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DISTANCE-*k* DOMINATION POLYNOMIAL OF SOME GRAPHS

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Abstract

In this paper, we study the distance-k domination number, distance-k domination polynomial and distance-k domination roots of some graphs.

1. Introduction

Let k be a positive integer and let $G(V, E)$ be a simple finite graph. A set $S \subseteq V$ is a distance-k dominating set if each vertex $v \in V - S$ is with in a distance k from some vertex of S. The distance-k domination number of G, denoted by $\gamma_k(G)$, is the minimum cardinality of the

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distance-k dominating sets in G. Let $\mathcal{D}_k(G, i)$ be the family of distance-k dominating sets of G with cardinality i and let $d_k(G, i) = |\mathcal{D}_k(G, i)|$. The polynomial

$$D_k(G, x) = \sum_{i=\gamma_k(G)}^{|V(G)|} d_k(G, i)x^i,$$

is defined as distance-k domination polynomial of G. A root of $D_k(G, x)$ is called a distance-k domination root of G. Set of all distance-k domination roots of G is denoted by $Z(D_k(G, x))$. It is easy to see that the distance-k domination polynomial is monic with no constant term. Consequently, 0 is a root of every distance-k domination polynomial whose multiplicity is equal to the distance-k domination number of the graph.

2. Distance-k Domination Polynomial of Some Graphs

When $k = 1$, this polynomial coincide with the domination polynomial $D(G, x)$. Throughout this paper, k denote a positive integer greater than one.

Theorem 1. *If G and H are isomorphic, then* $D_k(G, x) = D_k(H, x)$.

Remark. The converse of the above theorem is not true. There are numerous non-isomorphic graphs with the same distance-k domination polynomials.

Theorem 2. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$D_k(G, x) = D_k(G_1, x)D_k(G_2, x)\ldots D_k(G_m, x).$$

Proof. It suffices to prove this theorem for $m = 2$. For $l \geq \gamma_k(G)$, a distance-k dominating set of l vertices in G arises by choosing a distance-k dominating set of j vertices in G_1, for some $j \in \{\gamma_k(G_1), \gamma_k(G_1) + 1, \ldots, |V(G)|\}$, and a distance-$k$ dominating set of $l - j$ vertices of G_2. The number of ways of doing this over all $j = \gamma_k(G_1), \gamma_k(G_1) + 1, \ldots, |V(G)|$ is

exactly the coefficient of x^l in $D_k(G_1, x)D_k(G_2, x)$. Hence both side of the above equation have the same coefficient, so they are identical polynomial. □

For a positive integer k, the k-th power of a graph G is the graph with the same set of vertices as G and an edge between two vertices if and only if there is a path of length at most k between them, and that graph is denoted by G^k.

Theorem 3. *Let G be graph and let k be any positive integer, then* $D_k(G, x) = D(G^k, x)$.

Proof. It follows from the fact that every distance-k dominating set of G with cardinality i is exactly the dominating set of G^k with cardinality i. □

Theorem 4. *Let G be a graph having n vertices with diameter d. Then* $D_k(G, x) = (1 + x)^n - 1$ *if and only if* $k \geq d$.

Proof. Suppose $k \geq d$. Then all vertices of G are with in a distance k. This implies that, for $1 \leq i \leq n$, any subset of vertices of G of cardinality i is a distance-k dominating set. Therefore $D_k(G, x) = (1 + x)^n - 1$. Conversely, suppose that $D_k(G, x) = (1 + x)^n - 1$. Then $\gamma_k(G) = 1$ and $d_k(G, 1) = n$. This implies that all vertices of G are with in a distance k. Hence $k \geq d$. □

Corollary 5. *For any complete graph K_n,*

$$D_k(K_n, x) = (1 + x)^n - 1.$$

Corollary 6. *For any complete m-partite graph $K_{n_1, n_2, \ldots, n_m}$,*

$$D_k(K_{n_1, n_2, \ldots, n_m}, x) = (1 + x)^N - 1,$$

where $N = n_1 + n_2 + \ldots + n_m$.

Corollary 7. *For any complete bipartite graph* $K_{m,n}$,

$$D_k(K_{m,n}, x) = (1 + x)^{m+n} - 1.$$

Corollary 8. *For any star graph* S_n,

$$D_k(S_n, x) = (1 + x)^{n+1} - 1.$$

Corollary 9. *For any wheel graph* W_n,

$$D_k(W_n, x) = (1 + x)^n - 1.$$

If H and G are any two graphs, then $H + G$ is the graph obtained from $H \cup G$ by joining each vertex of H to every vertex of G.

