IJAMML 5:1 (2016) 25-42 September 2016
ISSN: 2394-2258
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/ijamml_7100121664

EIGENVALUES OF CAYLEY GRAPHS ON THE GENERALIZED DICYCLIC GROUP

Pedro Manuel Dominguez Wade

Department of Mathematics, Matanzas University, Cuba

Abstract

Let G be a finite group. In this paper, we introduce the notion of quasi-normal Cayley graph on G. We show that the eigenvalues of a quasi-normal Cayley graph can be computed as a sum of irreducible characters and eigenvalues of a Hermitian matrix. Moreover, descriptions are given of the spectrum of some quasi-normal Cayley graphs on the generalized dicyclic group.

Keywords: quasi-normal Cayley graph, eigenvalue, irreducible representation, generalized dicyclic group.

1. Introduction

Let G be a finite group and S be a subset of G, which is closed under inversion and does not contain the identity. The Cayley graph of G with respect to S, denoted by $\Gamma = Cay(G, S)$, is the graph whose vertices are

*Corresponding author.
E-mail address: pedroalgebralineal@gmail.com (Pedro Manuel Dominguez Wade).

2010 Mathematics Subject Classification: Primary 05C25; Secondary 20B25.
Submitted by Ali H. Hakami.
Received May 2, 2016; Revised July 21, 2016

the elements of G and two vertices g and $\overline{g}$ are adjacent if $g\overline{g}^{-1} \in S$. If S is closed under conjugation, then Γ is called a normal Cayley graph.

The eigenvalues of Γ are the eigenvalues of its adjacency matrix. If $\Gamma = Cay(G, S)$ is a normal Cayley graph, then, from [1], we get that the eigenvalues of Γ are given by

$$\lambda = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s), \tag{1.1}$$

where χ ranges over all irreducible characters of G.

Since all Cayley graphs on finite abelian groups are normal, we claim that the eigenvalues of these Cayley graphs can be computed applying the last result. Unfortunately, the Cayley graphs on nonabelian groups generally are not normal. Thus, we can deduce that to describe the spectrum of the Cayley graphs on nonabelian groups, it is a very hard problem.

Diaconis and Shahshahani [4] showed the direct connection that there is between the spectrum of any Cayley graph and the irreducible complex representations of the group. In other words, it was proved that the irreducible representations are effective tools in the study of the eigenvalues of any Cayley graph on a finite group. In this paper, we introduce the notion of quasi-normal Cayley graph on G. We show that the eigenvalues of a quasi-normal Cayley graph can be computed as a sum of irreducible characters and eigenvalues of a Hermitian matrix. Moreover, descriptions are given of the spectrum of some quasi-normal Cayley graphs on the generalized dicyclic group.

2. Preliminary Results

In this section, we mention some properties of the irreducible complex representations of finite groups. These properties are ones that will arise frequently in the problems that we are examining.

We say that a (linear) representation of G (over $\mathbb{C}$) is a group homomorphism

$$\rho : G \to GL(V).$$

In a situation where V is free as a $\mathbb{C}$-space, on taking a basis for V, we may write each element of $GL(V)$ as a matrix with entries in $\mathbb{C}$ and we obtain for each $g \in G$ a matrix $\rho(g)$. These matrices multiply together in the manner of the group and we have a matrix representation of G. In this situation, the dimension of the $\mathbb{C}$-space V is called the degree of the representation ρ.

If $\rho : G \to GL(V)$ is a representation of the finite group G, then V has the structure of a G-module. Let W be a subspace of V, which is closed under the action of G. In such case, we say that W is an invariant subspace of V. A representation ρ is said to be irreducible if V has no invariant subspaces other than 0 and V.

We note that the regular representation G_L can be write as a sum of irreducible representations.

An extremely important result, connected to the ordinary representations theory is the follows.

Proposition 2.1. *The number of irreducible representations of a finite group G is equal to the number of conjugacy classes of G.*

The following results are very important in this context.

