Abstract

In this paper, we propose an economic order quantity model of a retailer for deteriorating items with default risk consideration. The demand is assumed as a function of inventory level and credit period. The objective is to find the retailer's optimal credit period and cycle time while maximizing profit per unit time. For any given credit period, we first prove that the optimal cycle time not only exists, but is also unique in some conditions. Second, we show how the optimal credit period for any given replenishment cycle can be decided. Furthermore, we use the Lingo to obtain global maximum solutions to the optimal cycle time and the optimal credit period for the proposed model. Sensitivity analysis is conducted to provide some managerial insights.
Keywords: inventory, deteriorating items, trade credit, supply chain management.

1. Introduction

Along with the globalization of the market and increasing competition, the enterprises always take trade credit financing policy to promote sales, increase the market share, and reduce the current inventory levels. As a result, the trade credit financing play an important role in business as a source of funds after Banks or other financial institutions. Many researchers have studied inventory models in which the seller permits its buyers to delay payment without charging interest (i.e., trade credit). Goyal [3] was the first proponent for developing an EOQ model under the conditions of permissible delay in payments (i.e., credit period). Chu et al. [1] extended Goyal's model to consider deteriorating items. Jamal et al. [8] further generalized the EOQ model to allow for shortages. Teng [16] established an easy analytical closed-form solution to the problem. Huang [4] extended the trade credit problem to the case in which a supplier offers its retailer a credit period, and the retailer in turn provides another credit period to its customers. Liao [11] extended Huang's model to an EPQ model for deteriorating items. Teng [17] provided the optimal ordering policies for a retailer who offers distinct trade credits to its good and bad customers. Lately, Teng et al. [18] generalized traditional constant demand to non-decreasing demand. Several relevant articles related to this subject are Huang [5]; Kreng and Tan [9, 10]; Zhou et al. [24] and others.

The papers above discussed the EOQ or EPQ inventory models under trade credit financing based on the assumption that the demand rate is constant over time. However, in practice, the market demand is always changing rapidly and is affected by several factors such as price, time, inventory level, and delayed payment period. Some researchers realize
this phenomenon and extend their studies above to build the inventory models by assuming that the demand is variable. Chung and Liao [2] discuss the inventory replenishment problems with trade credit financing by considering a price-sensitive demand. Sarkar [14] and Teng et al. [18] build the economic quantity model with trade credit financing for time-dependent demand. Min et al. [13] develop an inventory model under conditions of permissible delay in payments, assuming that the items are replenished with the demand rate of the items dependent on the current inventory level. The inventory model with the credit-linked demand are discussed by Su et al. [15]; Jaggi et al. [6]; Jaggi et al. [7]. Thangam and Uthayakumar [20] discuss trade credit financing for perishable items in a supply chain when demand depends on both selling price and credit period. Lou and Wang [12] study optimal trade credit and order quantity by considering trade credit with a positive correlation of market sales, but are negatively correlated with credit risk. Teng et al. [19] discuss the optimal trade credit and lot size policies considering the demand and default risk sensitive to the credit period with learning curve production costs. Wu et al. [21] explore optimal credit period and lot size by considering demand dependence on delayed payment time with default risk for deteriorating items with expiration dates. Wu and Chan [22] analyse the lot sizing policies for deteriorating items with expiration dates and partial trade credit to credit risk customer by considering demand dependence on trade credit and default risk related to credit period. None of the paper discusses the optimal trade credit and order policy by considering demand dependence on trade credit and inventory level involving default risk.

In this paper, we explore the optimal order policy and the trade credit policy for deteriorating items with stock and credit period sensitive demand involving default risk. First, inventory model is established. Second, we show how to make optimal decisions based on the model. Furthermore, numerical examples and sensitivity analysis are conducted to provide some managerial insights.
2. Notation and Assumptions

The following notation and assumptions are used in the entire paper to develop the model:

\( A \) : The ordering cost per order in dollars.

\( h \) : The inventory holding cost per dollar per unit per year excluding interest charges.

\( c \) : The unit purchasing cost in dollars.

\( s \) : The unit selling price in dollars with \((s > c)\).

\( n \) : The credit period in years offered by the retailer to its customers (a decision variable).