Corollary 10. *For* $i = 1, 2,$ *let* G_i *be a graph of* n_i *vertices, then*

$$D_k(G_1 + G_2, x) = (1 + x)^{n_1+n_2} - 1.$$

If H and G are any two graphs, then the cartesian product $H \square G$ of H and G is a graph such that

- The vertex set of $H \square G$ is the cartesian product $V(H) \times V(G)$.

- Any two vertices (h, g) and (h', g') are adjacent if and only if either $h = h'$ and g is adjacent with g' in G or $g = g'$ and h is adjacent with h' in H.

Corollary 11. *For the complete graphs* K_m *and* K_n, $D_k((K_m \square K_n), x)$ $= (1 + x)^{mn} - 1$.

Corollary 12. *Let P be the Petersen graph, then*

$$D_k(P, x) = (1 + x)^{10} - 1.$$

The Dutch Windmill graph G_3^n is the graph obtained by selecting one vertex in each of n triangles and identifying them.

Corollary 13. *The distance-k domination polynomial of the Dutch Windmill graph G_3^n is*

$$D_k(G_3^n, x) = (1 + x)^{2n+1} - 1.$$

The Lollipop graph $L_{m,n}$ is the graph obtained by joining a complete graph K_m to a path P_n, with a bridge.

Corollary 14. *The distance-k domination polynomial of $L_{m,1}$ is*

$$D_k(L_{m,1}, x) = (1 + x)^{m+1} - 1.$$

The bipartite Cocktail party graph B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$.

Theorem 15. *Let B_n be the bipartite Cocktail party graph. Then for $n \geq 3$,*

$$D_2(B_n, x) = (1 + x)^{2n} - 2nx - 1, \quad \textit{and}$$

$$D_k(B_n, x) = (1 + x)^{2n} - 1, \qquad \textit{for } k \neq 2.$$

Proof. Clearly, the diameter of B_n is 3. Therefore for $k \neq 2$, the proof is trivial. For $k = 2$, it is clear that $\gamma_2(B_n) = 2$ and for $2 \leq i \leq n$, any subset of vertices of B_n of cardinality i is a distance-2 dominating set. Therefore $D_2(B_n, x) = (1 + x)^{2n} - 2nx - 1$. □

Corollary 16. *Let B_n be the bipartite Cocktail party graph. Then for $n \geq 3$, $D(B_n^2, x) = (1 + x)^{2n} - 2nx - 1$.*

Remark. $B_1 = 2K_1$ and $B_2 = 2K_2$, so $D_2(B_1, x) = x^2$ and $D_2(B_2, x) = x^2(x + 2)^2$.

The n-barbell graph $B_{n,1}$ is the simple graph obtained by connecting two copies of complete graph K_n by a bridge.

Theorem 17. *Let $B_{n,1}$ be n-barbell graph. Then for all n,*

$$D_2(B_{n,1}, x) = (1+x)^{2n} - 2(1+x)^{n-1} + 1, \quad \textit{and}$$

$$D_k(B_{n,1}, x) = (1+x)^{2n} - 1, \qquad \textit{for } k \neq 2.$$

Proof. Clearly, the diameter of $B_{n,1}$ is 3. Therefore for $k \neq 2$, the proof is trivial. For $k = 2$, let $V = \{v_1, v_2, \cdots, v_n\}$ and $U = \{u_1, u_2, \cdots, u_n\}$ be the vertices of $B_{n,1}$ such that if $i \neq j$ every vertices V are adjacent, every vertices U are adjacent and v_n and u_n is adjacent. Then $\{v_n\}$ and $\{u_n\}$ are the only distance-2 domination sets of cardinality 1 of $B_{n,1}$. Therefore $\gamma_2(B_{n,1}) = 1$ and $d_2(B_{n,1}, 1) = 2$. For $2 \leq i \leq 2n$, a subset S of vertices $B_{n,1}$ of cardinality i is not a distance-2 domination set if either $S \subset V - \{v_n\}$ or $S \subset U - \{u_n\}$. Therefore, $d_2(B_{n,1}, i) = \binom{2n}{i} - 2\binom{n-1}{i}$, for $2 \leq i \leq n-1$ and $d_2(B_{n,1}, i) = \binom{2n}{i}$, for $n \leq i \leq 2n$. This implies that $D_2(B_{n,1}, x) = (1+x)^{2n} - 2(1+x)^{n-1} + 1$. □

Corollary 18. *Let $B_{n,1}$ be n-barbell graph. Then for all n,* $D(B_{n,1}^2, x) = (1+x)^{2n} - 2(1+x)^{n-1} + 1$.