Lemma 2.2. *Let G be a finite group with r conjugacy classes. Assume that* $n_1, \ldots, n_r$ *is a complete list of the degrees of the irreducible representations. Then we have*

$$|G| = \sum_{i=1}^{r} n_i^2. \tag{2.1}$$

Lemma 2.3 (Schur's lemma)**.** *Let* $\mathbb{C}G$ *be the group algebra of G over* $\mathbb{C}$ *and let* V_1 *and* V_2 *be irreducible* $\mathbb{C}G$*-modules. Then* $Hom_{\mathbb{C}G}(V_1, V_2) = 0$ *unless* $V_1 \cong V_2$, *in which case the endomorphism ring* $End_{\mathbb{C}G}(V_1) \cong \mathbb{C}$.

Assume that $\rho : G \to GL(V)$ is a finite-dimensional representation of G over the complex numbers $\mathbb{C}$. We define the character χ of ρ to be the function $\chi : G \to \mathbb{C}$ given by

$$\chi(g) = \mathrm{tr}(\rho(g)),$$

the trace of the linear map $\rho(g)$. We say that the representation ρ and the representation space V afford the character χ, and we may write χ_ρ or χ_V when we wish to specify this character more precisely.

Finally, we mention some results about the circulant matrices. An $m \times m$ circulant matrix is formed from any m-vector by cyclically permuting the entries.

Let ε be a primitive m-th root of unity. For each m-th root of unity $\varepsilon^j (j = 0, \ldots, m-1)$ an associated eigenvector is given by

$$v_j = (1, \varepsilon^j, \varepsilon^{2j}, \ldots, \varepsilon^{(m-1)j}).$$

The eigenvalues are given by

$$\lambda_{j+1} = \sum_{y=0}^{m-1} \alpha_{1,\, y+1} \varepsilon^{jy}, \quad j = 0, 1, \ldots, m-1, \tag{2.2}$$

where $\alpha_{1,\, y+1}$ is the $1(y+1)$-entry of the circulant matrix.

3. Eigenvalues of Cayley Graphs

In this section, we introduce the notion of quasi-normal Cayley graph and discuss some of their properties.

Let G be a finite group and $\Gamma = Cay(G, S)$ be a Cayley graph on G. Assume that $A(\Gamma)$ is the adjacency matrix. Let us write m for $|G|$. Since $A(\Gamma)$ is a symmetric matrix, we may assert that there exists a matrix $Q \in O_m(\mathbb{R})(O_m(\mathbb{R})$ is the orthogonal group consisting of $m \times m$ real matrices Q satisfying $QQ^T = I)$ such that

$$QA(\Gamma)Q^{-1} = D(\Gamma),$$

where $D(\Gamma)$ is a diagonal matrix. Recall that, in such case, we say that $A(\Gamma)$ and $D(\Gamma)$ are similar matrices.

We know that, for all $g \in G$, define $f_g : G \to G$ by $f_g(\overline{g}) = g\overline{g}$. Then $G_L = \{f_g : g \in G\}$ is itself a group isomorphic to G, which is called regular representation of G. We claim that each element $f_g \in G_L$ is a permutation, so can be represented by a unique m-th-order permutation matrix $G_L(g) = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1, & \text{if } gg_j = g_i (g_i, g_j \in G); \\ 0, & \text{otherwise.} \end{cases}$$

Since the equation $gx = g_i$ has unique solution, we claim that the i-th row (j-th column) of $G_L(g)$ has only one entry equal to 1. Moreover, we may see that if $g \in S$, then g_i and g_j are adjacency vertices. In such case, we observe that $g^{-1} = g_j g_i^{-1}$. Thus, we have

$$A(\Gamma) = \sum_{s \in S} G_L(s). \tag{3.1}$$

We know that there exists an invertible matrix C such that $CG_L(g)C^{-1} = D_L(g)$, being D_L the regular representation of G into block matrix form, and $g \in G$.

Let ρ_t be an irreducible representation of G. Then, we denote the matrix $\sum_{s\in S}\rho_t(s)$ by $\rho_t(S)$.

In [4] was proved the following result:

Lemma 3.1. *Let G be a finite group and $\Gamma = Cay(G, S)$ be a Cayley graph on G. We assume that $\rho_1, \ldots, \rho_r$ is a complete list of irreducible representations of G.*

(1) *The eigenvalues of Γ are the roots of the polynomials given by*

$$\det|\rho_t(S) - xI|, \tag{3.2}$$

where $\rho_t(S) = \sum_{s\in S}\rho_t(s)$ and $t = 1, \ldots, r$.

