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distance-$k$ dominating sets in $G$. Let $\mathcal{D}_k(G, i)$ be the family of distance-$k$ dominating sets of $G$ with cardinality $i$ and let $d_k(G, i) = |\mathcal{D}_k(G, i)|$. The polynomial

$$D_k(G, x) = \sum_{i=\gamma_k(G)}^{V(G)} d_k(G, i)x^i,$$

is defined as distance-$k$ domination polynomial of $G$. A root of $D_k(G, x)$ is called a distance-$k$ domination root of $G$. Set of all distance-$k$ domination roots of $G$ is denoted by $Z(D_k(G, x))$. It is easy to see that the distance-$k$ domination polynomial is monic with no constant term. Consequently, 0 is a root of every distance-$k$ domination polynomial whose multiplicity is equal to the distance-$k$ domination number of the graph.

## 2. Distance-$k$ Domination Polynomial of Some Graphs

When $k = 1$, this polynomial coincide with the domination polynomial $D(G, x)$. Throughout this paper, $k$ denote a positive integer greater than one.

**Theorem 1.** If $G$ and $H$ are isomorphic, then $D_k(G, x) = D_k(H, x)$.

**Remark.** The converse of the above theorem is not true. There are numerous non-isomorphic graphs with the same distance-$k$ domination polynomials.

**Theorem 2.** If a graph $G$ consists of $m$ components $G_1, G_2, \ldots, G_m$, then

$$D_k(G, x) = D_k(G_1, x)D_k(G_2, x)\ldots D_k(G_m, x).$$

**Proof.** It suffices to prove this theorem for $m = 2$. For $l \geq \gamma_k(G)$, a distance-$k$ dominating set of $l$ vertices in $G$ arises by choosing a distance-$k$ dominating set of $j$ vertices in $G_1$, for some $j \in \{\gamma_k(G_1), \gamma_k(G_1) + 1, \ldots, |V(G)|\}$, and a distance-$k$ dominating set of $l - j$ vertices of $G_2$. The number of ways of doing this over all $j = \gamma_k(G_1), \gamma_k(G_1) + 1, \ldots, |V(G)|$ is
For a positive integer $k$, the $k$-th power of a graph $G$ is the graph with the same set of vertices as $G$ and an edge between two vertices if and only if there is a path of length at most $k$ between them, and that graph is denoted by $G^k$.

**Theorem 3.** Let $G$ be a graph and let $k$ be any positive integer, then $D_k(G, x) = D(G^k, x)$.  

**Proof.** It follows from the fact that every distance-$k$ dominating set of $G$ with cardinality $i$ is exactly the dominating set of $G^k$ with cardinality $i$. \hfill $\square$

**Theorem 4.** Let $G$ be a graph having $n$ vertices with diameter $d$. Then $D_k(G, x) = (1 + x)^n - 1$ if and only if $k \geq d$.  

**Proof.** Suppose $k \geq d$. Then all vertices of $G$ are with in a distance $k$. This implies that, for $1 \leq i \leq n$, any subset of vertices of $G$ of cardinality $i$ is a distance-$k$ dominating set. Therefore $D_k(G, x) = (1 + x)^n - 1$. Conversely, suppose that $D_k(G, x) = (1 + x)^n - 1$. Then $\gamma_k(G) = 1$ and $d_k(G, 1) = n$. This implies that all vertices of $G$ are with in a distance $k$. Hence $k \geq d$. \hfill $\square$

**Corollary 5.** For any complete graph $K_n$,  

$$D_k(K_n, x) = (1 + x)^n - 1.$$ 

**Corollary 6.** For any complete $m$-partite graph $K_{n_1, n_2, \ldots, n_m}$,  

$$D_k(K_{n_1, n_2, \ldots, n_m}, x) = (1 + x)^N - 1,$$

where $N = n_1 + n_2 + \ldots + n_m$.  