A bi-star graph $B_{(m,n)}$ is a tree obtained from the graph K_2 with two vertices u and v by attaching m pendant edges in u and n pendant edges in v.

Theorem 19. *Let $B_{(m,n)}$ be the bi-star graph. Then for all $m \leq n$,*

$$D_2(B_{(m,n)}, x) = (1+x)^{m+n+2} - (1+x)^n - (1+x)^m + 1, \quad \textit{and}$$

$$D_k(B_{(m,n)}, x) = (1+x)^{m+n+2} - 1, \qquad \textit{for } k \neq 2.$$

Proof. The proof is similar to the proof of the Theorem 17. □

Corollary 20. *Let* $B_{(m,n)}$ *be the bi-star graph. Then for all* $m \leq n$,

$$D(B^2_{(m,n)}, x) = (1+x)^{m+n+2} - (1+x)^n - (1+x)^m + 1.$$

The corona $H \circ G$ of two graphs H and G is the graph formed from one copy of H and $|V(H)|$ copies of G, where the i-th vertex of H is adjacent to every vertex in the i-th copy of G.

Theorem 21. *If* K_m *and* K_n *be the complete graphs with* m *and* n *vertices, respectively. Then for* $m \geq 2$,

$$D_2(K_m \circ K_n, x) = (1+x)^{m(n+1)} - m(1+x)^n + m - 1, \quad \textit{and}$$

$$D_k(K_m \circ K_n, x) = (1+x)^{m(n+1)} - 1, \qquad \textit{for } k \neq 2.$$

Proof. The proof is similar to the proof of the Theorem 17. □

Corollary 22. *If* K_m *and* K_n *be the complete graphs with* m *and* n *vertices, respectively. Then for* $m \geq 2$, $D((K_m \circ K_n)^2, x) = (1+x)^{m(n+1)} - m(1+x)^n + m - 1$.

Consider the graph K_m and m copies of K_n. The graph $Q(m, n)$ is obtained by identifying each vertex of K_m with a vertex of a unique K_n.

Corollary 23. *For* $m \geq 2$, *the distance-*k *domination polynomial of* $Q(m, n)$ *is*

$$D_2(Q(m, n), x) = (1+x)^{mn} - m(1+x)^{n-1} + m - 1, \quad \textit{and}$$

$$D_k(Q(m, n), x) = (1+x)^{mn} - 1, \qquad \textit{for } k \neq 2.$$

Proof. It follows from the fact that $Q(m, n)$ and $K_m \circ K_{n-1}$ are isomorphic.

Corollary 24. *For* $m \geq 2$, $D(Q^2(m, n), x) = (1+x)^{mn} - m(1+x)^{n-1} + m - 1$.

3. Distance-k Domination Roots of Some Graphs

For distance-k domination polynomial of a graph, it is clear that $(0, \infty)$ is zero-free interval. Brouwer [1] has shown that the number of dominating set of any graph is odd. Thus by Theorem 3, we have the following theorem:

Theorem 25. *For every graph G the number of distance-k dominating set is odd. That is, $D_k(G, 1)$ is odd.*

Corollary 26. *Let G be graph. Then for every odd integer n, $D_k(G, n)$ is odd.*

Proof. It follows from the fact that $D_k(G, m) \equiv D_k(G, n) \pmod 2$, for every odd integers m and n.

