet shows that in a dynamic process the area of the triangle increases monotonically. Using the link the applet can be activated and the dynamic process demonstrated.

Link 4: <https://www.geogebra.org/m/hVVMYndr>

The convergence proof of the dynamic process to an equilateral triangle is in fact the proof given in the previous section.

The question raised here is what is so special in this process, which is correct for any triplet of numbers (even if initially they do not satisfy the triangle inequality), where at the end of the process one obtains a triplet of equal numbers. The difference is the fact that at any stage there is progress towards an equilateral triangle and at any stage the area of the triangle increases, up to the maximal value, culminating in an equilateral triangle.

3. Proof of the Monotonic Increase of the Triangle's Area During the Process

It must be proven that at each stage: $\frac{S_{\Delta A_n B_n C_n}}{S_{\Delta A_{n-1} B_{n-1} C_{n-1}}} > 1$.

$$S_{\Delta A_0 B_0 C_0} = \sqrt{p(p - a_0)(p - b_0)(p - c_0)};$$

$$S_{\Delta A_1 B_1 C_1} = \sqrt{p\left(p - \frac{a_0 + b_0}{2}\right)\left(p - \frac{b_0 + c_0}{2}\right)\left(p - \frac{c_0 + b_0}{2}\right)} = \frac{\sqrt{p \cdot a_0 \cdot b_0 \cdot c_0}}{8},$$

and hence

$$\frac{S_{\Delta A_n B_n C_n}}{S_{\Delta A_{n-1} B_{n-1} C_{n-1}}} = \sqrt{\frac{a_0 \cdot b_0 \cdot c_0}{8(p - a_0)(p - b_0)(p - c_0)}}.$$

To prove convergence we must prove that $\frac{a_0 b_0 c_0}{(p - a_0)(p - b_0)(p - c_0)} > 8$.

Proof of the inequality – Method A

From the inequality of the averages: $\sqrt{(p - a_0)(p - b_0)} \leq \frac{(p - a_0) + (p - b_0)}{2} = \frac{c_0}{2}$. Therefore, in the same manner, we obtain

$$(1) \quad \frac{a_0}{2} \geq \sqrt{(p - b_0)(p - c_0)};$$

$$(2) \quad \frac{b_0}{2} \geq \sqrt{(p - a_0)(p - c_0)};$$

$$(3) \quad \frac{c_0}{2} \geq \sqrt{(p - a_0)(p - b_0)}.$$

When the last three inequalities are multiplied together, we obtain the proof of the required inequality, in other words, the area of the triangle increases in the transition $\Delta A_0 B_0 C_0 \rightarrow \Delta A_1 B_1 C_1$.

Proof of the inequality – Method B

From a well-known relation, we have $S_\Delta = \frac{a \cdot b \cdot c}{4R}$.

$$\text{We obtain } R_0 = \frac{a_0 \cdot b_0 \cdot c_0}{4S_{\Delta A_0 B_0 C_0}} = \frac{a_0 \cdot b_0 \cdot c_0}{4\sqrt{(p - a_0)(p - b_0)(p - c_0)}}.$$

$$\text{From a well-known relation, we have } r = \frac{S_\Delta}{p}.$$

$$\text{We obtain } r_0 = \frac{S_{\Delta A_0 B_0 C_0}}{p} = \frac{\sqrt{p(p - a_0)(p - b_0)(p - c_0)}}{p}.$$

$$\text{After dividing, we obtain } \frac{R_0}{r_0} = \frac{a_0 \cdot b_0 \cdot c_0}{4(p - a_0)(p - b_0)(p - c_0)}.$$

According to Euler's formula for the distance between the centers of the circumcircle and the incircle of a triangle, it is known that $\frac{R}{r} \geq 2$,

and therefore we obtain $\frac{a_0 \cdot b_0 \cdot c_0}{(p - a_0)(p - b_0)(p - c_0)} > 8$.

Equality is obtained only when all the side lengths of the triangle are equal. We obtain the same inequality at each stage, and therefore: $S_0 < S_1 < S_2 < \dots < S_n$. When $n \rightarrow \infty$, we shall obtain an equilateral triangle.