exactly the coefficient of $x^i$ in $D_k(G_1, x)D_k(G_2, x)$. Hence both side of the above equation have the same coefficient, so they are identical polynomial.
Corollary 7. For any complete bipartite graph $K_{m,n}$,

$$D_k(K_{m,n}, x) = (1 + x)^{m+n} - 1.$$  

Corollary 8. For any star graph $S_n$,

$$D_k(S_n, x) = (1 + x)^{n+1} - 1.$$  

Corollary 9. For any wheel graph $W_n$,

$$D_k(W_n, x) = (1 + x)^n - 1.$$  

If $H$ and $G$ are any two graphs, then $H + G$ is the graph obtained from $H \cup G$ by joining each vertex of $H$ to every vertex of $G$.

Corollary 10. For $i = 1, 2$, let $G_i$ be a graph of $n_i$ vertices, then

$$D_k(G_1 + G_2, x) = (1 + x)^{n_1+n_2} - 1.$$  

If $H$ and $G$ are any two graphs, then the cartesian product $H \square G$ of $H$ and $G$ is a graph such that

- The vertex set of $H \square G$ is the cartesian product $V(H) \times V(G)$.
- Any two vertices $(h, g)$ and $(h', g')$ are adjacent if and only if either $h = h'$ and $g$ is adjacent with $g'$ in $G$ or $g = g'$ and $h$ is adjacent with $h'$ in $H$.

Corollary 11. For the complete graphs $K_m$ and $K_n$, $D_k((K_m \square K_n), x) = (1 + x)^{mn} - 1$.

Corollary 12. Let $P$ be the Petersen graph, then

$$D_k(P, x) = (1 + x)^{10} - 1.$$  

The Dutch Windmill graph $G_2^n$ is the graph obtained by selecting one vertex in each of $n$ triangles and identifying them.
Corollary 13. The distance-\(k\) domination polynomial of the Dutch Windmill graph \(G^n_3\) is

\[
D_k(G^n_3, x) = (1 + x)^{2n+1} - 1.
\]

The Lollipop graph \(L_{m,n}\) is the graph obtained by joining a complete graph \(K_m\) to a path \(P_n\), with a bridge.

Corollary 14. The distance-\(k\) domination polynomial of \(L_{m,1}\) is

\[
D_k(L_{m,1}, x) = (1 + x)^{m+1} - 1.
\]

The bipartite Cocktail party graph \(B_n\) is the graph obtained by removing a perfect matching from the complete bipartite graph \(K_{n,n}\).

Theorem 15. Let \(B_n\) be the bipartite Cocktail party graph. Then for \(n \geq 3\),

\[
D_2(B_n, x) = (1 + x)^{2n} - 2nx - 1, \quad \text{and}
\]

\[
D_k(B_n, x) = (1 + x)^{2n} - 1, \quad \text{for } k \neq 2.
\]

Proof. Clearly, the diameter of \(B_n\) is 3. Therefore for \(k \neq 2\), the proof is trivial. For \(k = 2\), it is clear that \(\gamma_2(B_n) = 2\) and for \(2 \leq i \leq n\), any subset of vertices of \(B_n\) of cardinality \(i\) is a distance-2 dominating set. Therefore \(D_2(B_n, x) = (1 + x)^{2n} - 2nx - 1\).

Corollary 16. Let \(B_n\) be the bipartite Cocktail party graph. Then for \(n \geq 3\), \(D(B_n^2, x) = (1 + x)^{2n} - 2nx - 1\).

Remark. \(B_1 = 2K_1\) and \(B_2 = 2K_2\), so \(D_2(B_1, x) = x^2\) and \(D_2(B_2, x) = x^2(x + 2)^2\).