Corollary 27. *Every integer distance-k domination root of a graph is even.*

Theorem 28. *Let G be graph. Then zero is the only distance-k domination root of G if and only if G is a null graph.*

Theorem 29. *There is no connected graph with n vertices such that $Z(D_k(G, x)) = \{0, \frac{-3 \pm \sqrt{5}}{2}\}$.*

Proof. Let G be graph with n vertices such that $Z(D_k(G, x)) = \{0, \frac{-3 \pm \sqrt{5}}{2}\}$. Then by Theorem 3, $Z(D(G^k, x)) = \{0, \frac{-3 \pm \sqrt{5}}{2}\}$. Then $G^k = H \circ 2K_1$ for some graph H by Theorem 6.4.1 in [7]. But G^k has no leaf except $G = K_1, K_2$. So there is no graph H such that $G^k = H \circ 2K_1$, this is a contradiction. Therefore $Z(D_k(G, x)) = \{0, \frac{-3 \pm \sqrt{5}}{2}\}$. □

Theorem 30. *Suppose that a, b are rational numbers, $r \geq 2$ is an integer that is not a perfect square, and $a - |b|\sqrt{r} < 0$. Then $-a - |b|\sqrt{r}$ can not be a distance-k domination root.*

Proof. It follows from the fact that if $x = a + b\sqrt{r}$ is a root of some polynomial with integer coefficients, then so is $x^* = a - b\sqrt{r}$. □

Corollary 31. *Let b be a rational number, and let r be a positive rational number such that $\sqrt{r}$ is irrational. Then $-|b|\sqrt{r}$ can not be a distance-k domination root.*

Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Next we will prove that $-\tau^n$ for odd n, can not be a distance-k domination root. We need some relations between golden ratio τ and Fibonacci numbers F_n.

Theorem 32 (see [8])**.** *For every natural number n,*

$$F_n = \frac{1}{\sqrt{5}}(\tau^n - (1-\tau)^n).$$

Corollary 33. *For every natural number n, $\frac{F_n}{F_{n-1}} < \tau$, if n is even and $\frac{F_n}{F_{n-1}} > \tau$, if n is odd.*

Theorem 34 (Cassini's formula [8])**.** *For every natural number n,*

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

Theorem 35 (see [8])**.** *For every $n \geq 2$, $\tau^n = F_n\tau + F_{n-1}$.*

Now we ready to prove the following theorem:

Theorem 36. *Let n be an odd natural number. Then $-\tau^n$ can not be a distance-k domination root.*

Proof. For $n = 1$, it follows from the fact that $\frac{1-\sqrt{5}}{2} < 0$. For odd $n \geq 2$,

$$\tau^n = F_n\tau + F_{n-1} = \left(\frac{F_n + 2F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right).$$

Let G be a graph with $-\tau^n$ be a distance-k domination root. Then,

$$D_k\left(G, -\left(\frac{F_n + 2F_{n-1}}{2}\right) - \left(\frac{\sqrt{5}F_n}{2}\right)\right) = 0.$$

Then,

$$D_k\left(G, -\left(\frac{F_n + 2F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right)\right) = 0.$$

But

$$-\left(\frac{F_n + 2F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right) = \left(\frac{F_{n+1} + F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right)$$

$$= \tau^{-1}F_n - F_{n-1}.$$

As n is odd,

$$\frac{F_{n-1}}{F_{n-2}} < \tau < \frac{F_n}{F_{n-1}},$$

$$\frac{F_{n-1}}{F_n} < \tau^{-1} < \frac{F_{n-2}}{F_{n-1}},$$

$$F_{n-1} < \tau^{-1}F_n < \frac{F_{n-2}F_n}{F_{n-1}},$$

$$0 < \tau^{-1}F_n - F_{n-1} < \frac{F_{n-2}F_n - F_{n-1}^2}{F_{n-1}},$$

$$0 < \tau^{-1}F_n - F_{n-1} < \frac{1}{F_{n-1}},$$

$$\tau^{-1}F_n - F_{n-1} \in \left(0, \frac{1}{F_{n-1}}\right),$$

this is a contradiction. □

Theorem 37. *Let G be a graph having n vertices with diameter d. If $k \geq d$, then*

(1) $D_k(G, x)$ *has only one nonzero real root, if n is even.*

(2) $D_k(G, x)$ *has no nonzero real root, if n is odd.*

Proof. The result follows from the transformation $y = 1 + x$. □

Theorem 38. *Let G be a graph with diameter d. If $k \geq d$, then all distance-k domination roots of the graph G are lie on the unit circle with center* $(-1, 0)$.