(2) *The multiplicity of the eigenvalue λ in Γ is given by*

$$\sum_{t=1}^{r} n_t m_t(\lambda), \tag{3.3}$$

where n_t is the degree of ρ_t and $m_t(\lambda)$ is the multiplicity of λ in $\rho_t(S)$.

The following lemma is easy but very useful to our main results.

Lemma 3.2. *Let G be a finite group and $\Gamma = Cay(G, S)$ be a Cayley graph on G. We assume that $\rho_1, \ldots, \rho_r$ is a complete list of irreducible representations of G. If ρ_t is a representation of degree 1, then $\lambda = \sum_{s\in S}\rho_t(s)$ is an eigenvalue of Γ.*

Proof. We know that $\sum_{s\in S}\rho_t(s) \in \mathbb{C}$, so the result follows from part (1) of Lemma 3.1. □

We now introduce the notion of quasi-normal Cayley graph.

Definition 3.3. Let G be the a finite group, and let $\Gamma = Cay(G, S)$ be a Cayley graph. Assume that $S_1 \subseteq S(S_1 \neq \emptyset)$ is closed under conjugation. Then Γ is called a quasi-normal Cayley graph.

Remark 3.4. Observe that every normal Cayley graph is a quasi-normal Cayley graph.

Theorem 3.5. *Let G be a finite group and $\Gamma(G, S)$ be a quasi-normal Cayley graph on G. We assume that $\rho_1, \ldots, \rho_r$ is a complete list of irreducible representations of G.*

(1) *The eigenvalues of Γ are given by*

$$\lambda = \frac{1}{\chi_{\rho_t}(1)} \sum_{s \in S_1} \chi_{\rho_t}(s) + \beta^{(t)} (t = 1, \ldots, r), \tag{3.4}$$

where χ_{ρ_t} is the character of ρ_t and $\beta^{(t)}$ ranges over the set of all the eigenvalues of the Hermitian matrix $\rho(S_2) = \sum_{s \in S_2} \rho_t(s) (S_2 = S - S_1)$.

(2) *The multiplicity of λ in Γ is given by*

$$m(\lambda) = \sum_{t=1}^{r} n_t m_t(\lambda), \tag{3.5}$$

where n_t is the degree of ρ_t and $m_t(\lambda)$ is the multiplicity of λ in $\rho_t(S) = \sum_{s \in S} \rho_t(s)$.

Proof.

(1) We may write

$$\rho_t(S) = \sum_{s \in S} \rho_t(s). \tag{3.6}$$

From (3.6), it follows that

$$\rho_t(S) = \rho_t(S_1) + \rho_t(S_2), \tag{3.7}$$

where $\rho_t(S_1) = \sum_{s \in S_1} \rho_t(s)$ and $\rho_t(S_2) = \sum_{s \in S_2} \rho_t(s)$, being $S_2 = S - S_1$. By assumption, we may assert that $\rho_t(S_2)$ is a Hermitian matrix. Let U be a

unitary matrix such that $U\rho_t(S_2)U^*(U^* = \overline{U}^T)$ is a diagonal matrix. From (3.7), we obtain:

$$U\rho_t(S)U^* = U\rho_t(S_1)U^* + U\rho_t(S_2)U^*. \tag{3.8}$$

Since S_1 is closed under conjugation and ρ_t is an irreducible representation of G, we deduce that $\rho_t(S_1)$ is an element of the center of a complete matrix ring over $\mathbb{C}$. Therefore, the equality $\rho_t(S_1) = \frac{1}{\chi_{\rho_t}(1)} \sum_{s \in S_1} \chi_{\rho_t}(s) I_{n_t}$ holds, which is what we need to prove.

(2) The assertion follows by part (2) of Lemma 3.1. □

4. Irreducibles Complex Representations of the Generalized Dicyclic Group

In this section, we will discuss some properties of the generalized dicyclic group and compute their irreducible complex representations.