The \(n\)-barbell graph \(B_{n,1}\) is the simple graph obtained by connecting two copies of complete graph \(K_n\) by a bridge.
Theorem 17. Let $B_{n,1}$ be $n$-barbell graph. Then for all $n$,

$$D_2(B_{n,1}, x) = (1 + x)^{2n} - 2(1 + x)^{n-1} + 1,$$

and

$$D_k(B_{n,1}, x) = (1 + x)^{2n} - 1, \quad \text{for } k \neq 2.$$

Proof. Clearly, the diameter of $B_{n,1}$ is 3. Therefore for $k \neq 2$, the proof is trivial. For $k = 2$, let $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of $B_{n,1}$ such that if $i \neq j$ every vertices $V$ are adjacent, every vertices $U$ are adjacent and $v_n$ and $u_n$ is adjacent. Then $\{v_n\}$ and $\{u_n\}$ are the only distance-2 domination sets of cardinality 1 of $B_{n,1}$. Therefore $\gamma_2(B_{n,1}) = 1$ and $d_2(B_{n,1}, 1) = 2$. For $2 \leq i \leq 2n$, a subset $S$ of vertices $B_{n,1}$ of cardinality $i$ is not a distance-2 domination set if either $S \subset V - \{v_n\}$ or $S \subset U - \{u_n\}$. Therefore, $d_2(B_{n,1}, i) = \left(\begin{array}{c} 2n \\ i \end{array}\right) - 2 \left(\begin{array}{c} n-1 \\ i \end{array}\right)$, for $2 \leq i \leq n-1$ and $d_2(B_{n,1}, i) = \left(\begin{array}{c} 2n \\ i \end{array}\right)$, for $n \leq i \leq 2n$. This implies that $D_2(B_{n,1}, x) = (1 + x)^{2n} - 2(1 + x)^{n-1} + 1$. \quad \square

Corollary 18. Let $B_{n,1}$ be $n$-barbell graph. Then for all $n$,

$$D(B_{n,1}^2, x) = (1 + x)^{2n} - 2(1 + x)^{n-1} + 1.$$

A bi-star graph $B_{(m,n)}$ is a tree obtained from the graph $K_2$ with two vertices $u$ and $v$ by attaching $m$ pendant edges in $u$ and $n$ pendant edges in $v$.

Theorem 19. Let $B_{(m,n)}$ be the bi-star graph. Then for all $m \leq n$,

$$D_2(B_{(m,n)}, x) = (1 + x)^{m+n+2} - (1 + x)^{m} - (1 + x)^{n} + 1,$$

and

$$D_k(B_{(m,n)}, x) = (1 + x)^{m+n+2} - 1, \quad \text{for } k \neq 2.$$
**Proof.** The proof is similar to the proof of the Theorem 17. \qed

**Corollary 20.** Let $B_{(m,n)}$ be the bi-star graph. Then for all $m \leq n$,

$$D(B_{(m,n)}^2, x) = (1 + x)^{m+n+2} - (1 + x)^n - (1 + x)^m + 1.$$  

The corona $H \circ G$ of two graphs $H$ and $G$ is the graph formed from one copy of $H$ and $|V(H)|$ copies of $G$, where the $i$-th vertex of $H$ is adjacent to every vertex in the $i$-th copy of $G$.

**Theorem 21.** If $K_m$ and $K_n$ be the complete graphs with $m$ and $n$ vertices, respectively. Then for $m \geq 2$,

$$D_2(K_m \circ K_n, x) = (1 + x)^{m(n+1)} - m(1 + x)^n + m - 1,$$  

and

$$D_k(K_m \circ K_n, x) = (1 + x)^{m(n+1)} - 1, \quad \text{for } k \neq 2.$$  

**Proof.** The proof is similar to the proof of the Theorem 17. \qed

**Corollary 22.** If $K_m$ and $K_n$ be the complete graphs with $m$ and $n$ vertices, respectively. Then for $m \geq 2$, $D((K_m \circ K_n)^2, x) = (1 + x)^{m(n+1)} - m(1 + x)^n + m - 1$.