Proof. It follows from the fact that all n-th roots of unity are lies on the unit circle centered at $(0, 0)$. □

Corollary 39. *Let G be a graph having n vertices with diameter d. If $k \geq d$, then*

(1) $D_k(G, x)$ *has no nonzero integer root, if n is odd.*

(2) -2 *is the only nonzero integer root of* $D_k(G, x)$, *if n is even.*

Corollary 40. *All distance-k domination roots of the complete graph K_n are lie on the unit circle centered at* $(-1, 0)$.

Corollary 41. *All distance-k domination roots of the complete m-partite graph $K_{n_1, n_2, \ldots, n_m}$ are lie on the unit circle centered at* $(-1, 0)$.

Corollary 42. *All distance-k domination roots of the complete bipartite graph $K_{m,n}$ are lie on the unit circle centered at* $(-1, 0)$.

Corollary 43. *All distance-k domination roots of the star graph S_n are lie on the unit circle centered at* $(-1, 0)$.

Corollary 44. *All distance-k domination roots of the wheel graph W_n are lie on the unit circle centered at* $(-1, 0)$.

Corollary 45. *For any two graphs* G_1 *and* G_2, *all distance-*k *domination roots of the graph* $G_1 + G_2$ *are lie on the unit circle centered at* $(-1, 0)$.

Corollary 46. *For the complete graphs* K_m *and* K_n, *all distance-*k *domination roots of the graph* $K_m \square K_n$ *are lie on the unit circle centered at* $(-1, 0)$.

Corollary 47. *All distance-*k *domination roots of the Petersen graph P are lie on the unit circle centered at* $(-1, 0)$.

Corollary 48. *All distance-*k *domination roots of the Dutch Windmill graph* G_3^n *are lie on the unit circle centered at* $(-1, 0)$.

Corollary 49. *All distance-*k *domination roots of the Lollipop graph of* $L_{m,1}$ *are lie on the unit circle centered at* $(-1, 0)$.

To locate the distance-k domination roots of a graph, we need the following theorem.

Theorem 50 (see [10]). *Let* $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, *where* $a_i \in \mathbb{C}$. *Then, inside the circle* $|z| = 1 + \max_i |a_i|$, *there are exactly n roots of f, multiplicities counted.*

Theorem 51. *All distance-*k *domination roots of the bipartite Cocktail party graph* B_n *are lie inside the circle with center* $(-1, 0)$ *and radius* $2n + 1$.

Proof. For $k \neq 2$, it follows from the fact that all distance-k domination roots of the bipartite Cocktail party graph B_n are lie on the unit circle centered at $(-1, 0)$. For $k = 2$, $D_2(B_n, y - 1) = y^{2n} - 2ny + 2n - 1$. Here $\max_i |a_i| = 2n$. Then by Theorem 50 we have the result. □

Theorem 52. *If* $n \geq 3$, *all nonzero distance-*2 *domination roots of the bipartite Cocktail party graph* B_n *are complex.*

Proof. We have $D_2(B_n, x) = (1+x)^{2n} - 2nx - 1$. Put $y = 1 + x$, then $D_2(B_n, y-1) = f(y) = y^{2n} - 2ny + 2n - 1$. Since the number of variations of the signs of the coefficients of $f(y)$ is 2, by Descartes Rule [10], it has at most two positive real roots. Clearly, $y = 1$ is a double root of $f(y)$. Since there is no variations of the signs of the coefficients of $f(-y)$, $f(y)$ has no negative real roots. This implies that the only real distance-2 domination root of the bipartite Cocktail party graph B_n is zero. □

Remark. -2 is a distance-2 domination root of the bipartite Cocktail party graph B_2 with multiplicity 2.

Theorem 53. *All distance-k domination roots of the n-barbell graph $B_{n,1}$ are lie inside the circle with center $(-1, 0)$ and radius* 3.

Proof. The proof is similar to the proof of the Theorem 51. □

Theorem 54. *For $n \geq 2$, we have the following*:

(1) *If n is even, then $D_2(B_{n,1}, x)$ has only one nonzero real root.*

(2) *If n is odd, then $D_2(B_{n,1}, x)$ has exactly three nonzero real root and one of them is* -2.