It is important to note that due to the monotonic increase of the area of the triangle during the convergence, the radius of the incircle r also grows, due to the famous relation $r = \frac{S_\Delta}{p}$.

This process in fact exemplifies the theorem: “of all the triangles of equal perimeter, the one with the largest area is equilateral”. Of course, the proof of the theorem using differential calculus is not possible on the level of high school students, since the program of studies includes extremum problems with a single variable, while this problem has two variables. We shall therefore present a well-known proof based on the property of the inequality between the different means.

The task

Given a triangle with a fixed perimeter of $2p$, find the lengths of the sides of the triangle with the largest area S .

Solution

Known is Heron’s formula for the area of a triangle, $S = \sqrt{p(p - a)(p - b)(p - c)}$, (a, b, c are the sides of the triangle).

We consider the three (positive) numbers $p - a$, $p - b$, and $p - c$. According to the inequality between averages, we obtain

$$\sqrt[3]{(p - a)(p - b)(p - c)} \leq \frac{(p - a) + (p - b) + (p - c)}{3} = \frac{3p - 2p}{3} = \frac{p}{3},$$

or

$$(p - a)(p - b)(p - c) \leq \frac{p^3}{27},$$

and hence

$$S = \sqrt{p} \cdot \sqrt{(p-a)(p-b)(p-c)} \leq \sqrt{p \cdot \frac{p^3}{27}} = \frac{p^2}{\sqrt{27}}.$$

Equality holds if $p - c = p - a = p - b$, in other words $a = b = c$.

Conclusion: of all the triangles with a fixed perimeter of $2p$, the one with the largest area is equilateral

Note. It is important to note that the parameter that is conserved and does not change throughout the entire dynamic process is the perimeter of the triangle that is formed at each stage.

Also, with sequential construction of triangles according to the formula for the sides of the triangle,

$$a_{n+1} = ka_n + (1-k)b_n,$$

$$b_{n+1} = kb_n + (1-k)c_n,$$

$$c_{n+1} = kc_n + (1-k)a_n,$$

when $0 < k < 1$, the perimeter of the triangles in the sequence shall be conserved, and at the end they will converge to an equilateral triangle.

The value of k affects the convergence rate.

4. Summary

We presented dynamic activity in which a certain polygon inscribed in a circle converges to a perfect polygon, as well as dynamic activity of the convergence of an arbitrary triangle to an equilateral triangle. We also followed the transformation of the side lengths, angles, area of the triangle as well as the ratio between the radius of the circumcircle and the radius of the incircle.

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Verbal explanations for the applets**Link 1:****A quadrilateral that is inscribed in a circle that by dynamic process convergent to a square**

The applet demonstrates that for any quadrilateral inscribed in a circle, by dynamic process, its convergent to a square. By connecting the midpoints of the arcs of a quadrilateral that is inscribed in a circle, we created new quadrilaterals. The ruler allows the creation of a sequence of quadrilaterals, the vertices of each of which are located at the middles of the arcs of the previous quadrilateral. For each quadrilateral, the values of the angles of the new quadrilateral obtained, and their standard deviations appear on the screen. The standard deviation indeed decreases monotonically, which suggests that the angles converge to 90° .

Link 2:**Convergence of any pentagon that is inscribed in a circle to a perfect pentagon**

The applet demonstrates that for any pentagon inscribed in a circle, by dynamic process, its convergent to a perfect pentagon. By connecting the midpoints of the arcs of a pentagon that is inscribed in a circle, we created new pentagons. The ruler allows the creation of a sequence of pentagons, the vertices of each of which are located at the middles of the arcs of the previous pentagon. For each pentagon, the values of the angles of the new pentagon obtained, and their standard deviations appear on the screen. The standard deviation indeed decreases monotonically, which suggests that the angles converge to 108° .

Link 3:**Dynamic convergence of any triangle to an equilateral triangle**

The applet demonstrates a dynamic convergence process of any triangle to an equilateral triangle. There are four rulers, three of them allows to change the lengths of the sides of the original triangle. The fourth ruler allows to promote the convergence process in which each new stage the length of sides are the mathematical mean of the length of pair sides in the previous stage. In every step appears on the screen the following values: The lengths of the sides, triangle angles, the radius of the inner circle and outer circle radius, the standard deviations of the sides and angles.