Let $Dic(a, b) = \langle a, b : a^k = b^{\ln}, bab^{-1} = a^u, a^{dk} = b^{d\ln} = 1 \rangle$ be the finite group, where k, s, and u are integers with $k > 1$ and $m \geq 1$. The positive integer d is a divisor of $u - 1$ and l is the multiplicative order of u modulo dk. Let $j = \ln q + r'$, $0 \leq r' < \ln$. Observe that for all element $g = b^j a^i (0 \leq i \leq dk - 1, 0 \leq j \leq d\ln - 1)$ we have

$$g = b^j a^i = b^j a^i b^{-j} b^j = a^{u^j i} b^j = a^{u^j i} b^{\ln q + r'} = a^{u^j i + kq} b^{r'}.$$

Thus, for every element $g \in Dic(a, b)$, we may write $g = a^i b^j (0 \leq i \leq dk - 1, 0 \leq j \leq \ln - 1)$. Therefore, we may assert that $|Dic(a, b)| = dk\ln$. Such group is called generalized dicyclic group.

Remark 4.1. Observe that when $u = -1$ and $n = 1$, according to $d = 1$ or $d = 2$, the group is dihedral or general quaternion group.

Center of the group

We denote the center of the group by $Z(Dic(a, b))$. Let d^* be the greatest common divisor of k and $\frac{u-1}{d}$. In [5] was proved that

$$Z(Dic(a, b)) = \langle g \in Dic(a, b) : g = a^{\frac{k}{d^*}\beta} b^{l\delta} \rangle, \tag{4.1}$$

where $\beta = 0, \ldots, dd^* - 1$, $\delta = 0, \ldots, n-1$. Hence, we have

$$|Z(Dic(a, b))| = dd^* n. \tag{4.2}$$

Remark 4.2. Let $I_{nn}(Dic(a, b))$ be the inner automorphisms group of $Dic(a, b)$. Then, from (4.2), it follows that $|I_{nn}(Dic(a, b))| = k'l$, $k' = \frac{k}{d^*}$.

Commutator group

We will denote the commutator subgroup of $Dic(a, b)$ by $Dic(a, b)'$. Since $bab^{-1}a^{-1} = a^{u-1}$, it follows that

$$\langle a^{u-1} \rangle \subseteq Dic(a, b)'. \tag{4.3}$$

In order to prove the reverse inclusion. Let $g_1 = a^i b^j$ and $g_2 = a^{i'} b^{j'}$ be two elements in $D(a, b)$. Then we have

$$g_1 g_2 g_1^{-1} g_2^{-1} = a^{(u-1)[i'(u^{j'}+\cdots+1)-i(u^j+\cdots+1)]}.$$

Hence, we may assert that $g_1 g_2 g_1^{-1} g_2^{-1} \in \langle a^{u-1} \rangle$. Therefore,

$$Dic(a, b)' \subseteq \langle a^{u-1} \rangle. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$Dic(a, b)' = \langle a^{u-1} \rangle.$$

Lemma 4.3. (1) *The commutator quotient group* $\frac{Dic(a, b)}{Dic(a, b)'}$ *has order* dd^* ln.

(2) *The group* $Dic(a, b)$ *is solvable.*

Proof. (1) The result follows since $|Dic(a, b)'| = \frac{k}{d^*}$.

(2) We have $1 < Dic(a, b)' < Dic(a, b)$, i.e., $Dic(a, b)$ is resolvable of degree 2. So we are done. □

Proposition 4.4. *Let $Dic(a, b)$ be the generalized dicyclic group. Then, the commutator quotient group is cyclic if and only if $ba^x b^{-1} = a^{x+1}$, where x is the multiplicative inverse of $u-1$ modulo dk.*

Proof. If the commutator quotient is cyclic, then we have

$$Dic(a, b)' = \langle a^{u-1} \rangle = \langle a \rangle, \tag{4.5}$$

by assumption. From (4.5), we deduce that $d = d^* = 1$. Therefore, we may assert that $u-1$ is invertible modulo dk. Let x be the inverse of $u-1$ modulo k. Thus, we may write

$$ba^x b^{-1} = a^{ux} = a^{(u-1+1)x} = a^{x+1}.$$

Conversely, by assumption $ba^x b^{-1} = a^{ux} = a^{x+1}$. Thus, we may assert that $(u-1)x \equiv 1 \mod dk$ with $d = 1$. Therefore $\gcd(u-1, k) = 1$ and $D(a, b)' = \langle a \rangle = \langle a^{u-1} \rangle$, which proves what we want. □