Proof. The proof is similar to the proof of the Theorem 52. □

Corollary 55. *For $n \geq 1$, we have the following*:

(1) *If n is even, then $D_2(B_{n,1}, x)$ has no nonzero integer root.*

(2) *If n is odd, then -2 is the only nonzero integer root of $D_2(B_{n,1}, x)$.*

Remark. -2 *is the only nonzero distance-*2 *domination root of the* 1-*barbell graph $B_{1,1}$.*

Theorem 56. *All distance-k domination roots of the bi-star graph $B_{(m,n)}$ are lie inside the circle with center $(-1, 0)$ and radius* 2.

Proof. The proof is similar to the proof of the Theorem 51. □

Theorem 57. *For* $m \leq n$, *we have the following*:

(1) *If* m *and* n *are even, then* $D_2(B_{(m,n)}, x)$ *has exactly three nonzero real root and one of them is* -2.

(2) *If* m *and* n *are odd, then* $D_2(B_{(m,n)}, x)$ *has exactly one nonzero real root*.

(3) *If* m *and* n *have opposite parity, then* $D_2(B_{(m,n)}, x)$ *has exactly two nonzero real roots and one of them is* -2.

Proof. We have $D_2(B_{(m,n)}, y-1) = f(y) = y^{m+n+2} - y^n - y^m + 1$. The proof of the existence of positive real roots, the proof is similar to the proof of Theorem 52. Now consider $f(-y)$. If m and n have same parity, the proof is similar to the proof of Theorem 52. So we need only consider the remaining two cases:

Case 1. If m is odd and n is even.

$f(-y) = -y^{m+n+2} - y^n + y^m + 1$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 1, by Descartes Rule, it has at most one negative real root. Clearly, $y = -1$ is a negative root of $f(y)$. Therefore $D_2(B_{(m,n)}, x)$ has exactly two nonzero real roots and one of them is -2.

Case 2. If m is even and n is odd.

$f(-y) = -y^{m+n+2} + y^n - y^m + 1$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 3, by Descartes Rule, it has at most three negative real roots. Clearly, $y = -1$ is a negative root of $f(y)$. Since the graphs in Case 1 and Case 2 are isomorphic, we can conclude that $D_2(B_{(m,n)}, x)$ has exactly two nonzero real roots and one of them is -2.

□

Corollary 58. *For* $m \leq n$, *we have the following*:

(1) *If* m *and* n *are even, then* -2 *is the only nonzero integer root of* $D_2(B_{(m,n)}, x)$.

(2) *If* m *and* n *are odd, then* $D_2(B_{(m,n)}, x)$ *has no nonzero integer root.*

(3) *If* m *and* n *have opposite parity, then* -2 *is the only nonzero integer root of* $D_2(B_{(m,n)}, x)$.

Corollary 59. *For* $m \leq n$ *and* $k > 2$, *we have the following*:

(1) *If* m *and* n *have same parity, then* -2 *is the only nonzero integer root of* $D_k(B_{(m,n)}, x)$.

(2) *Otherwise* $D_k(B_{(m,n)}, x)$ *has no nonzero integer root.*

Theorem 60. *All distance-*k *domination roots of the corona* $K_m \circ K_n$ *are lie inside the circle with center* $(-1, 0)$ *and radius* $m+1$.

Proof. The proof is similar to the proof of the Theorem 51. □

Corollary 61. *All distance-*k *domination roots of the graph* $Q(m, n)$ *are lie inside the circle with center* $(-1, 0)$ *and radius* $m+1$.

Theorem 62. *For* $m \geq 2$ *and* $n \geq 1$, *we have the following*:

(1) *If* n *is odd, then* $D_2(K_m \circ K_n, x)$ *has only one nonzero real root.*

(2) *If* n *is even and* m *is odd, then* $D_2(K_m \circ K_n, x)$ *has exactly two nonzero real roots.*

(3) *If* n *and* m *are even, then* $D_2(K_m \circ K_n, x)$ *has exactly three nonzero real roots and one of them is* -2.

Proof. The proof is similar to the proof of the Theorem 57. □

Corollary 63. *For* $m \geq 2$ *and* $n \geq 1$, *we have the following*:

(1) *If n is even, then* $D_2(Q(m, n), x)$ *has only one nonzero real root.*

(2) *If n and m are odd, then* $D_2(Q(m, n), x)$ *has exactly two nonzero real roots.*

(3) *If n is odd and m is even, then* $D_2(Q(m, n), x)$ *has exactly three nonzero real roots and one of them is* -2.

Conjecture. The only nonzero integer distance-k domination root is -2.

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