\( T \) : The retailer’s replenishment cycle time in years (a decision variable).

\( I(t) \) : The inventory level in units at time \( t \).

\( \theta \) : The deterioration rate for the products, where \( 0 \leq \theta < 1 \).

\( D(t, n) \) : The annual market demand rate as a function of the inventory level and the trade credit period.

\( Q \) : The retailer’s economic order quantity in units, where \( Q = D(t, n)T \).

\( \Pi(n, T) \) : The annual total profit in dollars of inventory system, which is a function of \( n \) and \( T \).

Next, the mathematical models proposed in this paper are based on the following assumptions:

(1) The market demand faced by the retailer is increased with the current total inventory level and credit period. The two types of stock-dependent demand that are usually considered are linear form and power form. The credit-dependent demand is used in three forms, namely, linear form, polynomial or exponential form. In this paper, the demand
follows the form of \( D(t, n) = D + aI(t) + \lambda(n) \), where \( D > 0 \) is a scaling factor, \( I(t) \) is the current inventory level at time \( t(0 \leq t \leq T) \), \( a > 0 \) is a constant, governing the increasing rate of the demand with respect to the current inventory level; \( \lambda(N) > 0 \) reflects the demand increasing with the credit period.

(2) When the credit period to the customer is longer, the default risk to the retailer is higher. For simplicity, the rate of the default risk given the credit period \( n \) offered by the retailer is \( F(n) = 1 - e^{-bn} \), where \( b \) is a positive constant. This default risk pattern is used in some studies (Lou and Wang [12]; Zhang et al. [23]).

(3) The time to deterioration of a product follows an exponential distribution with parameter \( \theta \), i.e., the deterioration rate is a constant fraction of the on-hand inventory.

(4) Replenishment is instantaneous.

(5) In today’s time-based competition, we may assume that shortages are not allowed to occur.

(6) The time horizon is infinite.

3. Mathematical Model

Based on the above assumptions, the inventory system goes as follows. The retailer receives the order quantity \( Q \) at \( t = 0 \). Hence, the inventory starts with \( Q \) units at \( t = 0 \), and gradually reaches zero at \( t = T \) due to the combined influence of the demand and deterioration.

Therefore, inventory level \( I(t) \) with respect to time is governed by the following differential equation:

\[
\frac{dI(t)}{dt} + \theta I(t) = -D - aI(t) - \lambda(n), \quad 0 \leq t \leq T,
\]

\( (1) \)

with the boundary condition \( I(T) = 0 \).
By solving the differential equation (1), we can obtain
\[ I(t) = \frac{D + \frac{\lambda(n)}{\theta + \alpha}}{\theta + \alpha} \left( e^{(\theta + \alpha)(T - t)} - 1 \right), \quad 0 \leq t \leq T. \] (2)

Consequently, the retailer’s order quantity is
\[ Q = I(0) = \frac{D + \frac{\lambda(n)}{\theta + \alpha}}{\theta + \alpha} \left( e^{(\theta + \alpha)T} - 1 \right). \] (3)

The annual market demand is
\[ D(t, n) = \rho + \frac{\alpha \rho}{\eta} \left( e^{\eta(T - t)} - 1 \right), \text{ where } \eta = \theta + \alpha, \rho = D + \lambda(n). \] (4)

The retailer’s annual total profit consists of the following elements: net annual revenue after default risk, annual ordering cost, annual purchasing cost, and annual holding cost. The components are evaluated as follows:

(1) Annual ordering cost is \( \frac{A}{T} \).

(2) Annual purchasing cost is \( \frac{cQ}{T} = \frac{c \rho}{T \eta} \left( e^{\eta T} - 1 \right) \).