Lemma 4.5. *Let $Dic(a, b)$ be the generalized dicyclic group and let $g = a^i b^j \in Dic(a, b)$ with $|g| = \overline{d}$. Assume that $\overline{d}^*$ is the multiplicative order of u modulo $\overline{d}$. We denote the conjugacy class of g by $g^{Dic(a, b)}$.*

$$|g^{Dic(a, b)}| = \begin{cases} \overline{d}^*, & \text{if } j = 0; \\ 1, & \text{if } j \neq 0 \text{ and } ds \equiv 0 \mod (\overline{d}); \\ \frac{k}{d^*}, & \text{if } j \neq 0 \text{ and } ds \not\equiv 0 \mod (\overline{d}). \end{cases}$$

Proof. We check tree cases.

Case 1. $j = 0$.

By assumption, we have $g = a^i (i = 1, \ldots, dk - 1)$. Let $a^{\bar{i}} b^{\bar{j}}$ be any element of $D(a, b)$. Therefore,

$$(a^{\bar{i}} b^{\bar{j}}) a^i (a^{\bar{i}} b^{\bar{j}})^{-1} = a^{\bar{i}} (b^{\bar{j}}) a^i b^{-\bar{j}}) a^{-\bar{i}}$$

$$= a^{u^{\bar{j}} i}.$$

Thus, we may write

$$(a^{\bar{i}} b^{\bar{j}}) a^i (a^{\bar{i}} b^{\bar{j}})^{-1} = a^{u^{\bar{j}} i}. \tag{4.6}$$

So the result follows from (4.6).

Case 2. $j \neq 0$ and $dn \equiv 0 \bmod (\bar{d})$.

From (4.2), we deduce that $g \in Z(Dic(a, b))$. Therefore, the assertion follows by assumption.

Case 3. $j \neq 0$ and $dn \not\equiv 0 \bmod (\bar{d})$.

Let $g' = a^{i'} b^{j'}$ be any element of $Dic(a, b)$. Then $g' g g'^{-1} = a^{i'(1-u^j)+u^j i} b^j$. Since

$$(1 - u^j) = -(u^j - 1)$$

$$= -(u - 1)(u^{j-1} + u^{j-1} + \cdots + 1),$$

we may write

$$g' g g'^{-1} = a^{i' dd^* k' + u^j i} b^j, \tag{4.7}$$

where $k' = \dfrac{u - 1}{dd^*}$.

Hence, the result follows from (4.7).

□

Irreducibles representations

Remark 4.6. Let $Dic(a, b)$ be the generalized dicyclic group. We denote the positive divisors of dk by d_j. Let d_j^* be the multiplicative order of u modulo d_j. On the set of the primitive d_j-th roots of unity, we define the following equivalence relation:

$$\varepsilon \equiv \varepsilon' \text{ if and only if } \varepsilon^{u^{i-1}} = \varepsilon', \text{ for some } i\,(1 \leq i \leq d_j^*).$$

The number of equivalent classes is given by $\dfrac{\varphi(d_j)}{d_j^*}$. We denote a set of representatives of these equivalent classes by $A_j = \left\{\varepsilon_{1j}, \ldots, \varepsilon_{\frac{\varphi(d_j)}{d_j^*}j}\right\}$.

Set $W_j = \{\omega_h \in C \mid \omega_h^{\ln} = \varepsilon_{\overline{m}j}^{k}, \varepsilon_{\overline{m}j} \in A_j\}$. On the set B_j, we define the following equivalent relation:

$$\omega_h \equiv \omega_{h'} \text{ if and only if } (\omega_h \omega_{h'}^{-1})^{d_j^*} = 1.$$

In this case, the number of equivalent classes is $\dfrac{\ln}{d_j^*}$. We denote a set of representatives of these equivalent classes by $\hat{W}_j = \left\{\omega_{1j}, \ldots, \omega_{\frac{\ln}{d_j^*}j}\right\}$.

