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THE INTRODUCTION OF FI-$\overline{Z}$-LIFTING MODULES

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Abstract

In this note, we introduce FI-$\overline{Z}$-lifting modules and prove some properties of them. In particular, we show that if module R_R is FI-$\overline{Z}$-lifting, then R/I has a projective $\overline{Z}$-cover for every two sided ideal I of R.

1. Introduction

Throughout this paper, R will denote an arbitrary associative ring with identity and all modules will be unitary right R-modules. In [5], Talebi and Vanaja defined $\overline{Z}(M)$ as follows:

$$\overline{Z}(M) = \mathrm{Re}(M, \mathcal{S}) = \bigcap\{\mathrm{Ker}(g) \mid g \in \mathrm{Hom}(M, L),\ L \in \mathcal{S}\},$$

where $\mathcal{S}$ denotes the class of all small modules.

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They called M a *cosingular* (*noncosingular*) module if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$).

Let $M \in \text{Mod-}R$. We recall that A is a $\overline{Z}$*-coessential submodule* of B in M if $B/A \subseteq \overline{Z}(M/A)$. Recall that a submodule K of M is called *fully invariant* (denoted by $K \trianglelefteq M$) if $\lambda(K) \subseteq K$ for all $\lambda \in End_R(M)$. A module M is called $\overline{Z}$*-lifting* if for every submodule K of M, there is a decomposition $K = A \oplus B$, such that A is a direct summand of M and $B \subseteq \overline{Z}(M)$.

We mainly study FI-$\overline{Z}$-lifting modules in this paper. We call M is *FI-*$\overline{Z}$*-lifting* if for every fully invariant submodule K of M, there is a decomposition $K = A \oplus B$, such that A is a direct summand of M and $B \subseteq \overline{Z}(M)$. In this note, we show that FI-$\overline{Z}$-lifting modules are closed under finite direct sums. We prove that if module R_R is FI-$\overline{Z}$-lifting, then R/I has a projective $\overline{Z}$-cover for every two sided ideal I of R.

2. FI-$\overline{Z}$-Lifting Modules

Lemma 2.1 (See [3, Lemma 1.1]). *Let M be a module. Then*:

(1) *Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule of M* (*in fact, the fully invariant submodules form a complete modular sublattice of the lattice of submodules of M*).

(2) *If* $X \subseteq Y \subseteq M$ *such that Y is a fully invariant submodule of M and X is a fully invariant submodule of Y, then X is a fully invariant submodule of M.*

(3) *If* $M = \bigoplus_{i \in I} X_i$ *and S is a fully invariant submodule of M, then* $S = \bigoplus_{i \in I} \pi_i(S) = \bigoplus_{i \in I}(X_i, \cap S)$, *where* π_i *is the i-th projection homomorphism of M.*

(4) *If* $X \subseteq Y \subseteq M$ *such that X is a fully invariant submodule of M and* Y/X *is a fully invariant submodule of* M/X, *then Y is a fully invariant submodule of M.*

We note that if $M = \bigoplus_{i=1}^{n} M_i$ and *N* is a fully invariant submodule of *M*, then $N = \bigoplus_{i=1}^{n}(N \cap M_i)$ and $N \cap M_i$ is a fully invariant submodule of M_i.

Lemma 2.2 (See [7, Lemma 3.1])**.** *Let N be a module. Then the following are equivalent*:

(a) *For every submodule K of N, there is a decomposition* $K = A \oplus B$, *such that A is a direct summand of N and* $B \subseteq \overline{Z}(N)$.

(b) *For every submodule K of N, there is a direct summand A of N such that* $A \subseteq K$ *and* $K / A \subseteq \overline{Z}(N/A)$.

(c) *For every submodule K of N, there is a decomposition* $N = A \oplus B$ *such that* $A \subseteq K$ *and* $B \cap K \subseteq \overline{Z}(N)$.

A module *N* is called $\overline{Z}$*-lifting* if it satisfies one of the equivalent conditions of Lemma 2.2. It is clear that every semisimple module is $\overline{Z}$-lifting.

A module *N* is called $\overline{Z}$*-hollow* if for every $A \leq N$, $A \leq \overline{Z}(N)$. It is obvious that every $\overline{Z}$-hollow module is $\overline{Z}$-lifting but it is not lifting. If every simple submodule is small, then every $\overline{Z}$-lifting module is lifting and if every small submodule is simple, then every lifting module is $\overline{Z}$-lifting.

Proposition 2.3. *Let N be a module. The following are equivalent*:

(1) *For every fully invariant submodule K of N, there is a decomposition* $K = A \oplus B$, *such that A is a direct summand of N and* $B \subseteq \overline{Z}(N)$.