(3) Annual stock holding cost excluding the interest charges is
\[ c_h = \frac{h}{T} \int_0^T I(t) dt = \frac{h \rho}{T \eta^2} \left( e^{\eta T} - \eta T - 1 \right). \]

(4) The sales revenue considering default risk is
\[ R = \frac{s e^{-bn}}{T} \int_0^T D(t, n) dt = \frac{s e^{-bn}}{T \eta^2} \left[ T \eta + \alpha \left( e^{\eta T} - 1 \right) \right]. \]

Hence, the retailer’s annual total profit can be expressed as
\[ \Pi(n, T) = \text{net annual revenue after default risk} - \text{annual purchasing cost} - \text{annual ordering cost} - \text{annual holding cost} \]
\[ = s [1 - F(n)] \frac{1}{T} \int_0^T D(t, n) dt - \frac{cQ}{T} - \frac{A}{T} - \frac{h}{T} \int_0^T I(t) dt. \]
\[
\frac{\partial^2 \Pi(n, T)}{\partial T^2} = -\frac{2}{T^3}f(T|n) + \frac{1}{T^2}f'(T|n) = -\frac{2}{T^3}[f(T|n) - Tf'(T|n)].
\]

If \( \hat{T} \) is the root of \( \frac{\partial \Pi(n, T)}{\partial T} = 0 \) (this may or may not exist) and \( f'(\hat{T}) < 0 \), then

Therefore, the retailer’s objective is to determine the optimal solution \((n^*, T^*)\) such that \( \Pi(n, T) \) in (5) is maximized.

4. Theoretical Results and Optimal Solution

We first show the existence of a unique optimal solution value of \( T^* \) for any given \( n \), and then we decide the value of \( n^* \) for any known \( T \).

For any given \( n \geq 0 \), to find the optimal replenishment cycle time \( T^* \), we take the first-order partial derivative of \( \Pi(n, T) \) with respect to \( T \). We can obtain

\[
\frac{\partial \Pi(n, T)}{\partial T} = \frac{1}{T^2}f(T|n),
\]

where

\[
f(T|n) = A + \frac{spae^{-bn}}{\eta^2} - \frac{h \rho - cp \eta}{\eta \eta}(\eta T e^{-\eta T} - e^{-\eta T} + 1).
\]

It is easy to obtain

\[
f'(T|n) = [spae^{-bn} - h \rho - cp \eta]T e^{-\eta T}.
\]

Taking the second order partial derivative of \( \Pi(n, T) \) with respect to \( T \), we obtain

\[
\frac{\partial^2 \Pi(T|n)}{\partial T^2} = -\frac{2}{T^3}f(T|n) + \frac{1}{T^2}f'(T|n) = -\frac{2}{T^3}[f(T|n) - Tf'(T|n)].
\]
\[
\frac{\partial^2 \Pi(T|n)}{\partial T^2} = \frac{1}{T^2} f'(T|n) < 0.
\]

Based on Equation (7), if \( sa - h - c\eta < 0 \), then \( \frac{\partial^2 \Pi(T|n)}{\partial T^2} < 0 \). The retailer’s profit is a concave function in \( T \). Based on this condition, it is easy to obtain \( \lim_{T \to 0} f(T|n) = A > 0 \) and \( \lim_{T \to \infty} f(T|n) = -\infty < 0 \). Therefore, the intermediate value theorem yields that \( \hat{T} \) in \((0, \infty)\) not only exits, but is also unique. Given this condition, \( \lim_{T \to \infty} f(T|n) = -\infty \), \( \lim_{T \to 0} f(T|n) > 0 \).

Based on the analysis above, it is easy to obtain Theorem 1.

**Theorem 1.** For any given \( n \geq 0 \), if \( sa - h - c\eta < 0 \), \( \Pi(T|n) \) is a strictly concave function in \( T \), which is a unique maximum solution \( \hat{T} \) for \( T \in (0, \infty) \); the optimal value of \( T^* \) corresponds to \( \hat{T} \).

For any given \( T > 0 \), taking the first-order and second-order partial derivative of \( \Pi(n, T) \) with respect to \( n \), we can obtain

\[
\frac{\partial \Pi(n|T)}{\partial n} = \frac{se^{-bn}}{\eta^2 T} (\rho' - b\rho) \left[ \eta^2 T + \alpha (e^{\eta T} - 1 - \eta T) \right] - \frac{cp'}{T\eta} (e^{\eta T} - 1)
- \frac{hp'}{T\eta^2} (e^{\eta T} - 1 - \eta T);
\]