Consider the graph $K_m$ and $m$ copies of $K_n$. The graph $Q(m, n)$ is obtained by identifying each vertex of $K_m$ with a vertex of a unique $K_n$.

**Corollary 23.** For $m \geq 2$, the distance-$k$ domination polynomial of $Q(m, n)$ is

$$D_2(Q(m, n), x) = (1 + x)^{mn} - m(1 + x)^{n-1} + m - 1,$$  

and

$$D_k(Q(m, n), x) = (1 + x)^{mn} - 1, \quad \text{for } k \neq 2.$$  

**Proof.** It follows from the fact that $Q(m, n)$ and $K_m \circ K_{n-1}$ are isomorphic.

**Corollary 24.** For $m \geq 2$, $D(Q^2(m, n), x) = (1 + x)^{mn} - m(1 + x)^{n-1} + m - 1$.  


3. Distance-\(k\) Domination Roots of Some Graphs

For distance-\(k\) domination polynomial of a graph, it is clear that \((0, \infty)\) is zero-free interval. Brouwer [1] has shown that the number of dominating set of any graph is odd. Thus by Theorem 3, we have the following theorem:

**Theorem 25.** For every graph \(G\) the number of distance-\(k\) dominating set is odd. That is, \(D_k(G, 1)\) is odd.

**Corollary 26.** Let \(G\) be graph. Then for every odd integer \(n\), \(D_k(G, n)\) is odd.

**Proof.** It follows from the fact that \(D_k(G, m) \equiv D_k(G, n) \mod 2\), for every odd integers \(m\) and \(n\).

**Corollary 27.** Every integer distance-\(k\) domination root of a graph is even.

**Theorem 28.** Let \(G\) be graph. Then zero is the only distance-\(k\) domination root of \(G\) if and only if \(G\) is a null graph.

**Theorem 29.** There is no connected graph with \(n\) vertices such that \(Z(D_k(G, x)) = \{0, -\frac{3 \pm \sqrt{5}}{2}\}\).

**Proof.** Let \(G\) be graph with \(n\) vertices such that \(Z(D_k(G, x)) = \{0, -\frac{3 \pm \sqrt{5}}{2}\}\). Then by Theorem 3, \(Z(D(G^k, x)) = \{0, -\frac{3 \pm \sqrt{5}}{2}\}\). Then \(G^k = H \circ 2K_1\) for some graph \(H\) by Theorem 6.4.1 in [7]. But \(G^k\) has no leaf except \(G = K_1, K_2\). So there is no graph \(H\) such that \(G^k = H \circ 2K_1\), this is a contradiction. Therefore \(Z(D_k(G, x)) = \{0, -\frac{3 \pm \sqrt{5}}{2}\}\). \(\square\)

**Theorem 30.** Suppose that \(a, b\) are rational numbers, \(r \geq 2\) is an integer that is not a perfect square, and \(a - |b|\sqrt{r} < 0\). Then \(-a - |b|\sqrt{r}\) can not be a distance-\(k\) domination root.
**Proof.** It follows from the fact that if $x = a + b\sqrt{r}$ is a root of some polynomial with integer coefficients, then so is $x^* = a - b\sqrt{r}$. \qed

**Corollary 31.** Let $b$ be a rational number, and let $r$ be a positive rational number such that $\sqrt{r}$ is irrational. Then $-|b|\sqrt{r}$ can not be a distance-$k$ domination root.

Let $\tau = \frac{1 + \sqrt{5}}{2}$ be the golden ratio. Next we will prove that $-\tau^n$ for odd $n$, can not be a distance-$k$ domination root. We need some relations between golden ratio $\tau$ and Fibonacci numbers $F_n$.

**Theorem 32** (see [8]). For every natural number $n$,

$$F_n = \frac{1}{\sqrt{5}} (\tau^n - (1 - \tau)^n).$$

**Corollary 33.** For every natural number $n$, $\frac{F_n}{F_{n-1}} < \tau$, if $n$ is even and $\frac{F_n}{F_{n-1}} > \tau$, if $n$ is odd.