(2) *For every fully invariant submodule K of N, there is a direct summand A of N such that* $A \subseteq K$ *and* $K/A \subseteq \overline{Z}(N/A)$.

(3) *For every fully invariant submodule K of N, there is a decomposition* $N = A \oplus B$ *such that* $A \subseteq K$ *and* $B \cap K \subseteq \overline{Z}(N)$.

Proof. (1) $\Rightarrow$ (2) Let K be a fully invariant submodule of N. By hypothesis, there exists a direct summand A of N and $B \subseteq \overline{Z}(N)$ such that $K = A \oplus B$. Now $N = A \oplus A'$ for some submodule A' of N. Consider the natural epimorphism $\pi : N \to N/A$. Then $\pi(B) = (B + A)/A = K/A \subseteq \overline{Z}(N/A)$. Therefore N is FI-$\overline{Z}$-lifting module.

(2) $\Rightarrow$ (3) By [7, Lemma 3.1].

(3) $\Rightarrow$ (1) Let K be a fully invariant submodule of N. By hypothesis, there is a decomposition $N = A \oplus B$ such that $A \subseteq K$ and $B \cap K \subseteq \overline{Z}(N)$. Therefore $K = A \oplus (K \cap B)$, as required. □

A module N is called *FI-$\overline{Z}$-lifting* if it satisfies one of the equivalent conditions of Proposition 2.3. Clearly, semisimple modules and $\overline{Z}$-lifting modules are FI-$\overline{Z}$-lifting.

Theorem 2.4. *Let* $N = \bigoplus_{i=1}^{n} N_i$ *be a direct sum of FI-$\overline{Z}$-lifting modules. Then N is FI-$\overline{Z}$-lifting.*

Proof. Let $K \trianglelefteq N$. Then $K = \bigoplus_{i=1}^{n}(K \cap N_i)$ and $K \cap N_i$ is a fully invariant submodule of N_i. As each N_i is FI-$\overline{Z}$-lifting we have $K \cap N_i = A_i \oplus B_i$, where A_i is a direct summand of N_i and $B_i \subseteq \overline{Z}(N_i)$. Put $A = \bigoplus_{i=1}^{n} A_i$ and $B = \bigoplus_{i=1}^{n} B_i$. Then $K = A \oplus B$, where A is a direct summand of N and $B = \bigoplus_{i=1}^{n} B_i \subseteq \bigoplus_{i=1}^{n} \overline{Z}(N_i) = \overline{Z}(\bigoplus_{i=1}^{n} N_i) = \overline{Z}(N)$. □

Corollary 2.5. *If N is a finite direct sum of* $\overline{Z}$*-lifting modules, then N is FI-*$\overline{Z}$*-lifting.*

Let N be a module. We call an epimorphism $f : P \to N$ a projective $\overline{Z}$-cover of N if P is projective and $\mathrm{Ker}(f) \subseteq \overline{Z}(P)$.

Theorem 2.6. *Let P be a projective module. If P is FI-*$\overline{Z}$*-lifting, then* P/A *has a projective* $\overline{Z}$*-cover for every fully invariant submodule A of P.*

Proof. Suppose P is a projective FI-$\overline{Z}$-lifting module and A is a fully invariant submodule of P. Then $A = X \oplus S$, where X is a direct summand of P and $S \subseteq \overline{Z}(P)$. Suppose $P = X \oplus Y$. As $S \subseteq \overline{Z}(P)$, $(X + S)/X \subseteq (X + \overline{Z}(P))/X \subseteq \overline{Z}(P/X)$. Hence, the natural map $f : P/X \to P/(X + S) = P/A$ is a projective $\overline{Z}$-cover. □

Corollary 2.7. *Suppose R is a ring. If module* R_R *is FI-*$\overline{Z}$*-lifting, then* R/I *has a projective* $\overline{Z}$*-cover for every two sided ideal I of R.*

Proposition 2.8. *Let N be a FI-*$\overline{Z}$*-lifting module. Then every fully invariant submodule of* $N/\overline{Z}(N)$ *is a direct summand.*

Proof. Let $K / \overline{Z}(N)$ be a fully invariant submodule of $N/\overline{Z}(N)$. Then K is a fully invariant submodule by Lemma 2.1. By hypothesis, there is a decomposition $N = N_1 \oplus N_2$ such that $N_1 \subseteq K$ and $K \cap N_2 \subseteq \overline{Z}(N)$. Thus $N/\overline{Z}(N) = (K/\overline{Z}(N)) \oplus ((N_2 + \overline{Z}(N))/\overline{Z}(N))$, as required. □

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