(9)

\[
\frac{\partial^2 \Pi(n|T)}{\partial n^2} = \frac{se^{-bn}}{\eta^2 T} (\rho'^2 - 2b\rho' - b^2 \rho) \left[ \eta^2 T + \alpha (e^{\eta T} - 1 - \eta T) \right] - \frac{cp'}{T\eta} (e^{\eta T} - 1)
- \frac{hp'}{T\eta^2} (e^{\eta T} - 1 - \eta T) - \frac{hp''}{T\eta^2} (e^{\eta T} - 1 - \eta T),
\]

(10)

where \( \rho' = \frac{d\lambda(n)}{dn} \) and \( \rho^* = \frac{d^2\lambda(n)}{dn^2} \).
It is easy to find that \( \Pi(n, T) \) is a continuous function of \( n \) in \( (0, \infty) \). To identify whether \( n \) is 0 or positive, we define the following discrimination term:

\[
\Lambda_n = \frac{\partial \Pi(n|T)}{\partial n} \bigg|_{n=0} = \frac{s}{\eta^2T} (\rho' - b\rho) \left[ \eta^2T + \alpha(e^nT - 1 - \eta T) \right] - \frac{cp' T}{T\eta^2} (e^nT - 1) - \frac{b\rho'}{T\eta^2} (e^nT - 1 - \eta T). \tag{11}
\]

If \( \frac{\partial^2 \Pi(n|T)}{\partial n^2} < 0 \), \( \Pi(n|T) \) is a strictly concave function in \( n \), hence exists a unique maximum solution \( \hat{n} \). If \( \Delta_n \leq 0 \), then \( \Pi(n|T) \) is maximized at \( n^* = 0 \); if \( \Delta_n > 0 \), \( \Pi(n|T) \) is maximized with \( n^* = \hat{n} > 0 \). Based on Equation (10), it is easy to obtain if \( \rho^* - 2b\rho' - b^2\rho \leq 0 \), \( \Pi(n, T) \) is strictly concave function in \( n \). Hence a unique maximum solution exists.

If \( \frac{\partial^2 \Pi(n|T)}{\partial n^2} \geq 0 \), then \( \Pi(n|T) \) is a convex function of \( n \). Therefore, the optimal solution of \( \Pi(n, T) \) is at one of the two boundary points (0 or \( \infty \)).

As \( \lim_{n \to \infty} \frac{\partial \Pi(n|T)}{\partial n} < 0 \), \( n = +\infty \) is not an optimal solution. This implies that the optimal solution is \( n^* = 0 \). Consequently, the following theoretical results can be derived.

**Theorem 2.** For any given \( T > 0 \), if \( -b^2\rho - 2b\rho' + \rho^* \leq 0 \), then

1. \( \Pi(n|T) \) is strictly concave function in \( n \), hence exists a unique maximum solution.

2. If \( \Delta_n \leq 0 \), then \( \Pi(n|T) \) is maximized at \( n^* = 0 \).
(3) If $\Delta_n > 0$, then there exists a unique $\hat{n} > 0$ such that $\prod(n|T)$ is maximized at $n^* = \hat{n} > 0$.

Now, it is time to present some numerical examples in the next section.

5. Numerical Examples

In this section, we provide several numerical examples to illustrate theoretical results as well as to gain some managerial insights.

Example 1. Let us assume $\lambda(n) = ke^{an}$, $k = 5$ / year, $a = 5$ / year, $\theta = 0.1$ / year, $s = $30 / unit, $c = $10 / unit, $b = 0.5$ / year, $A = $10 / order, $D = 100$ unit / year, $h = $5 / unit / year. By using software Mathematica 7.0, we have the optimal solution as follow:

$n^* = 1.7456$ years, $T^* = 0.0236$ years, and $\prod(n^*, T^*) = $26868.32.