**Theorem 34** (Cassini’s formula [8]). For every natural number $n$,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

**Theorem 35** (see [8]). For every $n \geq 2$, $\tau^n = F_n\tau + F_{n-1}$.

Now we ready to prove the following theorem:

**Theorem 36.** Let $n$ be an odd natural number. Then $-\tau^n$ can not be a distance-$k$ domination root.

**Proof.** For $n = 1$, it follows from the fact that $\frac{1 - \sqrt{5}}{2} < 0$. For odd $n \geq 2$,

$$\tau^n = F_n\tau + F_{n-1} = \left(\frac{F_n + 2F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right).$$
Let $G$ be a graph with $-\tau^n$ be a distance-$k$ domination root. Then,

$$D_k\left(G, -\left(\frac{F_n + 2F_{n-1}}{2}\right) - \left(\frac{\sqrt{5}F_n}{2}\right)\right) = 0.$$ 

Then,

$$D_k\left(G, -\left(\frac{F_n + 2F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right)\right) = 0.$$ 

But

$$-\left(\frac{F_n + 2F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right) = \left(\frac{F_{n+1} + F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right)$$

$$= \tau^{-1}F_n - F_{n-1}.$$ 

As $n$ is odd,

$$\frac{F_{n-1}}{F_{n-2}} < \tau < \frac{F_n}{F_{n-1}},$$

$$\frac{F_{n-1}}{F_n} < \tau^{-1} < \frac{F_{n-2}}{F_{n-1}},$$

$$F_{n-1} < \tau^{-1}F_n < \frac{F_{n-2}F_n}{F_{n-1}},$$

$$0 < \tau^{-1}F_n - F_{n-1} < \frac{F_{n-2}F_n - F_{n-1}^2}{F_{n-1}},$$

$$0 < \tau^{-1}F_n - F_{n-1} < \frac{1}{F_{n-1}},$$

$$\tau^{-1}F_n - F_{n-1} \in \left(0, \frac{1}{F_{n-1}}\right),$$

this is a contradiction. $\square$
Theorem 37. Let $G$ be a graph having $n$ vertices with diameter $d$. If $k \geq d$, then

1. $D_k(G, x)$ has only one nonzero real root, if $n$ is even.
2. $D_k(G, x)$ has no nonzero real root, if $n$ is odd.

Proof. The result follows from the transformation $y = 1 + x$. □

Theorem 38. Let $G$ be a graph with diameter $d$. If $k \geq d$, then all distance-$k$ domination roots of the graph $G$ are lie on the unit circle with center $(-1, 0)$.

Proof. It follows from the fact that all $n$-th roots of unity are lies on the unit circle centered at $(0, 0)$. □

Corollary 39. Let $G$ be a graph having $n$ vertices with diameter $d$. If $k \geq d$, then

1. $D_k(G, x)$ has no nonzero integer root, if $n$ is odd.
2. $-2$ is the only nonzero integer root of $D_k(G, x)$, if $n$ is even.

Corollary 40. All distance-$k$ domination roots of the complete graph $K_n$ are lie on the unit circle centered at $(-1, 0)$.

Corollary 41. All distance-$k$ domination roots of the complete $m$-partite graph $K_{n_1, n_2, \ldots, n_m}$ are lie on the unit circle centered at $(-1, 0)$.

Corollary 42. All distance-$k$ domination roots of the complete bipartite graph $K_{m, n}$ are lie on the unit circle centered at $(-1, 0)$.

Corollary 43. All distance-$k$ domination roots of the star graph $S_n$ are lie on the unit circle centered at $(-1, 0)$.