We now give a description of the irreducible complex representations of $D(a, b)$.

Theorem 4.7. *Let* $Dic(a, b)$ *be the generalized dicyclic group. Assume that* V *is a* $\mathbb{C}$*-vector space of dimension* d_j^* *with basis* $X = \{x_1, \ldots, x_{d_j^*}\}$ *and an action of* $D(a, b)$ *given by*

$$ax_i = \varepsilon_{\overline{m}j}^{u^{i-1}} x_i,\ bx_1 = \omega_{\overline{n}j} x_{d_j^*},\ bx_i = \omega_{\overline{n}j} x_{i-1} (2 \le i \le d_j^*), \tag{4.8}$$

where $\varepsilon_j \in A_j$, $\omega_{\overline{n}j} \in \hat{W}_j$.

(1) V *is an irreducible* $\mathbb{C}D(a, b)$*-module.*

(2) *The number of non-equivalent irreducible representations is given by*

$$\sum_{j=1}^{\beta} \frac{\varphi(d_j)}{d_j^{2*}} \ln,$$

where β *equals the number of positive divisors of* dk.

Proof. Let us write G for $Dic(a, b)$.

(1) We may check that is indeed a representation of G by verifying that $a^k x = b^{\ln} x$, $a^{dk} x = b^{d \ln} x = x$, $bab^{-1} x = a^u x$ for all $x \in V$, which is immediate. We will show that V is an irreducible $\mathbb{C}G$-module. Thus, according to Lemma 2.3, we will prove that $End_{\mathbb{C}\mathbb{G}}(V) \cong \mathbb{C}$.

Let $\theta \in End_{\mathbb{C}\mathbb{G}}(V)$. Then, we may write

$$\theta(gx) = g(\theta(x)) \text{ for all } g \in G \text{ and } x \in V. \tag{4.9}$$

Combining (4.8) and (4.9), we may deduce that $\theta = \alpha I$, with $\alpha \in \mathbb{C}$, which proves what we want.

(2) Let d_j be a divisor of dk. Since $|A_j| = \dfrac{\varphi(d_j)}{d_j^*}$ and $|\hat{W}_j| = \dfrac{\ln}{d_j^*}$, from (4.8), we deduce that for each d_j can be computed $\dfrac{\varphi(d_j)\ln}{d_j^{2*}}$

representations of degree d_j^*. Let V_1 and V_2 be two of these representations. We prove that $V_1 \not\cong V_2$. Let $X = \{x_1, \ldots, x_{d_j^*}\}$ and $Y = \{y_1, \ldots, y_{d_j^*}\}$ be bases of V_1 and V_2, respectively. Assume that $\theta \in Hom_{\mathbb{C}G}(V_1, V_2)$. Therefore,

$$\theta(ax_i) = a\theta(x_i) \text{ and } \theta(bx_i) = b\theta(x_i) \text{ for all } x_i \in V_1. \tag{4.10}$$

Combining (4.8) and (4.10), we deduce that $\theta = 0$. Therefore, by Lemma 2.3, we obtain $V_1 \not\cong V_2$. Thus, we have

$$\begin{aligned}\sum_{j=1}^{\beta} \frac{\varphi(d_j)}{d_j^{2*}} \ln d_j^{2*} &= (\varphi(d_1) + \cdots + \varphi(d_\beta)) \ln \\ &= dk \ln \\ &= |G|.\end{aligned}$$

The result follows by Lemma 2.2. □

We now show one immediate consequence of the last theorem.

Corollary 4.8. *Let $Dic(a, b)$ be the generalized dicyclic group and d_j be a divisor of dk. Assume that ρ_t is an irreducible representation of degree d_j^* corresponding to d_j. Then, we may write*

(1)

$$\rho_t(a) = (\varepsilon^{u^{i-1}} \delta_{i\bar{j}})(i, \bar{j} = 1, \ldots, d_j^*) \text{ and } \rho_t(b) = \omega_{\overline{nj}}\sigma,$$

where ε is a primitive d_j-th roots of unity, $\delta_{y\hat{j}}$ is the Kronecker delta, and σ is the permutation matrix corresponding to the cycle $(12 \cdots d_j^*)$.