Example 2. Using the same data as in Example 1, we study the sensitivity analysis on the optimal solution with respect to each parameter in appropriate unit. The computational results are shown in Table 1.
Table 1. Sensitivity analysis of the parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( n^* )</th>
<th>( T^* )</th>
<th>( \Pi^*(n, T) )</th>
<th>Parameter</th>
<th>( n^* )</th>
<th>( T^* )</th>
<th>( \Pi^*(n, T) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>10 1.7456</td>
<td>0.0236</td>
<td>26868.26</td>
<td>100 1.7456</td>
<td>0.0236</td>
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<tr>
<td></td>
<td>14 1.7444</td>
<td>0.0259</td>
<td>26574.32</td>
<td>150 1.7449</td>
<td>0.0235</td>
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<td></td>
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<td>0.0279</td>
<td>26320.54</td>
<td>200 1.7443</td>
<td>0.0235</td>
<td>27030.82</td>
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<tr>
<td></td>
<td>22 1.7424</td>
<td>0.0292</td>
<td>26114.42</td>
<td>250 1.7437</td>
<td>0.0236</td>
<td>27111.56</td>
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</tr>
<tr>
<td>( s )</td>
<td>30 1.7456</td>
<td>0.0236</td>
<td>26868.54</td>
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<td>0.0236</td>
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<td>0.0177</td>
<td>91352.82</td>
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<td></td>
<td>40 2.2192</td>
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<td></td>
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<td></td>
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<td>0.5 1.7453</td>
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<tr>
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<td>0.0236</td>
<td>26868.04</td>
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</table>

The sensitivity analysis reveals that:

1. If the value of \( A \) increases, then the optimal order cycle \( T^* \) increases while the values of \( n^* \) and the optimal profit \( \Pi^*(n, T) \) decrease. When the ordering cost is higher, the retailer orders the products in the longer replenishment period to reduce the order frequency. Thus, he pays less ordering cost.

2. If the value of \( D \) increases, then the values of \( n^* \) and the optimal order cycle \( T^* \) decrease while the value of the optimal profit \( \Pi^*(n, T) \) increases. When the initial market demand is greater, the retailer can make more profit.
(3) If the value of $\alpha$ increases, then the values of $n^*$ decrease, but the optimal order cycle $T^*$ and the value of the optimal profit $\Pi^*(n, T)$ increase. When the market demand is more sensitive to the inventory level, the retailer provides shorter delayed payment time and longer order cycle to make more profit.

(4) If the value of $h$ or $\theta$ increases, then the values of $n^*$, the optimal order cycle $T^*$, and the value of the optimal profit $\Pi^*(n, T)$ decreases. The retailer can adopt some measurements to reduce the deteriorating rate, the purchasing cost, or the holding cost to make more profit.

(5) If the value of $c$ or $b$ increases, then the values of $n^*$ and the optimal profit $\Pi^*(n, T)$ decrease while the value of the order cycle $T^*$ increases. When the unit purchasing price of the retailer is higher, then the retailer makes less profit. When the default risk of the customers is higher, then the retailer should offer a shorter delayed payment time to his customers. The retailer can take some measurements to reduce the default risk of the customer, such as by adopting the partial delayed payment policy, to make more profit.

(6) If the value of $s$, $k$ or $a$ increases, then the values of $n^*$ and the value of the optimal profit $\Pi(n, T)$ increase while the optimal order cycle $T^*$ decreases. When the sales price is higher, the retailer offers a shorter delayed payment time and longer order cycle for his customers to make more profit. When the market demand is more sensitive to the trade credit, the retailer should provide a longer delayed payment time to make more profit.

6. Conclusion

How to determine the retailer’s optimal credit period and optimal replenishment time for deteriorating items and considering demands that are dependent on stock and credit period involving default risk has received a very little attention by the researchers. In this paper, we have
formulated an EOQ model for the retailer to obtain its optimal credit period and cycle time by incorporating the following important and relevant facts: (1) the selling items are perishable such as fruits, fresh fishes, gasoline, photographic films, etc., (2) the demand is a function of inventory level and credit period involving default risk. The objective is to find optimal replenishment and trade credit policies while maximizing profit per unit time. For any given credit period, we first prove that the optimal replenishment policy not only exists, but is also unique in some conditions. Second, we show how the optimal credit period for any given replenishment cycle can be decided. Furthermore, we use the Lingo to obtain global maximum solutions to the optimal replenishment cycle and an optimal credit period for the proposed model. Sensitivity analysis is conducted to provide some managerial insights.

For future research, one can extend the mathematical model in several ways. For example, one immediate possible extension could be allowable shortages, quantity discounts, etc. Also, one can extend the fully trade credit policy to the partial trade credit policy in which a retailer requests its credit-risk customers to pay a fraction of the purchasing amount at the time of placing an order as a collateral deposit, and then grants a permissible delay on the rest of the purchase amount.

References


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