Corollary 44. All distance-$k$ domination roots of the wheel graph $W_n$ are lie on the unit circle centered at $(-1, 0)$. 
Corollary 45. For any two graphs $G_1$ and $G_2$, all distance-$k$ domination roots of the graph $G_1 + G_2$ are lie on the unit circle centered at $(-1, 0)$.

Corollary 46. For the complete graphs $K_m$ and $K_n$, all distance-$k$ domination roots of the graph $K_m \square K_n$ are lie on the unit circle centered at $(-1, 0)$.

Corollary 47. All distance-$k$ domination roots of the Petersen graph $P$ are lie on the unit circle centered at $(-1, 0)$.

Corollary 48. All distance-$k$ domination roots of the Dutch Windmill graph $G^n_3$ are lie on the unit circle centered at $(-1, 0)$.

Corollary 49. All distance-$k$ domination roots of the Lollipop graph of $L_{m, 1}$ are lie on the unit circle centered at $(-1, 0)$.

To locate the distance-$k$ domination roots of a graph, we need the following theorem.

Theorem 50 (see [10]). Let $f(z) = z^n + a_1z^{n-1} + \ldots + a_n$, where $a_i \in \mathbb{C}$. Then, inside the circle $|z| = 1 + \max_i |a_i|$, there are exactly $n$ roots of $f$, multiplicities counted.

Theorem 51. All distance-$k$ domination roots of the bipartite Cocktail party graph $B_n$ are lie inside the circle with center $(-1, 0)$ and radius $2n + 1$.

Proof. For $k \neq 2$, it follows from the fact that all distance-$k$ domination roots of the bipartite Cocktail party graph $B_n$ are lie on the unit circle centered at $(-1, 0)$. For $k = 2$, $D_2(B_n, y - 1) = y^{2n} - 2ny + 2n - 1$. Here $\max_i |\alpha_i| = 2n$. Then by Theorem 50 we have the result. □

Theorem 52. If $n \geq 3$, all nonzero distance-2 domination roots of the bipartite Cocktail party graph $B_n$ are complex.
Proof. We have $D_2(B_n, x) = (1 + x)^{2n} - 2nx - 1$. Put $y = 1 + x$, then $D_2(B_n, y - 1) = f(y) = y^{2n} - 2ny + 2n - 1$. Since the number of variations of the signs of the coefficients of $f(y)$ is 2, by Descartes Rule [10], it has at most two positive real roots. Clearly, $y = 1$ is a double root of $f(y)$. Since there is no variations of the signs of the coefficients of $f(-y)$, $f(y)$ has no negative real roots. This implies that the only real distance-2 domination root of the bipartite Cocktail party graph $B_n$ is zero.

Remark. $-2$ is a distance-2 domination root of the bipartite Cocktail party graph $B_2$ with multiplicity 2.

Theorem 53. All distance-$k$ domination roots of the $n$-barbell graph $B_{n,1}$ are lie inside the circle with center $(-1, 0)$ and radius 3.

Proof. The proof is similar to the proof of the Theorem 51. $\square$

Theorem 54. For $n \geq 2$, we have the following:

1. If $n$ is even, then $D_2(B_{n,1}, x)$ has only one nonzero real root.

2. If $n$ is odd, then $D_2(B_{n,1}, x)$ has exactly three nonzero real root and one of them is $-2$.

Proof. The proof is similar to the proof of the Theorem 52. $\square$

Corollary 55. For $n \geq 1$, we have the following:

1. If $n$ is even, then $D_2(B_{n,1}, x)$ has no nonzero integer root.

2. If $n$ is odd, then $-2$ is the only nonzero integer root of $D_2(B_{n,1}, x)$.

Remark. $-2$ is the only nonzero distance-2 domination root of the 1-barbell graph $B_{1,1}$.

Theorem 56. All distance-$k$ domination roots of the bi-star graph $B_{(m,n)}$ are lie inside the circle with center $(-1, 0)$ and radius 2.
Proof. The proof is similar to the proof of the Theorem 51.