(2)

$$\chi_{\rho_t}(a) = \sum_{i=1}^{d_j^*} \varepsilon^{u^{i-1}} \text{ and } \chi_{\rho_t}(b) = \omega_{\overline{nj}}\mu,$$

where

$$\mu = \begin{cases} 1, & \text{if } dd^* \equiv 0 \bmod(d_j); \\ 0, & \text{if } dd^* \not\equiv 0 \bmod (d_j). \end{cases}$$

Proof. (1) Taking matrices for the representation ρ_t, the result follows from (4.8).

(2) The result follows by part (1). □

5. Eigenvalues of the Cayley Graphs on $D(a, b)$

In this section, we present our main results about the spectrum of Cayley graphs on $D(a, b)$.

Theorem 5.1. *Let $Dic(a, b)$ be the generalized dicyclic group, and let $\Gamma = Cay(Dic(a, b), S)$ be a quasi-normal Cayley graph such that $S_2 = \{s \in S | s = b^y, 0 \le y \le \ln - 1, y \not\equiv 0 \bmod(l)\}$. Assume that $\rho_1, \ldots, \rho_N$ is a complete list of irreducible representations of $D(a, b)$. Then, the eigenvalues of Γ are given by*

$$\lambda = \begin{cases} \sum_{s \in S} \chi_{\rho_t}(s), & \text{if } dd^* \equiv 0 \bmod(d_j); \\ \sum_{h=0}^{d_j^*-1} \alpha_{1h+1} \varepsilon^{\bar{j}h} (\bar{j} = 0, 1, \ldots, d_j^* - 1), & \text{if } dk \equiv 0 \bmod(d_j) \text{ and } d_j > dd^*. \end{cases}$$

Here ε is a primitive d_j^*-th root of unity and α_{1h+1} is the $1h+1$-entry of the matrix $\rho_t(S)$.

Proof. Let d_j be a positive divisor of dk. According to part (2) of Theorem 4.7, we can associate $\frac{\varphi(d_j)}{d_j^{*2}} \ln$ irreducible representations of degree d_j^* to d_j. Let ρ_t be an irreducible representation corresponding to d_j. We check two cases.

Case 1. $dd^* \equiv 0 \bmod(d_j)$.

In such case, we may assert that d_j is a divisor of $u-1$. Therefore, by assumption, it follows that ρ_t is a representation of degree 1. Thus, the result follows from Lemma 3.2.

Case 2. $dd^* \not\equiv 0 \bmod(d_j)$.

In this case, we claim that ρ_t is an irreducible representation of degree $d_j^* > 1$. By assumption, we may assert that $\rho_t(S_1) = \frac{1}{d_j^*} \sum_{s \in S_1} \chi_{\rho_t}(s) I_{d_j^*}$. From part (1) of Corollary 4.8, we deduce that $\rho_t(S_2)$ is a circulant matrix. Therefore $\rho_t(S) = \rho_t(S_1) + \rho_t(S_2)$ is a circulant matrix also. Thus, the result follows by (2.2). □

Theorem 5.2. *Let $Dic(a, b)$ be the generalized dicyclic group, and let $\Gamma = Cay(Dic(a, b), S)$ be a quasi-normal Cayley graph such that $S_2 = \{s \in S | s = a^x, 0 \le x \le dk-1, x \not\equiv 0 \bmod(d)\}$. Assume that $\rho_1, \ldots, \rho_N$ is a complete list of irreducible representations of $D(a, b)$. Then, the eigenvalues of Γ are given by*

$$\lambda = \begin{cases} \sum_{s \in S} \chi_{\rho_t}(s), & \text{if } dd^* \equiv 0 \bmod(d_j); \\ \frac{1}{d_j^*} \sum_{s \in S_1} \chi_{\rho_t}(s) + \rho_{ii}^{(t)}(S_2) (i = 1, \ldots, d_j^*), & \text{if } dk \equiv 0 \bmod(d_j) \text{ and } d_j > dd^*. \end{cases}$$

Here $\rho_{ii}^{(t)}(S_2)$ is the ii-entry of the matrix $\rho_t(S_2)$.