Theorem 57. For $m \leq n$, we have the following:

(1) If $m$ and $n$ are even, then $D_2(B_{(m,n)}, x)$ has exactly three nonzero real root and one of them is $-2$.

(2) If $m$ and $n$ are odd, then $D_2(B_{(m,n)}, x)$ has exactly one nonzero real root.

(3) If $m$ and $n$ have opposite parity, then $D_2(B_{(m,n)}, x)$ has exactly two nonzero real roots and one of them is $-2$.

Proof. We have $D_2(B_{(m,n)}, y-1) = f(y) = y^{m+n+2} - y^n - y^m + 1$.

The proof of the existence of positive real roots, the proof is similar to the proof of Theorem 52. Now consider $f(-y)$. If $m$ and $n$ have same parity, the proof is similar to the proof of Theorem 52. So we need only consider the remaining two cases:

Case 1. If $m$ is odd and $n$ is even.

$f(-y) = -y^{m+n+2} - y^n + y^m + 1$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 1, by Descartes Rule, it has at most one negative real root. Clearly, $y = -1$ is a negative root of $f(y)$. Therefore $D_2(B_{(m,n)}, x)$ has exactly two nonzero real roots and one of them is $-2$.

Case 2. If $m$ is even and $n$ is odd.

$f(-y) = -y^{m+n+2} + y^n - y^m + 1$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 3, by Descartes Rule, it has at most three negative real roots. Clearly, $y = -1$ is a negative root of $f(y)$. Since the graphs in Case 1 and Case 2 are isomorphic, we can conclude that $D_2(B_{(m,n)}, x)$ has exactly two nonzero real roots and one of them is $-2$. 

\[\square\]
Corollary 58. For $m \leq n$, we have the following:

1. If $m$ and $n$ are even, then $-2$ is the only nonzero integer root of $D_2(B_{(m,n)}, x)$.

2. If $m$ and $n$ are odd, then $D_2(B_{(m,n)}, x)$ has no nonzero integer root.

3. If $m$ and $n$ have opposite parity, then $-2$ is the only nonzero integer root of $D_2(B_{(m,n)}, x)$.

Corollary 59. For $m \leq n$ and $k > 2$, we have the following:

1. If $m$ and $n$ have same parity, then $-2$ is the only nonzero integer root of $D_k(B_{(m,n)}, x)$.

2. Otherwise $D_k(B_{(m,n)}, x)$ has no nonzero integer root.

Theorem 60. All distance-$k$ domination roots of the corona $K_m \circ K_n$ are lie inside the circle with center $(-1, 0)$ and radius $m + 1$.

Proof. The proof is similar to the proof of the Theorem 51. \qed

Corollary 61. All distance-$k$ domination roots of the graph $Q(m, n)$ are lie inside the circle with center $(-1, 0)$ and radius $m + 1$.

Theorem 62. For $m \geq 2$ and $n \geq 1$, we have the following:

1. If $n$ is odd, then $D_2(K_m \circ K_n, x)$ has only one nonzero real root.

2. If $n$ is even and $m$ is odd, then $D_2(K_m \circ K_n, x)$ has exactly two nonzero real roots.

3. If $n$ and $m$ are even, then $D_2(K_m \circ K_n, x)$ has exactly three nonzero real roots and one of them is $-2$.

Proof. The proof is similar to the proof of the Theorem 57. \qed
Corollary 63. For $m \geq 2$ and $n \geq 1$, we have the following:

1. If $n$ is even, then $D_2(Q(m, n), x)$ has only one nonzero real root.

2. If $n$ and $m$ are odd, then $D_2(Q(m, n), x)$ has exactly two nonzero real roots.

3. If $n$ is odd and $m$ is even, then $D_2(Q(m, n), x)$ has exactly three nonzero real roots and one of them is $-2$.

Conjecture. The only nonzero integer distance-$k$ domination root is $-2$.

References