Proof. Let ρ_t be an irreducible representation corresponding to d_j. We check two cases.

Case 1. $dd^* \equiv 0 \bmod(d_j)$.

Proceeding as in Case 1 of Theorem 5.2, the result follows.

Case 2. $dd^* \not\equiv 0 \bmod(d_j)$.

In such case, we claim that ρ_t is an irreducible representation of degree $d_j^* > 1$. From part (1) of Corollary 4.8, we deduce that $\rho_t(S_2)$ is a real diagonal matrix. Therefore, we are done by Theorem 3.5.

Remark 5.3. Note that in the following theorem, Γ is any Cayley graph (i.e., Γ is not necessarily a quasi-normal Cayley graph).

Theorem 5.4. *Let $Dic(a, b)$ be the generalized dicyclic group with $l = 2$, and let $\Gamma = Cay(Dic(a, b), S)$ be a Cayley graph. Assume that $\rho_1, \ldots, \rho_N$ is a complete list of irreducible representations of $D(a, b)$. Then, the eigen-values of Γ are given by*

$$\lambda = \begin{cases} \sum_{s \in S} \chi_{\rho_t}(s), & \text{if } dd^* \equiv 0 \bmod(d_j); \\ \frac{1}{2} \sum_{s \in S_1} \chi_{\rho_t}(s) \pm \tau, & \text{if } dk \equiv 0 \bmod(d_j) \text{ and } d_j > dd^*. \end{cases}$$

Here τ is the absolute value of 12-entry of the matrix $\rho_t(S_2)$.

Proof. Firstly, we observe that if $s = a^x (0 \le x \le dk - 1)$, then $s \in S_1$. Thus, we deduce that $S_2 = \{s \in S | s = a^x b^y, 0 \le x \le dk - 1, 0 < y \le \ln - 1, y \not\equiv 0 \bmod l\}$. We know that d_j is a positive divisor of dk. We check two cases.

Case 1. $dd^* \equiv 0 \bmod(d_j)$.

Again the result follows proceeding as in Case 1 of Theorem 5.2.

Case 2. $dd^* \not\equiv 0 \bmod(d_j)$.

By assumption, the Cayley graph Γ is not necessarily quasi-normal. Hence, we make the assumption that $S_1 = \emptyset$, then $\frac{1}{2} \sum_{s \in S_1} \chi_{\rho_t}(s) = 0$. We claim that ρ_t is an irreducible representation of degree 2. From part (1)

of Corollary 4.8, we may assert that the diagonal entries of $\rho_t(S_2)$ are all zeros. Since $\rho_t(S_2)$ is a Hermitian matrix, we deduce that $x^2 - \tau^2$ is its characteristic polynomial, which proves what we want.

References

[1] L. Babai, Spectra of Cayley graphs, J. Combin. Theory B 2 (1979), 180-189; 29 (1977), 329-336.

[2] C. Y. Chao, On the classification of symmetric graphs with a prime number of vertices, Trans. Amer. Math. Soc. 158 (1971), 247-256.

[3] Y. Cheng and J. Oxley, On weakly symmetric graphs of order twice a prime, J. Combin. Theory Ser. B 42(2) (1987), 196-211.

[4] P. Diaconis and M. Shahshahani, Generating a random permutation with random transpositions, Z. Wahsch. Verw. Gebiete 57 (1981), 159-179.

[5] P. M. Dominguez Wade, Modular representations of the group MQ over K_m, Asian Journal of Mathematics, International Press 10(4) (2006), 665-677.

[6] B. Elspas and J. Turner, Graphs with circulant adjacency matrices, J. Combin. Theory Ser. B 9 (1970), 297-307.

[7] C. D. Godsil and G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.

[8] G. R. Robinson, R. Geoffrey and R. Staszewski, On the representation theory of π-separable groups, J. Algebra 119(1) (1988), 226-232.

[9] J.-P. Serre, Linear Representations of Finite Groups, Graduate Texts in Mathematics 42, Springer-Verlag, 1977.

■